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ORIGINAL RESEARCH

Some applications of fixed point results for generalized two classes of Boyd–Wong's F-contraction in partial b-metric spaces

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Abstract
In this paper, we will present some fixed point results for two classes of generalized contractions of Boyd–Wong type in partial b-metric spaces. More precisely, the structure of the paper is the following. In section one, we present some useful notions and results. The aim of section two is to introduce the concepts of Boyd–Wong F-contractions of **type A** and of **type B** and establish some new common fixed point results in partial b-metric spaces. We show the validity and superiority of our main results by suitable examples which are visualized by corresponding surfaces and related graphs. In section three, we correct some slip-ups in some recent papers. Finally, in section four, two applications to integral equation and periodic boundary value problem are included which make effective the new concepts and results.

Keywords Common fixed point · Partial metric spaces · b-metric spaces · F-contraction · α-Admissible

Mathematics subject classification 47H10 · 54H25

Introduction and preliminaries

There are lots of extensions and generalizations of metric space. In 1989, Bakhtin [1] introduced the notion of b-metric space, and in 1993, Czerwik [2, 3] extensively used the concept of b-metric space. On the other hand, the concept of partial metric space was introduced by Mathews [4]. In recent times, Shukla [5] generalized both the concept of b-metric and partial metric space by introducing the partial b-metric space. After that, in [6], Mustafa et al. introduced a modified version of partial b-metric space. On the other hand, in 2012, Wardowski [7] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Very recently, Piri et al. [9] improve the result of Wardowski [7] by launching the concept of an F-Suzuki contraction and proved some curious fixed point results. The results of Wardowski [7] were generalized by several authors (see, e.g., [10–14]).

The purpose of this article is to extend the concept of F-contraction by introducing Boyd–Wong **type A** and **type B** F-contraction in partial b-metric space, motivated and inspired by the ideas of Wardowski [7] and Mustafa et al. [6]. Our results substantially generalize and extend the corresponding results contained in Shukla et al. [19, 20],

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Alsulami et al. [21], Singh et al. [22], and many others. We also point out some slip-ups of recent papers present in the literature. Some examples and applications are presented to highlight the realized improvement.

In the sequel, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{N}^*$ will represent the set of all real numbers, natural numbers, and positive integers, respectively. Some elementary definitions and fundamental results, which will be used in the sequel, are described here.

Definition 1.1 [2] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Definition 1.2 [4] A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$, such that for all $x, y, z \in X$

(p_1) $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$;
(p_2) $p(x, x) \leq p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p), such that X is a nonempty set and p is a partial metric on X.

Definition 1.3 [5] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(p_{b_1}) $x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b_2}) $p_b(x, x) \leq p_b(x, y)$;
(p_{b_3}) $p_b(x, y) = p_b(y, x)$;
(p_{b_4}) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

In the following definition, Mustafa et al. [6] modified the Definition 1.3 to find that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 1.4 [6] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(p_{b_1}) $x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b_2}) $p_b(x, x) \leq p_b(x, y)$;
(p_{b_3}) $p_b(x, y) = p_b(y, x)$;
(p_{b_4}) $p_b(x, y) \leq s(p_b(x, z) + p_b(z, y) - p_b(z, z)) + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

Example 1.1 [5] Let $X = \mathbb{R}^+$, $q > 1$ be a constant and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x, y) = [\max\{x, y\}]^q + |x - y|^q,$$

for all $x, y \in X$. Then, (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Remark 1.1 The class of partial b-metric space (X, p_b) is effectively larger that the class of partial metric space, since a partial metric space is a special case of a partial b-metric space (X, p_b) when $s = 1$. In addition, the class of partial b-metric space (X, p_b) is effectively larger that the class of b-metric space, since a b-metric space is a special case of a partial b-metric space (X, p_b) when the self distance $p(x, x) = 0$.

Proposition 1.1 [5] *Let X be a nonempty set, and let p be a partial metric and d be a b-metric with the coefficient* $s \geq 1$ *on X. Then, the function* $p_b : X \times X \to [0, \infty)$ *defined by* $p_b(x, y) = p(x, y) + d(x, y)$ *for all* $x, y \in X$ *is a partial b-metric on X with the coefficient s.*

Proposition 1.2 [6] *Every partial b-metric* p_b *defines a b-metric* d_{p_b}, *where*

$$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y) \quad \text{for all } x, y \in X.$$

For p_b-convergent, p_b-Cauchy sequence, and p_b-complete, we refer [6].

Lemma 1.1 [6] *Let* (X, p_b) *be a partial b-metric space. Then*

1. *A sequence* $\{x_n\}$ *is a* p_b*-Cauchy sequence in* (X, p_b) *if and only if it is a b-Cauchy sequence in the b-metric space* (X, d_{p_b});
2. (X, p_b) *is* p_b*-complete if and only if the b-metric space* (X, d_{p_b}) *is complete. Moreover,* $\lim_{n\to\infty} d_{p_b}(x_n, x) = 0$ *if and only if* $p_b(x, x) = \lim_{n\to\infty} p_b(x_n, x) = \lim_{n,m\to\infty} p_b(x_n, x_m)$.

Definition 1.5 [15] Let f be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. We say that f is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \implies \alpha(fx, fy) \geq 1.$$

Definition 1.6 [16] Let $f, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$. The mapping f is g-α-admissible if for all $x, y \in X$, such that $\alpha(gx, gy) \geq 1$, we have $\alpha(fx, fy) \geq 1$.

If g is identity mapping, then f is called α-admissible.

Definition 1.7 [17] An α-admissible map f is said to be triangular α-admissible if $x, y, z \in X$, $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1$.

Definition 1.8 [18] Let $(X, \preceq)$ be a partially ordered set and let f, g be two maps on X. Then

(i) the pair (f, g) is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$;
(ii) f is said to be g-weakly isotone increasing if $fx \preceq gfx \preceq fgfx$ for all $x \in X$.

Note that if $f, g : X \to X$ are weakly increasing, then f is a g-weakly isotone increasing.

Lemma 1.2 [17] *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$, such that $\alpha(x_0, fx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \quad \text{for all} \quad m, n \in \mathbb{N} \quad \text{with} \quad m < n.$$

Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$, such that

1. *ϕ is monotonic increasing, i.e., $t_1 \leq t_2 \Longrightarrow \phi(t_1) \leq \phi(t_2)$.*
2. *ϕ is continuous and $\phi(t) < t$ for each $t > 0$.*

Let Ψ denote the set of all decreasing functions $\psi : (0, \infty) \to (0, \infty)$.

On the other hand, Wardowski [7] introduced the F-contraction as follows:

Definition 1.9 Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$, such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$.
(F2) For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.
(F3) There exists $k \in (0, 1)$, such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying $(F1)$–$(F3)$ by $\mathcal{F}$. On the other hand, Secelean [8] proved the following lemma.

Lemma 1.3 [8] *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold:*

(a) *If $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$.*
(b) *If $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

Secelean [8] reintegrated the condition $(F2)$ by more elementary condition $(F2')$.

$(F2')$ $\inf F = -\infty$,

or, also by

$(F2'')$, there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers, such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Most recently, Piri et al. [9] used the following condition $(F3')$ instead of $(F3)$.

$(F3')$ F is continuous on $(0, \infty)$.

We denote the set of all functions satisfying $(F1)$, $(F2')$, and $(F3')$ by Δ_F.

Main results

Common fixed point results for Boyd–Wong type A *F*-contraction

In this section, we present our essential results. For this, we introduce the following definition.

Definition 2.1 Let (X, p_b) be a partial b-metric space and $f, g : X \to X$ two mappings. In addition, suppose that $\alpha : X \times X \to [0, \infty)$ be a function, where α is defined as in Definition 1.7. We say that the pair (f, g) is a Boyd–Wong **type A** F-contraction on a partial b-metric space X if there exists $F \in \Delta_F$, $\psi \in \Psi$ and $\phi \in \Phi$, such that for all $x, y \in X$ and $s > 1$ with $p_b(fx, gy) > 0$

$$\alpha(x, y)F(s^{\epsilon} p_b(fx, gy)) \leq F(M_s(x, y)) - \psi(p_b(x, y)), \tag{1}$$

where

$$M_s(x, y) = \max\Big\{\phi(p_b(x, y)), \phi(p_b(x, fx)), \phi(p_b(y, gy)), \phi\Big(\frac{p_b(x, gy) + p_b(y, fx)}{2s}\Big)\Big\}, \tag{2}$$

and $\epsilon > 1$ is a constant. If $\alpha(x, y) = 1$ for all $x, y \in X$, then the pair (f, g) is called Boyd–Wong **type** A^* F-contraction.

