Mathematical Sciences (2018) 12:157–161
https://doi.org/10.1007/s40096-018-0254-4

ORIGINAL PAPER

Constructing signed strongly regular graphs via star complement technique

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Received: 12 March 2018 / Accepted: 13 June 2018 / Published online: 2 July 2018

Abstract
We consider signed graphs, i.e., graphs with positive or negative signs on their edges. The notion of signed strongly regular graph is recently defined by the author (Signed strongly regular graphs, Proceeding of 48th Annual Iranian Mathematical Conference, 2017). We construct some families of signed strongly regular graphs with only two distinct eigenvalues. The construction is based on the well-known method known as star complement technique.

Keywords Signed graphs · Eigenvalues · Signed strongly regular graphs

Mathematics Subject Classification 05C50

Introduction

A simple graph $G = (V, E)$ together with an assignment of $\{1, -1\}$ to its edges is called a *signed graph*, and will be denoted by (G, Σ), where Σ is the set of negative edges. For two vertices u, v in a signed graph by $u \nsim v$, $u \sim^+ v$, or $u \sim^- v$, we mean there is no edge, positive edge, or negative edge between them, respectively. The *adjacency matrix*, A^s of the signed graph (G, Σ) on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, is an $n \times n$ matrix whose entries are defined as below.

$$A^s(i,j) = \begin{cases} 1, & v_i \sim^+ v_j; \\ -1, & v_i \sim^- v_j; \\ 0, & v_i \nsim v_j. \end{cases}$$

Note that any symmetric $(0, \pm 1)$-matrix A with zero entries on the diagonal can be considered as a signature of the graph with adjacency matrix $|A|$. Some times we use the symmetric matrix A instead of the signature which is in its correspondence. The *spectrum* of a signed graph is the eigenvalues of its adjacency matrix. In a signed graph (G, Σ) by *resigning* at a vertex $v \in V(G)$, we mean multiplying the signs of all the edges incident to v by -1. Two signed graphs (G, Σ) and (G, Σ') are called *equivalent* if one is obtained from the other by a sequence of resigning around prescribed vertices. Otherwise, we call them *distinct*. Most properties of signed graphs specially the spectrum are invariant in equivalent signed graphs. In [6], it is proved that two signed graphs (G, Σ) and (G, Σ') are equivalent if and only if the symmetric difference of Σ and Σ' is an edge cut of the graph G. Recently, several studies have been done on the spectrum of signed graphs. We may refer to [1] where the author has determined the signed graphs sharing the spectrum with the path graph. A k-regular graph G with n vertices is said to be strongly regular if there are also integers λ and μ such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. A graph of this kind is denoted by $SRG(n, k, \lambda, \mu)$. The notion of signed strongly regular graphs is defined in [4] recently. A *signed strongly regular graph* is a signed k-regular graph (G, Σ) if there are also integers λ^s and μ^s such that the following conditions hold.

- For any two adjacent vertices u, v, the following hold,
$$\sum_{z \in N(u) \cap N(v)} s(zu)s(zv) = \lambda^s s(uv).$$
- For any two non-adjacent vertices u, v,
$$\sum_{z \in N(u) \cap N(v)} s(zu)s(zv) = \mu^s, -\mu^s.$$

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A signed strongly regular graph with corresponding parameters is denoted by $SRG^s(n, k, \lambda^s, \mu^s)$. In [4], several families of signed strongly regular graphs are constructed. In this paper, we apply the well-known star complement technique and Kronecker product of matrices to construct infinitely many signed strongly regular graphs. We recall the star complement technique from [5]. Let A be an $n \times n$ matrix, with an eigenvalue μ of multiplicity k. A k-subset X of $1, 2, \ldots, n$ is called a star set if the matrix obtained from A by removing rows and columns corresponding to X does not have μ as an eigenvalue. In graph theory context, a star set for an eigenvalue μ in G is a subset X of vertices such that μ is not an eigenvalue of $G\backslash X$. The graph $G\backslash X$ is called a star complement for μ in G. Note that the same technique can be applied for the signed graphs as the only required property is the symmetry in the adjacency matrix of graphs. During the preparation of this paper, we encounter interesting results from other combinatorial areas such as regular two graphs, Hadamard matrices, partial Hadamard matrices, conference matrices, and orthogonal codes.