It needs mentioning that the following lemma will be useful in proving our main results.

Lemma 2.1 *Let (X, p_b) be a complete partial b-metric space. Let f and g are self-mappings on X, such that (f, g) is a Boyd–Wong* **typeA** *F-contraction on (X, p_b). If f or g has a fixed point u in X, then u is a unique common fixed point of f and g and $p_b(u, u) = 0$.*

Proof Let $u \in X$ be a fixed point of f, i.e., $fu = u$. We will prove that u is also a fixed point of g on the contrary assume that $p_b(fu, gu) > 0$. From inequality (1), we have

$$F(p_b(fu,gu)) \le F(s^\epsilon p_b(fu,gu)) \le \alpha(u,u)F(s^\epsilon p_b(fu,gu))$$
$$\le F(\max\{\phi(p_b(u,u)),\phi(p_b(u,fu)),\phi(p_b(u,gu)),$$
$$\phi\left(\frac{p_b(u,gu)+p_b(u,fu)}{2s}\right)\}) - \psi(p_b(u,u))$$
$$< F(p_b(u,gu)),$$

a contradiction. This yields $p_b(fu,gu)=0$, i.e., $fu=gu=u$. Thus, u is a common fixed point of f and g. Furthermore, if $p_b(u,u)>0$,, then by repeating the same process as mentioned above, one can easily get

$$F(p_b(u,u)) = F(p_b(fu,gu)) \le F(M_s(u,u))) - \psi(p_b(u,u)),$$

which gives

$$F(p(u,u)) < F(p(u,u)),$$

a contradiction. Thus, $p_b(u,u)=0$. To prove uniqueness, let v be another common fixed point of f and g, that is, $fv=gv=v$. If $p_b(u,v)>0$, then from (1), we arrive at

$$F(p_b(fu,gv)) \le F(s^\epsilon p_b(fu,gv)) \le \alpha(u,v)F(s^\epsilon p_b(fu,gv))$$
$$\le F(\max\{\phi(p_b(u,v)),\phi(p_b(u,fu)),\phi(p_b(v,gv)),$$
$$\phi\left(\frac{p_b(u,gv)+p_b(v,fu)}{2s}\right)\}) - \psi(p_b(u,v))$$
$$< F(p_b(u,v)),$$

which gives

$$F(p_b(u,v)) < F(p_b(u,v)),$$

a contradiction. Hence, we conclude that $p_b(u,v)=0$, that is, $u=v$. Thus, the common fixed point of f and g is unique. On the other hand, if u is the fixed point of g, then by following the same procedure one can show that u is the unique common fixed point of f and g.

One of our main result of this paper is the following one.

Theorem 2.1 *Let (X,p_b) be a complete partial b-metric space. Let f and g are self-mappings on X satisfying the following conditions:*

1. *f is α-admissible.*
2. *There exists $x_0 \in X$, such that $\alpha(x_0,fx_0) \ge 1$.*
3. *(f,g) is a Boyd–Wong* **type A** *F-contraction on (X,p_b).*

Then, f and g have a unique common fixed point $u \in X$ with $p_b(u,u)=0$.

Proof Let $x_0 \in X$, such that $\alpha(x_0,fx_0) \ge 1$. Define a sequence $\{x_n\}$ in the following way:

$$fx_{2n} = x_{2n+1} \text{ and } gx_{2n+1} = x_{2n+2} \text{ for all } n \in \mathbb{N}^*. \tag{3}$$

Since f is α-admissible, therefore, $\alpha(x_0,x_1)=\alpha(x_0,fx_0)\ge 1 \Longrightarrow \alpha(x_1,x_2)=\alpha(fx_0,fx_1)\ge 1$. By induction, we get

$$\alpha(x_n,x_{n+1}) \ge 1 \text{ for all } n \in \mathbb{N}^*. \tag{4}$$

If $x_{m+1}=x_m$ for any $m \in \mathbb{N}$, without loss of generality let $x_{2n+1}=x_{2n}$ and $p_b(x_{2n},x_{2n+1})>0$. Therefore, from $(\Delta_{(F1)})$, property of ϕ and in account of inequality (1), we arrive at

$$\begin{aligned} F(p_b(x_{2n+1},x_{2n+2})) &\le F(s^\epsilon p_b(fx_{2n},gx_{2n+1})) \\ &\le \alpha(x_{2n},x_{2n+1})F(s^\epsilon p_b(fx_{2n},gx_{2n+1})) \\ &\le F(M_s(x_{2n},x_{2n+1})) - \psi(p_b(x_{2n},x_{2n+1})), \end{aligned} \tag{5}$$

where

$$\begin{aligned} M_s(x_{2n},x_{2n+1}) &= \max\Big\{\phi(p_b(x_{2n},x_{2n+1})),\phi(p_b(x_{2n},x_{2n+1})), \\ &\phi(p_b(x_{2n+1},x_{2n+2})),\phi\left(\frac{p_b(x_{2n},x_{2n+2})+p_b(x_{2n+1},x_{2n+1})}{2s}\right)\Big\} \\ &= \max\{\phi(p_b(x_{2n},x_{2n})),\phi(p_b(x_{2n},x_{2n+2})), \\ &\phi\left(\frac{p_b(x_{2n},x_{2n+2})+p_b(x_{2n},x_{2n})}{2s}\right)\} \\ &= \max\{\phi(p_b(x_{2n},x_{2n})),\phi(p_b(x_{2n},x_{2n+2}))\} \\ &= \phi(p_b(x_{2n},x_{2n+2})). \end{aligned}$$

Putting the value of $M_s(x_{2n},x_{2n+1})$ in (5) and using the property of ϕ and ψ, we have $F(p_b(x_{2n+1},x_{2n+2})) < F(p_b(x_{2n+1},x_{2n+2}))$, which gives a contradiction. Thus $p_b(x_{2n},x_{2n+1})=0$, which yields $x_{2n}=x_{2n+1}=x_{2n+2}$. Consequently, we get $x_{2n}=x_{2n+1}=x_{2n+2}=x_{2n+3}=\ldots$, which implies $x_{2n}=fx_{2n}=gx_{2n}$. Hence, x_{2n} is a common fixed point of f and g.

Assume that $x_m \ne x_{m+1}$ for all $m \in \mathbb{N}$, that is, $p_b(x_{2n+1},x_{2n})>0$, then by inequality (1), we have

$$\begin{aligned} F(p_b(x_{2n+1},x_{2n})) &\le F(s^\epsilon.p_b(fx_{2n},gx_{2n-1})) \\ &\le \alpha(x_{2n},x_{2n-1})F(s^\epsilon p_b(fx_{2n},gx_{2n-1})) \\ &\le F(M_s(x_{2n},x_{2n-1})) - \psi(p_b(x_{2n},x_{2n-1})), \end{aligned} \tag{6}$$

where

$$\begin{aligned} &M_s(x_{2n},x_{2n-1}) \\ &= \max\{\phi(p_b(x_{2n},x_{2n-1})),\phi(p_b(x_{2n},x_{2n+1})), \\ &\quad \phi(p_b(x_{2n-1},x_{2n})),\phi\left(\frac{p_b(x_{2n},x_{2n})+p_b(x_{2n-1},x_{2n+1})}{2s}\right)\Big\} \\ &= \max\{\phi(p_b(x_{2n},x_{2n-1})),\phi(p_b(x_{2n},x_{2n+1})), \\ &\quad \phi\left(\frac{s(p_b(x_{2n-1},x_{2n})+p_b(x_{2n},x_{2n+1}))-p_b(x_{2n},x_{2n})+p_b(x_{2n},x_{2n}))}{2s}\right)\Big\} \\ &= \max\{\phi(p_b(x_{2n},x_{2n-1})),\phi(p_b(x_{2n},x_{2n+1}))\}. \end{aligned}$$

If $M_s(x_{2n},x_{2n-1})=\phi(p_b(x_{2n},x_{2n+1}))$ for all $n \in \mathbb{N}^*$. Taking (6) into account, we arrive at

$$F(p_b(x_{2n+1},x_{2n})) \le F(\phi(p_b(x_{2n},x_{2n+1}))) - \psi(p_b(x_{2n},x_{2n-1})).$$

By the property of ϕ, ψ and from $(F1)$, one can easily get

$$F(p_b(x_{2n+1},x_{2n}))<F(p_b(x_{2n+1},x_{2n})).$$

We get a contradiction. Thus, $M_s(x_{2n},x_{2n-1}) = \phi(p_b(x_{2n},x_{2n-1}))$. Therefore, from (6) and taking the property of F, ϕ and ψ in account of, we have

$$F(p_b(x_{2n+1},x_{2n}))\leq F(\phi(p_b(x_{2n},x_{2n-1}))-\psi(p_b(x_{2n},x_{2n-1})), \tag{7}$$

which yields $p_b(x_{2n+1},x_{2n})<p_b(x_{2n},x_{2n-1})$. Thus, $\{p_b(x_{2n+1},x_{2n})\}$ is a decreasing sequence of positive real numbers. Using the property of ϕ and repeated use of (7) gives

$$\begin{aligned}F(p_b(x_{2n+1},x_{2n}))&<F(p_b(x_{2n},x_{2n-1}))-\psi(p_b(x_{2n},x_{2n-1}))\\ &\leq F(\phi(p_b(x_{2n-1},x_{2n-2})))-\psi(p_b(x_{2n-1},x_{2n-2}))\\ &\quad-\psi(p_b(x_{2n},x_{2n-1}))\\ &<F(p_b(x_{2n-1},x_{2n-2}))-\psi(p_b(x_{2n-1},x_{2n-2}))\\ &\quad-\psi(p_b(x_{2n},x_{2n-1})).\end{aligned}$$

Since ψ is an decreasing function, therefore, it follows from the above inequality that

$$F(p_b(x_{2n+1},x_{2n}))<F(p_b(x_{2n-1},x_{2n-2}))-2\psi(p_b(x_{2n-1},x_{2n-2}))$$

from the successive application, we arrive at

$$F(p_b(x_{2n+1},x_{2n}))<F(p_b(x_0,x_1))-2n\psi(p_b(x_0,x_1)). \tag{8}$$

Similarly

$$F(p_b(x_{2n+2},x_{2n+1}))<F(p_b(x_0,x_1))-(2n+1)\psi(p_b(x_0,x_1)). \tag{9}$$

As $F\in\Delta_F$, letting the limit as $n\to\infty$ in (8) and (9), we get $\lim\limits_{n\to\infty}F(p_b(x_n,x_{n+1}))=-\infty$. It follows from $(F2')$ and Lemma 1.3 that

$$\lim_{n\to\infty}p_b(x_n,x_{n+1})=0. \tag{10}$$

Moreover, from (p_{b_2}), we have the following:

$$\lim_{n\to\infty}p_b(x_n,x_n)=0. \tag{11}$$

To prove that $\{x_n\}$ is a p_b-Cauchy sequence in X, it is sufficient to show that $\{x_{2n}\}$ is a p_b-Cauchy sequence in X. From Lemma 1.1, we need to prove that $\{x_{2n}\}$ is a b-Cauchy sequence in the b-metric space (X,d_{p_b}). Suppose to the contrary that, there exists $\delta>0$, such that for an integer k, there exist integer $n(k)>m(k)\geq k$, such that

$$d_{p_b}(x_{2m(k)},x_{2n(k)})\geq\delta. \tag{12}$$

For every integer k, let $m(k)$ is the least positive integer exceeding $n(k)$ satisfying (12), such that

$$d_{p_b}(x_{2m(k)},x_{2n(k)-1})<\delta. \tag{13}$$

Due to triangle inequality and from (12), we arrive at

$$\delta\leq d_p(x_{2m(k)},x_{2n(k)})\leq sd_p(x_{2m(k)},x_{2n(k)-1})+sd_p(x_{2n(k)-1},x_{2m(k)}),$$

which on making $k\to\infty$ and using (13) gives rise to

$$\frac{\delta}{s}\leq\lim_{k\to\infty}\inf d_{p_b}(x_{2m(k)},x_{2n(k)-1})\leq\lim_{k\to\infty}\sup d_{p_b}(x_{2m(k)},x_{2n(k)-1})\leq\delta. \tag{14}$$

In addition, from (13) and (14), we get

$$\delta\leq\lim_{k\to\infty}\sup d_{p_b}(x_{2m(k)},x_{2n(k)})\leq s\delta. \tag{15}$$

In addition

$$\begin{aligned}d_{p_b}(x_{2m(k)+1},x_{2n(k)})&\leq sd_{p_b}(x_{2m(k)+1},x_{2m(k)})+sd_{p_b}(x_{2m(k)},x_{2n(k)})\\ &\leq sd_{p_b}(x_{2m(k)+1},x_{2m(k)})+s^2d_{p_b}(x_{2m(k)},x_{2n(k)-1})+s^2d_{p_b}(x_{2n(k)-1},x_{2n(k)})\\ &\leq sd_{p_b}(x_{2m(k)+1},x_{2m(k)})+s^2\delta+s^2d_{p_b}(x_{2n(k)-1},x_{2n(k)}),\end{aligned}$$

which yields

$$\lim_{k\to\infty}\sup d_{p_b}(x_{2m(k)+1},x_{2n(k)})\leq s^2\delta.$$

Moreover

$$\begin{aligned}d_{p_b}(x_{2m(k)+1},x_{2n(k)-1})&\leq sd_{p_b}(x_{2m(k)+1},x_{2m(k)})\\ &\quad+sd_{p_b}(x_{2m(k)},x_{2n(k)-1}),\end{aligned}$$

which gives

$$\lim_{k\to\infty}\sup d_{p_b}(x_{2m(k)+1},x_{2n(k)-1})\leq s\delta.$$

From Proposition 1.2, we must have

$$\lim_{k\to\infty}\sup d_{p_b}(x_{2m(k)},x_{2n(k)-1})=2\lim_{k\to\infty}\sup p_b(x_{2m(k)},x_{2n(k)-1}).$$

Then, from above inequality along with (14), we comes at

$$\begin{aligned}\frac{\delta}{2s}&\leq\lim_{k\to\infty}\inf p_b(x_{2m(k)},x_{2n(k)-1})\\ &\leq\lim_{k\to\infty}\sup p_b(x_{2m(k)},x_{2n(k)-1})\leq\frac{\delta}{2}.\end{aligned} \tag{16}$$

Analogously, we derive

$$\lim_{k\to\infty}\sup p_b(x_{2m(k)},x_{2n(k)})\leq\frac{s\delta}{2}. \tag{17}$$

$$\frac{\delta}{2s}\leq\lim_{k\to\infty}\sup p_b(x_{2m(k)+1},x_{2n(k)}). \tag{18}$$

$$\lim_{k\to\infty}\sup p_b(x_{2m(k)+1},x_{2n(k)-1})\leq\frac{s\delta}{2}. \tag{19}$$

As $F(p_b(fx_{2m(k)},gx_{2n(k)-1}))=F(p_b(x_{2m(k)+1},x_{2n(k)}))>0$, therefore, the contractive condition (1) and property of ψ imply that

$$\begin{aligned}F(p_b(x_{2m(k)+1},x_{2n(k)}))&\le F(s^{\epsilon}p_b(fx_{2m(k)},gx_{2n(k)-1}))\\&\le \alpha(x_{2m(k)},x_{2n(k)-1})F(s^{\epsilon}p_b(fx_{2m(k)},gx_{2n(k)-1}))\\&\le F(M_s(x_{2m(k)},x_{2n(k)-1}))-\psi(p_b(x_{2m(k)},x_{2n(k)-1}))\\&\le F(M_s(x_{2m(k)},x_{2n(k)-1})).\end{aligned}\tag{20}$$

Using the definition of $M_s(x,y)$ and keeping the inequalities (16)-(19) in mind, it is easy to see that

$$\lim_{k\to\infty}\sup M_s(x_{2m(k)},x_{2n(k)-1})<\frac{\delta}{2}.\tag{21}$$

Indeed

$$\begin{aligned}&M_s(x_{2m(k)},x_{2n(k)-1})=\\&\max\Big\{\phi(p_b(x_{2m(k)},x_{2n(k)-1})),\phi(p_b(x_{2m(k)},x_{2m(k)+1})),\\&\phi(p_b(x_{2n(k)-1},x_{2n(k)})),\phi\Big(\frac{p_b(x_{2m(k)},x_{2n(k)})+p_b(x_{2n(k)-1},x_{2m(k)+1})}{2s}\Big)\Big\}.\\&<\max\Big\{p_b(x_{2m(k)},x_{2n(k)-1}),p_b(x_{2m(k)},x_{2m(k)+1}),\\&p_b(x_{2n(k)-1},x_{2n(k)}),\frac{p_b(x_{2m(k)},x_{2n(k)})+p_b(x_{2n(k)-1},x_{2m(k)+1})}{2s}\Big\}.\end{aligned}$$