Preliminaries

In this section, we present some preliminary results which will be used through out the paper. For an $m \times n$ matrix A, and a $p \times q$ matrix B, the Kronecker product $A \otimes B$ is defined to be the following $mp \times nq$ block matrix.

$$A \otimes B = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

It is well known that any $n \times n$ symmetric matrix has n real eigenvalues. If $\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_k^{m_k}$ are the eigenvalues of a symmetric matrix considering their multiplicity, then the minimal polynomial of it will be $M(A, x) = \prod_{i=1}^{k}(x - \lambda_i)$. Therefore, any symmetric matrix having only two distinct eigenvalues has a quadratic minimal polynomial, say $x^2 + ax + b = 0$. In the following proposition, we prove that any signed graph with just two distinct eigenvalues is in fact signed strongly regular graphs.

Theorem A *Suppose the signed graph (G, Σ) has just two distinct eigenvalues, then the graph G is a signed strongly regular graph with $\mu^s = 0$.*

Proof The entries in diagonal of the matrix $(A^s)^2$ are the vertex degrees of G. On the other hand if α, β are the two distinct eigenvalues of (G, Σ), then A^s provides the equality $(A^s)^2 - (\alpha + \beta)A^s + \alpha\beta I = O$. The matrix A^s has zero diagonal; hence, the main diagonal entries of $(A^s)^2$ (and therefore the vertex degrees of G) are equal to $-\alpha\beta$, and this implies the regularity of G. Any off diagonal entry in place (i, j) of $(A^s)^2$ equals $\sum_{k=1}^{n} A^s(i,k)A^s(k,j)$, by the ordinary matrix multiplication rule. Hence, it equals $\sum_{z \in N(u) \cap N(v)} s(zu)s(zv) = \mu^s$. But from the above quadratic equation for A^s, the entry (i, j) of $(A^s)^2$ must be equal to $-(\alpha + \beta)A^s_{ij}$; therefore, the summation is zero if the vertices v_i and v_j are not adjacent and $-(\alpha + \beta)s(v_i v_j)$ if they are adjacent. Therefore, (G, Σ) is an $SRG^s(n, k, -(\alpha + \beta), 0)$ as desired. □

In [3], the author has introduced several families of signed graphs with only two distinct eigenvalues. As an example, signed complete graphs with only two distinct eigenvalues have been mentioned. For more details about two graphs, see [2]. As an other family of signed strongly regular graphs, which is mentioned in [4], the line graph of the complete graph, say $L(K_n)$, has been considered. We recall the construction from [4]. Let the vertices of the complete graph K_n are labeled with $\{1, 2, \ldots, n\}$. It is known that the graph $L(K_n)$ is an $SRG\left(\binom{n}{2}, 2n-4, n-2, 4\right)$. The vertices of the graph $L(K_n)$ are the $\binom{n}{2}$ two element sets $\{a, b\}$, where $a < b$, considering the following signature Σ on the graph $L(K_n)$, which actually make it a signed strongly regular graph with parameters $SRG^s\left(\binom{n}{2}, 2n-4, n-3, 0\right)$.

The edge between the vertices $\{a, b\}$ and $\{c, d\}$ is

- Positive if $a = c$ or $b = d$.
- Negative if $a = d$ or $b = c$.

Main result

Here using the well-known method, i.e., star complement technique, we construct signed strongly regular graphs on 6 and 8 vertices; then, by the Kronecker product of matrices, we introduce some signed strongly regular graphs of larger order. We recall the star complement technique from [5]. Let A be an $n \times n$ matrix, with an eigenvalue μ of multiplicity k. A k-subset X of $1, 2, \ldots, n$ is called a star set if the matrix obtained from A by removing rows and columns corresponding to X does not have μ as an eigenvalue. In graph theory context, a star set for an eigenvalue μ in G is a subset X of vertices such that μ is not an eigenvalue of $G\backslash X$. The graph $G\backslash X$ is called a star complement for μ in G.

Theorem 1 *Let X be a set of k vertices in a signed graph $\sum$ and suppose that A^s, the adjacency matrix of $\sum$, is of the form*

$$A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in $\sum$ if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^T(\mu I - C)^{-1}B.$$

Corollary 1 *Let $\sum$ be a signed graph with n vertices, having the eigenvalue μ with multiplicity k. Let H be an induced subgraph of $\sum$ with $n-k$ vertices not having the eigenvalue μ in its spectrum. Let u be a vertex in $\sum \backslash H$, then the adjacency matrix of the graph $H+u$ which is the induced subgraph of $\sum$ on the vertex set $V(H) \cup \{u\}$ will be of the following form*

$$A = \begin{pmatrix} 0 & \mathbf{b}^T \\ \mathbf{b} & C \end{pmatrix},$$

where C is the adjacency matrix of H and $\mathbf{b}^T(\mu I - C)^{-1}\mathbf{b} = \mu$.