Which yields

$$\lim_{k\to\infty}\sup M_s(x_{2m(k)},x_{2n(k)-1})<\max\left\{\frac{\delta}{2},0,0,\frac{1}{2s}\left[\frac{s\delta}{2}+\frac{s\delta}{2}\right]\right\}<\frac{\delta}{2}.$$

Furthermore, from (18), (20), and (21), one can easily get

$$\begin{aligned}F\left(s\frac{\delta}{2s}\right)&\le\lim_{k\to\infty}(\alpha(x_{2m(k)},x_{2n(k)-1})F(s^{\epsilon}\sup p_b(fx_{2m(k)},gx_{2n(k)-1})))\\&\le F(\lim_{k\to\infty}\sup M_s(x_{2m(k)},x_{2n(k)-1})),\end{aligned}$$

which means that $F(\frac{\delta}{2})<F(\frac{\delta}{2})$, which is impossible. This contradiction proves that $\{x_n\}$ is a b-Cauchy sequence in the b-metric space (X,d_{p_b}), then from Lemma 1.1, $\{x_n\}$ is a p_b-Cauchy sequence in the partial b-metric space (X,p_b). Since, (X,p_b) is complete then by Lemma 1.1 b-metric space (X,d_{p_b}) is b-complete. Therefore, the sequence $\{x_n\}$ converges to some point $u\in X$, that is, $\lim_{n\to\infty}d_{p_b}(x_n,u)=0$.

Again, from Lemma 1.1

$$\lim_{n\to\infty}p_b(x_n,u)=\lim_{n,m\to\infty}p_b(x_n,x_m)=p(u,u)=0.\tag{22}$$

Next, we will show that u is a common fixed point of f and g. Since $F\in\Delta_F$, therefore, from (Δ_{F3}), F is continuous. Then, we have to consider the following two cases:

Case (i): For each $n\in\mathbb{N}$, there exists $j_n\in\mathbb{N}$, such that $p_b(x_{j_n+1},fu)=0$, i.e., $x_{j_n+1}=fu$ and $j_n>j_{n-1}$, where $j_0=1$. Then, we get

$$u=\lim_{n\to\infty}x_{j_n+1}=\lim_{n\to\infty}fu=fu.$$

This yields that u is a fixed point of f.

Case (ii): There exists $n_2\in\mathbb{N}$, such that $p_b(x_{n+1},fu)\neq 0$ for all $n\ge n_2$, which means $p_b(fx_n,fu)>0$ for all $n\ge n_2$. It follows from the inequality (1) that

$$\begin{aligned}F(p_b(fu,x_{2n+2}))&\le\alpha(u,x_{2n+1})F(s^{\epsilon}p_b(fu,gx_{2n+1}))\\&\le F(M_s(u,x_{2n+1}))-\psi(p_b(u,x_{2n+1})),\end{aligned}\tag{23}$$

where

$$\begin{aligned}&M_s(u,x_{2n+1})=\max\\&\Big\{\phi(p_b(u,x_{2n+1})),\phi(p_b(u,fu)),\phi(p_b(x_{2n+1},x_{2n+2})),\\&\phi\Big(\frac{p_b(u,x_{2n+2})+p_b(x_{2n+1},fu)}{2s}\Big)\Big\}.\end{aligned}\tag{24}$$

If $p_b(u,fu)>0$, then by the fact

$$\lim_{n\to\infty}p_b(u,x_{2n+1})=\lim_{n\to\infty}p_b(u,x_{2n+2})=\lim_{n\to\infty}p_b(x_{2n+1},x_{2n+2})=0$$

there exists $n_3\in\mathbb{N}$, such that for all $n\ge n_3$, we have

$$\begin{aligned}&\max\Big\{\phi(p_b(u,x_{2n+1})),\phi(p_b(u,fu)),\phi(p_b(x_{2n+1},x_{2n+2})),\\&\phi\Big(\frac{p_b(u,x_{2n+2})+p_b(x_{2n+1},fu)}{2s}\Big)\Big\}=\phi(p_b(u,fu)).\end{aligned}\tag{25}$$

From (23) and (25), we conclude that

$$F(p_b(fu,x_{2n+2}))\le F(\phi(p_b(u,fu))),$$

for all $n\ge\{n_2,n_3\}$. Since F is continuous, taking the limit as $n\to\infty$ in above inequality, we have $F(p_b(fu,u))<F(p_b(fu,u))$, a contradiction, which yields that our assumption is wrong, which means that u is the fixed point of f. From Lemma 2.1, u is an unique common fixed point of f and g with $p_b(u,u)=0$.

The following examples show the superiority of our assertions. □

Example 2.1 Let $X=[0,\infty)$ be equipped with the partial-order relation $\preceq$ defined by, $x\preceq y \iff x>y$ and the function $p_b:X\times X\to[0,\infty)$ is defined by $p_b(x,y)=[\max\{x,y\}]^2$ for all $x,y\in X$, where $s=2$. It is obvious that (X,p_b) is a complete partial b-metric space. Let the mappings $f,g:X\to X$ are defined by

$$fx=\frac{1}{1024}[\log(1+x^2)+x^2];\quad gx=\frac{1}{32}x^2e^{-x}.$$

We define the mapping $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1, & (x,y)\in[0,30],\\0, & \text{otherwise}.\end{cases}$$

We also define $\psi:(0,\infty)\to(0,\infty)$ by $\psi(t)=\frac{1}{50(t+1)}$ and $\phi:[0,\infty)\to[0,\infty)$ be given by $\phi(t)=\frac{10t+1}{12}$. Let $F(t)=\log t$ for all $t\in\mathbb{R}^+$.

Now, we will show that f is α-admissible. Let $x, y \in X$, such that $\alpha(x,y) \geq 1$. By the definition of f and α, we have $\alpha(fx, fy) \geq 1$, for all $x, y \in [0,30]$. Hence, f is an α-admissible. On the other hand, there exists $x_0 = 0 \in X$, such that $\alpha(0, f0) = \alpha(0,0) = 1 \geq 1$.

Without loss of generality, we may take $x, y \in X$, such that $x > y$. To check the contractive condition (1) of Theorem 2.1, we have to consider the following cases:

Case I. If $x, y \in [0,30]$, then from (1), we get

$$\begin{aligned} \text{L.H.S.} &= \alpha(x,y) F(s^{\epsilon} p_b(fx, gy)) \\ &= F\left(2^{\epsilon} \max\left\{\frac{1}{1024}[\log(1+x^2)+x^2], \frac{1}{32} y^2 e^{-y}\right\}^2\right) \\ &\leq F\left(2^{\epsilon-5} \max\left\{\frac{1}{32}(\log(1+x^2)+x^2), y^2 e^{-y}\right\}^2\right) \\ &\leq F(2^{\epsilon-5}\max\{x,y\}^2) \\ &= \log(2^{\epsilon-5}x^2). \end{aligned} \tag{26}$$

For R.H.S., by the definition of $M_s(x,y)$, one can easily obtained that $M_s(x,y) = \phi(x^2)$. It follows form (1) that

$$\begin{aligned} \text{R.H.S.} &= F(\phi(x^2)) - \psi(\max\{x,y\}^2) \\ &= \log\left(\frac{10x^2+1}{12}\right) - \frac{1}{50(x^2+1)}. \end{aligned} \tag{27}$$

Figures 1 and 2 demonstrate that L.H.S. with blue surface dominates by the red surface, i.e., R.H.S.

From Figs. 1 and 2, we obtain that inequality (1) holds for all $x, y \in [0,30]$ with $\epsilon \in (1, 4.5]$.

Case II. If $x, y \in (30, \infty]$, then $\alpha(x,y) = 0$. One can verify that

$$M_s(x,y) = \begin{cases} \phi(x^2), & x, y \in (30, 1023.994], \\ \phi\left(\left(\frac{\log(1+x^2)+x^2}{1024}\right)^2\right), & x, y \in (1023.994, \infty], \\ \phi\left(\left(\frac{\log(1+x^2)+x^2}{1024}\right)^2\right), & y \in (30, 1023.994], x \in (1023.994, \infty], \end{cases}$$

Thus, one has to consider the following sub-cases:

Case II(i). If $x, y \in (30, 1023.994]$ then from (1), we have

$$\text{L.H.S.} = \alpha(x,y) F(s^{\epsilon} p_b(fx, gy)) =$$

$$0 \leq \log\left(\frac{10x^2+1}{12}\right) - \frac{1}{50(x^2+1)}.$$

Case II(ii). If $x, y \in (1023.994, \infty]$, then from (1), we arrive at

$$\text{L.H.S.} = \alpha(x,y) F(s^{\epsilon} p_b(fx, gy)) =$$

$$0 \leq \log\left(\frac{10\left(\frac{\log(1+x^2)+x^2}{1024}\right)^2 + 1}{12}\right) - \frac{1}{50(x^2+1)}.$$

Case II(iii). If $y \in (30, 1023.994]$ and $x \in (1023.994, \infty]$, then Case II(iii) is similar to the Case II(ii); therefore, we skip the details.

Figures 3 and 4 show that L.H.S. with green surface dominates by the blue surface, i.e., R.H.S.

From Figs. 3 and 4, it is easy to verify that inequality (32) holds for all $x, y \in (30, \infty]$.

Case III. If $y \in [0,30]$ and $x \in (30, \infty]$, then $\alpha(x,y) = 0$. We can easily conclude that

$$M_s(x,y) = \begin{cases} \phi(x^2), & x \in (30, 1023.994], \\ \phi\left(\left(\frac{\log(1+x^2)+x^2}{1024}\right)^2\right), & x \in (1023.994, \infty], \end{cases}$$

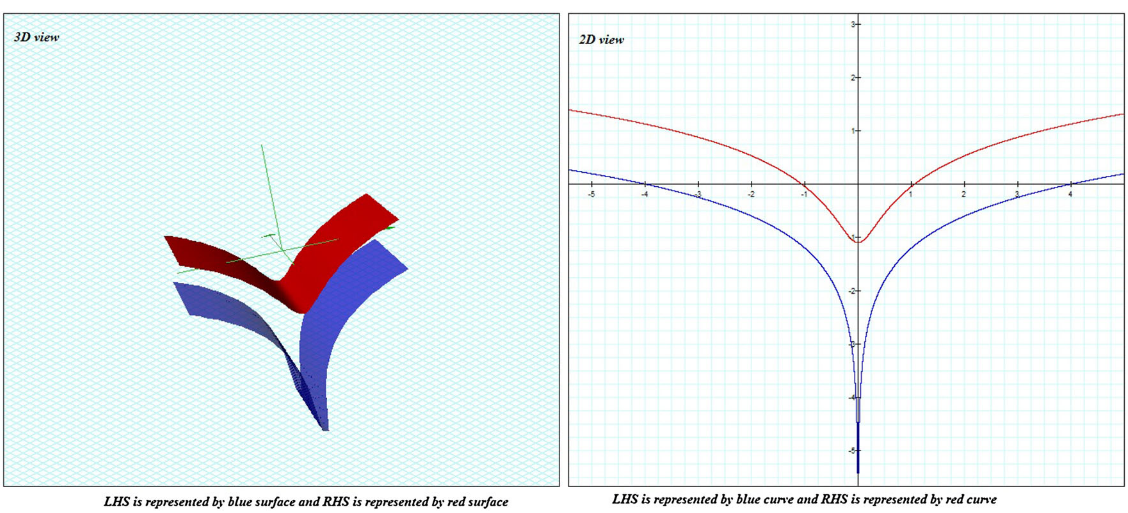


Fig. 1 Plot of inequality for $x, y \in [0,30]$ with $\epsilon = 1.1$

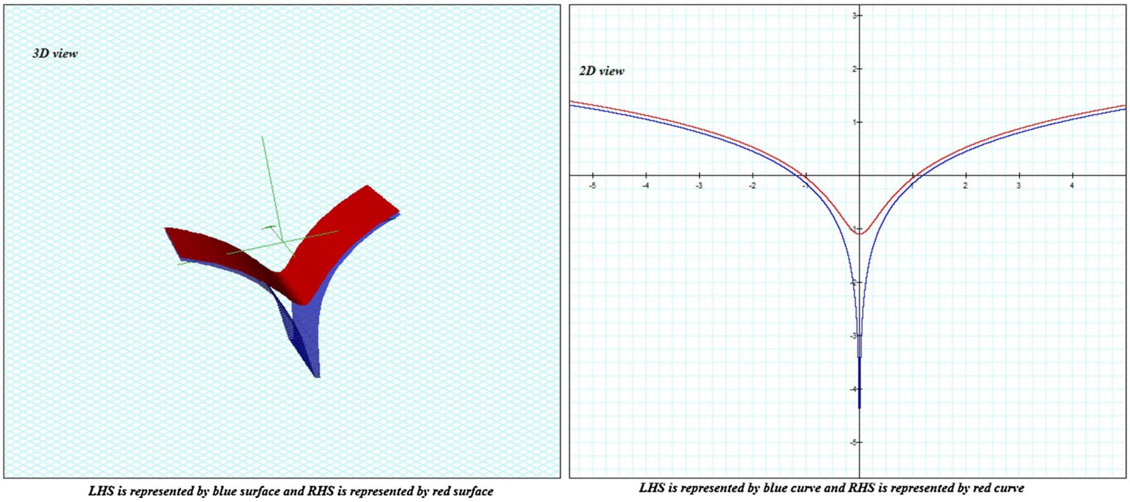


Fig. 2 Plot of inequality for $x, y \in [0, 30]$ with $\epsilon = 4.5$

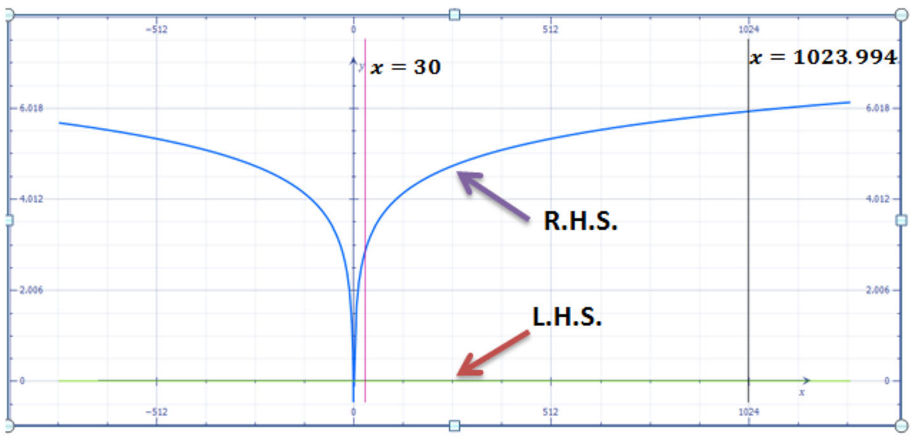


Fig. 3 Plot of inequality for Case II(i)

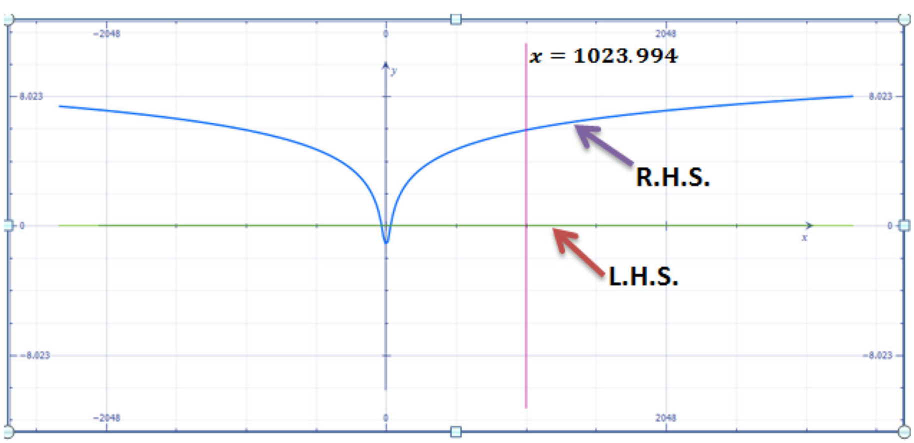


Fig. 4 Plot of inequality for Case II(ii)

By repeating the same procedure as in Case II, one can find that Case III holds for all $y \in [0, 30]$ and $x \in (30, \infty]$.