Note that the above results hold for any symmetric matrices, and specially, it applies to the signed graphs. For more convenient for two t vectors $\mathbf{b}_1, \mathbf{b}_2$, we denote $\mathbf{b}_1^T(\mu I - C)^{-1}\mathbf{b}_2$ by $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ which is an inner product on the set of t vectors.

Corollary 2 *Let $\sum, H, C$ be as above. Let u, v be two vertices in $\sum \backslash H$, then the adjacency matrix of the graph $H+u+v$ which is the subgraph of $\sum$ on the vertex set $V(H) \cup \{u, v\}$ will be of the following form*

$$A = \begin{pmatrix} 0 & A_{1,2} & \mathbf{b}_1^T \\ A_{2,1} & 0 & \mathbf{b}_2^T \\ \mathbf{b}_1 & \mathbf{b}_2 & C \end{pmatrix},$$

where $A_{1,2} = \langle \mathbf{b}_1, \mathbf{b}_2 \rangle$; moreover, we have the following equalities.

$$\begin{cases} 0, & u \nsim v; \\ -1, & u \sim^{+} v; \\ 1, & u \sim^{-} v. \end{cases}$$

Let H be a signed graph on t vertices and C be the adjacency of H, and we aim to construct a maximal signed graph that contains H as a star complement for the eigenvalue μ. We define the compatibility graph of H and μ as follows: The vertices are $(0, \pm 1)$ vectors $\mathbf{b}_i$ of dimension t such that $\langle \mathbf{b}_i, \mathbf{b}_i \rangle = \mu$, and two vertices $\mathbf{b}_i$ and $\mathbf{b}_j$ are adjacent if and only if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0, \pm 1$. It is not hard to prove that the compatibility graph is finite (see [5]). Now the problem of finding maximal signed graphs having H as a star complement for eigenvalue μ is equivalent to finding the maximal cliques in the compatibility graph of H and μ.

Examples

In this part, we present some examples to illustrate the method.

Example 1 Consider the signed graph $H = (P_3, \emptyset)$. We would like to find a signing on the complete three partite graph $K_{2,2,2}$, say Σ such that H becomes an star complement for the eigenvalue 2 of $\sum = (K_{2,2,2}, \Sigma)$. By the above corollary, any subgraph of $\sum$ on 4 vertices containing H should have the adjacency matrix as below:

$$\begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & 1 & 0 \\ b_2 & 1 & 0 & 1 \\ b_3 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \langle \mathbf{b}, \mathbf{b} \rangle = (b_1, b_2, b_3) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \mu$$

On the other hand, we have

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix};$$

now using the above Corollary, we obtain

$$\langle \mathbf{b}, \mathbf{b} \rangle = (b_1, b_2, b_3) \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 2.$$

Hence, the following equalities hold:

$$\begin{aligned} & b_1\left(\frac{3}{4}b_1 + \frac{1}{2}b_2 + \frac{1}{4}b_3\right) + b_2\left(\frac{1}{2}b_1 + b_2 + \frac{1}{2}b_3\right) \\ & \quad + b_3\left(\frac{1}{4}b_1 + \frac{1}{2}b_2 + \frac{3}{4}b_3\right) = 2 \\ & = \frac{3}{4}\left(b_1^2 + b_2^2 + b_3^2\right) + b_1b_2 + b_2b_3 + \frac{1}{4}\left(2b_1b_3 + b_2^2\right) = 2. \end{aligned}$$

Now the following cases may occur:

- *Case 1* The vertex u is in the same part of $K_{2,2,2}$ where the vertex of degree two in H belongs. In this case, $b_2 = 0$, while b_1, b_3 are ± 1. Hence, the above equation becomes

  $$\frac{3}{4}(b_1^2 + b_3^2) + \frac{1}{2}b_1b_3 = 2.$$

  Since $b_1^2 + b_3^2 = 2$, hence $b_1.b_3 = 1$, therefore one of the followings hold,

$$b_1 = 1, b_3 = 1 \quad \text{or} \quad b_1 = -1, b_3 = -1.$$

- *Case 2* In this case, u belongs to the third part of the graph $\sum$ which is apart from the vertices of H. In this case, we have $b_1^2 + b_2^2 + b_3^2 = 3$. Hence,

$$\frac{3}{4}.3 + b_1b_2 + b_2b_3 + \frac{1}{4}(2b_1b_3 + 1) = 2$$
$$\Rightarrow 2(b_1b_2 + b_2b_3) + b_1b_3 = -1$$

Hence, $b_1 = -b_3$ and $b_2 = 1$ or -1. Thus, vectors which hold on the required equalities are shown in the following table. Note we have found the vectors via MATLAB programming.

b_1	0	0	1	1	-1	-1
b_2	1	-1	1	-1	1	-1
b_3	1	-1	-1	-1	1	1

Using Corollary 2, the following example of signed strongly regular graph has been found.