Hence, we conclude that inequality (1) of Theorem 2.1 holds for all $x, y \in X$. Notice that, in the foregoing example, all the conditions of Theorem 2.1 are fulfilled and $x = 0$ is a unique common fixed point of the mappings f and g (see Fig. 5).

Example 2.2 Let $X = \{0, 1, 2\}$. Inspired by [18], let we define a partial b-metric $p_b : X \times X \to [0.\infty)$ by $p_b(x, x) = 0$ for all $x \in X$, $p_b(0, 1) = p_b(1, 0) = p_b(1, 2) =$

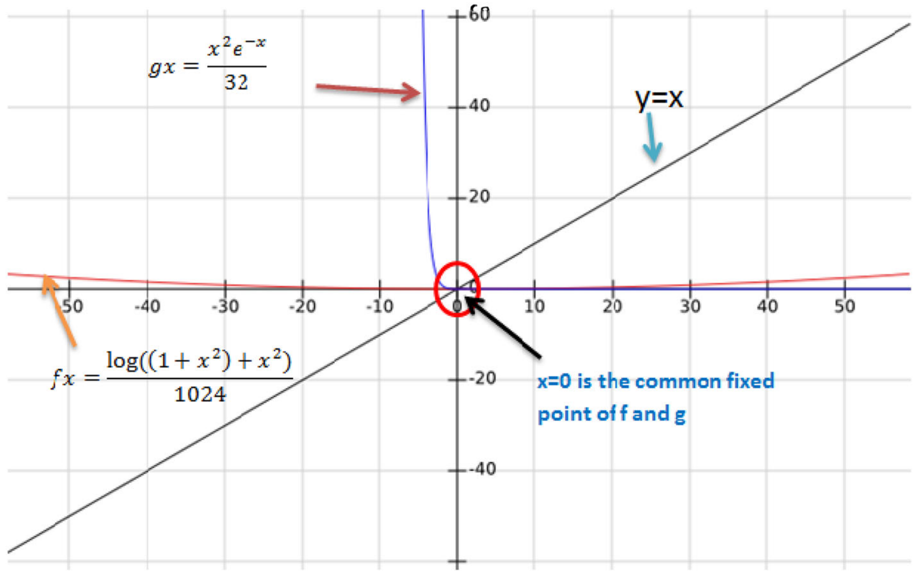


Fig. 5 Common fixed point of f and g

$p_b(2,1) = 1,\ p_b(0,2) = p_b(2,0) = 9/4$ with the partial-order relation $x \preceq y \iff x < y$

It is easy to obtain that (X, p_b) is a complete partial b-metric space with $s = 9/8$. Define self maps f and g by $f0 = 1; f1 = 1; f2 = 0$ and $g0 = g2 = 0;\ g1 = 1$. Clearly, the mappings f and g are continuous. Let $\alpha(x, y) = 1$ for all $x, y \in X$. Taking $x_0 = 2$, we have $\alpha(2, f2) = \alpha(2, 0) = 1 \geq 1$.

Let $\phi(t) = \frac{19t+3}{23}$ and $\psi(t) = \frac{1}{50(t+1)}$.

It is easy to see that the contractive condition (1) of Theorem 2.1 is satisfied for the points $x = 1, y = 2$ and $x = 0, y = 2$ with $1 < \epsilon < 5$ and $F(t) = \log t$. However, it is not holding for the point $x = 0, y = 1$. Thus, $x = 1$ is not the unique common fixed point of the mappings f and g.

Example 2.3 Let $X = [0, \frac{23}{100}]$. Clearly, X is a complete partial b-metric space with partial b-metric $p_b : X \times X \to [0, \infty)$ defined by

$$p_b(x, y) = [\max\{x, y\}]^2, \quad \forall x, y \in X,$$

where $s = 2$. Let mappings $f, g : X \to X$ are defined by

$$fx = \begin{cases} \dfrac{x + 0.01}{4}, & x \in \left[0, \dfrac{1}{10}\right]; \\ \dfrac{2}{10}, & \text{otherwise.} \end{cases} \quad \text{and}$$

$$gx = \begin{cases} \dfrac{3x}{4}, & x \in \left[0, \dfrac{1}{10}\right]; \\ \dfrac{2}{10}, & \text{otherwise.} \end{cases}$$

Now, we show that (f, g) is a Boyd–Wong **type** A F-contraction with the functions $\psi(t) = \frac{1}{200t(t^2+900)}$, $\phi(t) = \frac{123t+2}{126}$, $F(t) = \log t$ and

$$\alpha(x, y) = \begin{cases} 1, & x, y \in \left[0, \dfrac{1}{10}\right]; \\ \dfrac{\log 1.35(e^5 - e^4)}{e^3 - e^2}, & \text{otherwise.} \end{cases}$$

By the definitions of functions f and α, it is clear that f is an α-admissible map. Without loss of generality, we may assume that $x < y$. For $x, y \in X$, we distinguish the following cases:

Case I: $x, y \in [0, \frac{1}{10}]$. By repeating the same technique as mentioned in Example 2.1, one can obtain that

$$\begin{aligned} \text{L.H.S.} &= \alpha(x, y) F\left(s^{\epsilon} p_b(fx, gy)\right) \\ &\leq F\left(2^{\epsilon-4}(9y^2)\right). \end{aligned}$$

For R.H.S., it follows from the definition of $M_s(x, y)$ that $M_s(x, y) = \phi(y^2)$. By (1), we have

$$\begin{aligned} \text{R.H.S.} &= F(\phi(y^2)) - \psi(y^2) \\ &= \log\left(\frac{123y^2 + 2}{126}\right) - \frac{1}{200y^2(y^4 + 900)}. \end{aligned}$$

It is easy to verify that condition (1) is holding for all $x, y \in [0, \frac{1}{10}]$ with $\epsilon = 1.01$.

Case II: If $x, y \in (\frac{1}{10}, \frac{23}{100}]$, then $\alpha(x, y) = \frac{\log 1.35(e^5 - e^4)}{e^3 - e^2}$. By repeating the same process as in case I, one can easily say that (1) is satisfied for all $x, y \in (\frac{1}{10}, \frac{23}{100}]$.

From all cases, we conclude that (f, g) is a Boyd–Wong type A F-contraction on X. Notice that, all the conditions of Theorem 2.1 are satisfied and $x = \frac{2}{10}$ is the unique common fixed point of the mappings f and g.

The following result is an immediate consequence of Theorem 2.1 using $g=f$ for all $x\in X$, $\psi(t)=\tau>0$, $\alpha(x,y)=1$ for all $x,y\in X$, and $\phi(t)=t$ for all $t\in[0,\infty)$.

Corollary 2.1 *Let (X,p_b) be a complete partial b-metric space with $s\geq 1$. Let f be a self-mapping on X. If there exists $F\in\Delta_F$ and $\tau>0$, such that for all $x,y\in X$ with $p_b(fx,fy)>0$:*

$$\tau+F(s^{\epsilon}p_b(fx,fy))\leq F(M_s(x,y)), \tag{28}$$

where

$$M_s=\max\{p_b(x,y),p_b(x,fx),p_b(y,fy),\frac{1}{2s}[p_b(x,fy)+p_b(y,fx)]\}.$$

Then, f has a unique fixed point and $p_b(u,u)=0$.

Common fixed point results for Boyd–Wong type A^* F-contraction

Theorem 2.2 *Let $(X,p_b,\preceq)$ be a complete partially ordered partial b-metric space. Let f and g are self-mappings on X satisfying the following conditions:*

1. *The pair (f, g) is weakly increasing.*
2. *For every two comparable elements $x,y\in X$, (f, g) is a Boyd–Wong* **type** *A^* F-contraction on (X,p_b).*

Then, f and g have a unique common fixed point $u\in X$ with $p_b(u,u)=0$.

Proof Let $x_0\in X$ be arbitrary. Define a sequence $\{x_n\}$ in the following way:

$fx_{2n}=x_{2n+1}$ and $gx_{2n+1}=x_{2n+2}$ for all $n\in\mathbb{N}^*$.

Since f and g are weakly increasing with respect to $\preceq$, therefore

$$x_1=fx_0\preceq gfx_0=x_2=gx_1\preceq fgx_1=x_3\preceq\cdots$$
$$\preceq x_{2n+1}=fx_{2n}\preceq gfx_{2n}=x_{2n+1}\preceq\cdots.$$

By induction, we obtain that $x_n\preceq x_{n+1}$ for all $n\geq 1$.

The rest of the proof run on the lines of the proof of Theorem 2.1. This conclude the proof. □

Common fixed point results for Boyd–Wong type B F-contraction

In this section, we launch the following definition:

Definition 2.2 Let (X,p_b) be a partial b-metric space and $f,g:X\to X$ two mappings. In addition, suppose that $\alpha:X\times X\to[0,\infty)$ be a function, where α is defined as in Definition 1.8. We say that f is a Boyd–Wong **type B** F-contraction with respect to g, on a partial b-metric space X, if there exists $F\in\Delta_F$, $\psi\in\Psi$ and $\phi\in\Phi$, such that for all $x,y\in X$ and $s>1$ with $p_b(fx,gy)>0$

$$\alpha(x,y)F(s^{\epsilon}p_b(fx,fy))\leq F(\phi(M_s(x,y)))-\psi(p_b(x,y)), \tag{29}$$

where

$$M_s(x,y)=\max\Big\{p_b(gx,gy),p_b(gx,fx),p_b(gy,fy),\frac{p_b(gx,fy)+p_b(gy,fx)}{2s}\Big\}, \tag{30}$$

and $\epsilon>1$ is a constant.

Theorem 2.3 *Let (X,p_b) be a complete partial b-metric space. Let f and g are self-mappings on X, such that $fX\subseteq gX$. Suppose that gX is closed and the following conditions hold:*

1. *f is g-α-admissible and triangular α-admissible.*
2. *There exists $x_0\in X$, such that $\alpha(gx_0,fx_0)\geq 1$.*
3. *f is a Boyd–Wong* **type B** *F-contraction with respect to g* on *(X,p_b).*
4. *Either f or g is continuous. Then, f and g have a coincidence point in X. Moreover, f and g have a unique common fixed point if the following conditions hold:*
5. *The pair $\{f,g\}$ is weakly compatible.*
6. *Either $\alpha(u,v)\geq 1$ or $\alpha(v,u)\geq 1$ whenever $fu=gu$* and *$fv=gv$*

Proof By assumption (2), there exists a point $x_0\in X$, such that $\alpha(gx_0,fx_0)\geq 1$. As $fX\subseteq gX$, we can find a point $x_1\in X$, such that $fx_0=gx_1$. By induction, we construct a sequence $\{x_n\}\in X$, such that

$$gx_{n+1}=fx_n, n=0,1,2,\ldots \tag{31}$$

Since f is g-α-admissible and we have $\alpha(gx_0,fx_0)\geq 1$, therefore

$$\alpha(gx_0,gx_1)=\alpha(gx_0,fx_0)\geq 1\Longrightarrow\alpha(gx_1,gx_2)=\alpha(fx_0,fx_1)\geq 1,$$

$$\alpha(gx_1,gx_2)\geq 1\Longrightarrow\alpha(gx_2,gx_3)=\alpha(fx_1,fx_2)\geq 1.$$

Similar to above, we get

$$\alpha(gx_n,gx_{n+1})\geq 1, n=0,1,2,\ldots. \tag{32}$$

By assumption (1), f is triangular α-admissible; therefore, from Lemma 1.2, we obtain

$$\alpha(gx_m,gx_n)\geq 1 \text{ with } m<n.$$

If for some $n_0\in\mathbb{N}$, we have $fx_{n_0}=fx_{n_0+1}$, then from (31), we have

$$gx_{n_0+1}=fx_{n_0}=fx_{n_0+1}.$$

Then, f and g have a coincidence point at $x = x_{n_0+1}$. Therefore, in what follows, we assume that for each $n \geq 0$, $fx_n \neq fx_{n+1}$ holds. On using inequality (29) and property (p_{b_4}) [5], we have

$$\begin{aligned} F(p_b(gx_n, gx_{n+1})) &\leq F(s^{\epsilon} p_b(fx_{n-1}, fx_n)) \\ &\leq \alpha(x_{n-1}, x_n) F(s^{\epsilon} p_b(fx_{n-1}, fx_n)) \\ &\leq F(\phi(M_s(x_{n-1}, x_n))) - \psi(p_b(x_{n-1}, x_n)), \end{aligned} \tag{33}$$

in which

$$\begin{aligned} &M_s(x_{n-1}, x_n) = \\ &\max\{p_b(gx_{n-1}, gx_n), p_b(gx_{n-1}, fx_{n-1}), p_b(gx_n, fx_n), \\ &\frac{p_b(gx_{n-1}, fx_n) + p_b(gx_n, fx_{n-1})}{2s}\} \\ &= \max\left\{p_b(gx_{n-1}, gx_n), p_b(gx_n, gx_{n+1}), \right. \\ &\left. \frac{p_b(gx_{n-1}, gx_{n+1}) + p_b(gx_n, gx_n)}{2s}\right\} \\ &= \max\left\{p_b(gx_{n-1}, gx_n), p_b(gx_n, gx_{n+1})\right\}. \end{aligned}$$

If for some $n \in \mathbb{N}^*$, $M_s(x_{n-1}, x_n) = p_b(gx_n, gx_{n+1})$. Then, from (33), we get

$$F(p_b(gx_n, gx_{n+1})) \leq F(\phi(p_b(gx_n, gx_{n+1}))) - \psi(p_b(gx_{n-1}, gx_n)).$$

Utilizing the property of the function ψ, ϕ and from (Δ_{F1}), we arrive at

$$F(p_b(gx_n, gx_{n+1})) < F(p_b(gx_n, gx_{n+1})),$$

which gives a contradiction. Thus, $M_s(x_{n-1}, x_n) = p_b(gx_{n-1}, x_n)$. Therefore, from (6) and by the property of F, ϕ and ψ, we acquire

$$F(p_b(gx_n, gx_{n+1})) \leq F(\phi(p_b(gx_{n-1}, gx_n)) - \psi(p_b(x_{n-1}, x_n)), \tag{34}$$

which yields $p_b(gx_n, gx_{n+1}) < p_b(gx_{n-1}, gx_n)$. Thus, $\{p_b(gx_n, gx_{n+1})\}$ is a decreasing sequence of positive real numbers. Using the property of ϕ and repeated use of (34), we deduce that

$$\begin{aligned} &F(p_b(gx_n, gx_{n+1})) < F(p_b(gx_{n-1}, gx_n)) - \psi(p_b(x_{n-1}, x_n)) \\ &< F(p_b(gx_{n-2}, gx_{n-1})) - \psi(p_b(x_{n-2}, x_{n-1})) - \psi(p_b(x_{n-1}, x_n)). \end{aligned}$$

As ψ is an decreasing function. Therefore, from the above inequality, we arrive at

$$F(p_b(gx_n, gx_{n+1})) < F(p_b(gx_{n-2}, gx_{n-1})) - 2\psi(p_b(x_{n-2}, x_{n-1})).$$

It follows from the successive application that

$$F(p_b(gx_n, gx_{n+1})) < F(p_b(gx_0, gx_1)) - n\psi(p_b(x_0, x_1)). \tag{35}$$

Since $F \in \Delta_F$, making the limit as $n \to \infty$ in (35) and in account of property $(F2')$, Lemma 1.3, we have

$$\lim_{n\to\infty} F(p_b(gx_n, gx_{n+1})) = -\infty \iff \lim_{n\to\infty} p_b(gx_n, gx_{n+1}) = 0. \tag{36}$$

Furthermore, from (p_{b2}), we get

$$\lim_{n\to\infty} p_b(gx_n, gx_n) = 0. \tag{37}$$

Next, we shall show that $\{gx_n\}$ is a p_b-Cauchy sequence in X. By repeating the same technique as mentioned in Theorem 2.1, one can show that

$$\frac{\delta}{2s} \leq \lim_{k\to\infty} \inf p_b(x_{m(k)}, x_{n(k)-1}) \leq \lim_{k\to\infty} \sup p_b(x_{m(k)}, x_{n(k)-1}) \leq \frac{\delta}{2}. \tag{38}$$

$$\lim_{k\to\infty} \sup p_b(x_{m(k)}, x_{n(k)}) \leq \frac{s\delta}{2}. \tag{39}$$

$$\frac{\delta}{2s} \leq \lim_{k\to\infty} \sup p_b(x_{m(k)+1}, x_{n(k)}). \tag{40}$$

$$\lim_{k\to\infty} \sup p_b(x_{m(k)+1}, x_{n(k)-1}) \leq \frac{s\delta}{2}. \tag{41}$$