Example 2 We want to construct a signed graph with spectrum $2^4, -2^4$. Let $H = (K_{1,3}, \emptyset)$ and $\mu = 2$. Therefore, the matrix C and $(\mu I - C)^{-1}$ are as below (Fig. 1).

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} (\mu I - C)^{-1} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 1 & 0.5 \\ 1 & 0.5 & 0.5 & 1 \end{pmatrix}$$

Now adding a vertex v to H an eigenvalue 2 should appear. If the matrix of the signed graph $H + u$ has $\mathbf{b}_1 = [b_1, b_2, b_3, b_4]$ as the raw corresponding to u, then by the above description we have the following equality,

$$\langle \mathbf{b}_1, \mathbf{b}_1 \rangle = \mathbf{b}_1(\mu I - C)^{-1}\mathbf{b}_1^t = 2,$$

which implies the following equation.

$$2b_1(b_1 + b_2 + b_3 + b_4) + b_2(b_2 + b_3 + b_4) + b_3(b_3 + b_4) + b_4^2 = 2. \quad (1)$$

Admissible solutions of Eq. (1) are listed in the following table.

Note that for any vector $\mathbf{b}$ from above table, we have $\langle \mathbf{b}, -\mathbf{b} \rangle = -2$; hence, these two vectors cannot appear simultaneously as incidence of two vertices of the graph Γ. Therefore, from any two vectors of opposite signs in the above table, we may choose one of them. Hence, there are only 8 possible vertices in the corresponding compatibility graph. With no loss of generality, we choose vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_7, \mathbf{b}_9, \mathbf{b}_{10}, \mathbf{b}_{11}, \mathbf{b}_{12}$ in which none of them are parallel. Now we repeat the procedure for $\mu = -2$ so that the obtained graph have both eigenvalues $2, -2$. The vectors which are compatible with both the desired eigenvalues are $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_{10}$. In this case, the compatibility graph will be complete and we obtain the signed graph of Fig. 2. Note that the vertices are labeled with corresponding vectors.

En	Ve															
	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{b}_4$	$\mathbf{b}_5$	$\mathbf{b}_6$	$\mathbf{b}_7$	$\mathbf{b}_8$	$\mathbf{b}_9$	$\mathbf{b}_{10}$	$\mathbf{b}_{11}$	$\mathbf{b}_{12}$	$\mathbf{b}_{13}$	$\mathbf{b}_{14}$	$\mathbf{b}_{15}$	$\mathbf{b}_{16}$
b_1	0	0	0	0	0	0	-1	1	1	1	1	1	-1	-1	-1	-1
b_2	1	1	-1	-1	-1	1	1	-1	1	0	-1	-1	1	1	-1	0
b_3	1	-1	1	-1	1	-1	1	-1	-1	0	1	-1	1	-1	1	0
b_4	-1	1	1	1	-1	-1	1	-1	-1	0	-1	1	-1	1	1	0

The above example can be used to construct infinitely many complete tripartite signed graphs having only two distinct eigenvalues, or equivalently a signed strongly regular graph with $\mu^s = 0$.

Theorem B *Let $\sum$ be a signed graph with only two eigenvalues which are opposite, say $\pm\lambda$, then the Kronecker product of the matrix $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ by the adjacency matrix of $\sum$ gives the adjacency matrix of a signed graph with only two distinct eigenvalues $\pm\sqrt{2}\lambda$.*

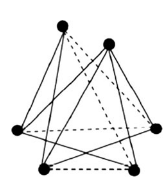

Fig. 1 An $\mathrm{SRG}^s(6,4,0,0)$ with spectrum $[2^3, -2^3]$

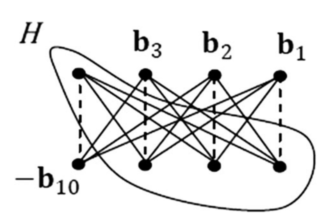

Fig. 2 An $\mathrm{SRG}^s(8,4,0,0)$ with spectrum $[2^4, -2^4]$

Conclusion

Repeatedly applying the Kronecker product of the matrix $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ by the adjacency matrix of $\sum$, we find infinitely many complete tripartite signed strongly regular graphs, whereas the same operation on the signed graph Γ will give infinitely many bipartite signed strongly regular graphs; the method can be used for several other examples to construct new families of signed graphs with the desired property.

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