Since $F(p_b(fx_{m(k)}, fx_{n(k)-1})) = F(p_b(gx_{m(k)+1}, gx_{n(k)})) > 0$, therefore, by inequality (29) and using the property of ψ, we have

$$\begin{aligned} F(p_b(gx_{m(k)+1}, gx_{n(k)})) &\leq F(s^{\epsilon} p_b(fx_{m(k)}, fx_{n(k)-1})) \\ &\leq \alpha(x_{m(k)}, x_{n(k)-1}) F(s^{\epsilon} p_b(fx_{m(k)}, gx_{n(k)-1})) \\ &\leq F(\phi(M_s(x_{m(k)}, x_{n(k)-1}))) - \psi(p_b(x_{m(k)}, x_{n(k)-1})) \\ &\leq F(\phi(M_s(x_{m(k)}, x_{n(k)-1}))). \end{aligned} \tag{42}$$

On using inequalities (38)–(41), we arrive at

$$\lim_{k\to\infty} \sup M_s(x_{m(k)}, x_{n(k)-1}) \leq \frac{\delta}{2}. \tag{43}$$

Moreover, from (40),(42), and (43), we get

$$\begin{aligned} F\left(s\frac{\delta}{2s}\right) &\leq \lim_{k\to\infty} (\alpha(x_{m(k)}, x_{n(k)-1}) F(s^{\epsilon} \sup p_b(fx_{m(k)}, gx_{n(k)-1}))) \\ &\leq F\left(\phi\left(\lim_{k\to\infty} \sup M_s(x_{m(k)}, x_{n(k)-1})\right)\right), \end{aligned}$$

which yields $F\left(\frac{\delta}{2}\right) \leq F\left(\phi\left(\frac{\delta}{2}\right)\right) < F\left(\frac{\delta}{2}\right)$, a contradiction. Hence, $\{gx_n\}$ is a b-Cauchy sequence in the b-metric space (X, d_{p_b}), then form Lemma 1.1, $\{gx_n\}$ is a p_b-Cauchy sequence in the partial b-metric space (X, p_b). Since gX is closed, there exists $u \in X$, such that

$$\lim_{n\to\infty} gx_n = gu. \tag{44}$$

From Lemma 1.1, we have

$$\lim_{n\to\infty} p_b(gx_n, gu) = \lim_{n,m\to\infty} p_b(gx_n, gx_m) = p(gu, gu) = 0. \tag{45}$$

Now, assume that, g is continuous. Next, we will show that u is a coincidence point of f and g. On the contrary, suppose that, that is not true then $p_b(fu, gu) > 0$. From (29), we get

$$\begin{aligned} F(p_b(fu, gx_{n+1})) = F(p_b(fu,fx_n)) &\le \alpha(u, x_n)F(s^\epsilon p_b(fu,fx_n)) \\ &\le F(\phi(M_s(u, x_n))) - \psi(p_b(u, x_n)), \end{aligned} \tag{46}$$

in which

$$\begin{aligned} &M_s(u, x_n) = \max \\ &\left\{p_b(gu, gx_n), p_b(gu,fu), p_b(gx_n, gx_{n+1}), \frac{p_b(gu, gx_{n+1}) + p_b(gx_n,fu)}{2s}\right\}. \\ &\lim_{n\to\infty} M_s(u, x_n) \\ &\max\left\{p_b(gu, gu), p_b(gu,fu), p_b(gu, gu), \frac{p_b(gu, gu) + p_b(gu,fu)}{2s}\right\} \\ &= p_b(gu,fu). \end{aligned} \tag{47}$$

Passing to the limit when $n \to \infty$ in (46) and using (47), we get that

$$\begin{aligned} &F(p_b(fu, gu)) \le F(\phi(p_b(gu,fu))) \\ &F(p_b(fu, gu)) < F(p_b(fu, gu)), \end{aligned}$$

a contradiction. Therefore, $p_b(fu, gu) = 0$, and hence, $fu = gu$, i.e., u is a coincidence point of f and g. Similarly, one can show that u is a coincidence point of f and g when f is continuous.

Next, we will prove that $gu = gv$. From hypotheses (5) and (6), let $fu = gu$ and $fv = gv$ whenever $\alpha(u, v) \ge 1$. Thus, by the routine calculation and from (29), we have

$$\begin{aligned} F(p_b(gu, gv)) &\le \alpha(u, v)F(s^\epsilon p_b(fu,fv)) \\ &\le F(\phi(gu, gv)) - \psi(p_b(u, v)) \\ F(p_b(gu, gv)) &< F(p_b(gu, gv)), \end{aligned}$$

a contradiction. Hence, we conclude that $gu = gv$. Similarly, for $\alpha(v, u) \ge 1$, one can easily show that $gu = gv$. By hypotheses, the pair $\{f, g\}$ is weakly compatible, i.e., $fw = f(gu) = g(fu) = gw$, whenever $fu = gu = w$. Hence, w is a coincidence point of f and g, which yields $gu = gw = w = fw$. Thus, w is a common fixed point of f and g. Furthermore, in a similar way as in the proof of Theorem 2.1, one can show the uniqueness of the common fixed point.

The following example demonstrates the usability of Theorem 2.3. □

Example 2.4 Let $X = [0, 5]$ be equipped with the partial-order relation $\preceq$ defined by $x \preceq y \iff x > y$ and the function $p_b : X \times X \to [0,\infty)$ is given by $p_b(x, y) = [\max\{x, y\}]^2$ for all $x, y \in X$, where $s = 2$. It is obvious that (X, p_b) is a complete partial b-metric space. Consider the mappings $f, g : X \to X$ defined by $fx = \frac{1}{5}\log(1 + 0.1x + 0.2x^2)$ and $gx = \frac{x}{2}$. The mapping $\alpha : X \times X \to [0,\infty)$ is defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in gX, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\psi : (0,\infty) \to (0,\infty)$ by $\psi(t) = \frac{1}{100(t+3)}$. In addition, let $\phi : [0,\infty) \to [0,\infty)$ be given by $\phi(t) = \frac{120t+2}{123}$. Let $F(t) = \log t$ for all $t \in \mathbb{R}^+$.

Clearly, f, g are continuous mappings and $fX \subseteq gX$. To prove that f is g-α-admissible mapping, let $x, y \in X$, such that $\alpha(gx, gy) \ge 1$, then by the definition of α and $fX \subseteq gX$, we have $\alpha(fx,fy) \ge 1$. Thus, we conclude that f is g-α-admissible mapping. Taking $x_0 = 0 \in X$, we have $\alpha(gx_0,fx_0) = \alpha(g0,f0) = \alpha(0, 0) = 1 \ge 1$. Let $x, y, z \in X$, such that $\alpha(x, z) \ge 1$ and $\alpha(z, y) \ge 1$, from the definition of α, we have $\alpha(x, y) \ge 1$, i.e., f is triangular α-admissible. To verify the inequality (29) of Theorem 2.3, we have to consider the following cases:

Case I. If $x, y \in [0, 2.5]$, then we obtain that

$$\begin{aligned} \text{L.H.S.} &= \alpha(x, y)F(s^\epsilon p_b(fx,fy)) \\ &= F\left(2^\epsilon \max\left\{\frac{1}{5}\log(1 + 0.1x + 0.2x^2), \frac{1}{5}\log(1 + 0.1y + 0.2y^2)\right\}^2\right) \\ &\le F\left(2^{\epsilon-4} \max\left\{\log(1 + 0.1x + 0.2x^2), \log(1 + 0.1y + 0.2y^2)\right\}^2\right) \\ &\le F(2^{\epsilon-4} \max\{x, y\}^2) \\ &= \log(2^{\epsilon-4}x^2). \end{aligned} \tag{48}$$

For R.H.S., using the definitions of $M_s(x, y)$, we have $M_s(x, y) = \frac{x^2}{4}$. From (29), we arrive at

$$\begin{aligned} \text{R.H.S.} &= F\left(\phi\left(\frac{x^2}{4}\right)\right) - \psi(\max\{x, y\}^2) \\ &= \log\left(\frac{120x^2 + 8}{492}\right) - \frac{1}{100(x^2 + 3)}. \end{aligned} \tag{49}$$

Figures 6 and 7 demonstrate that R.H.S. with red surface dominates the blue surface, i.e., L.H.S.

From Figs. 6 and 7, we have inequality (29) holds for all $x, y \in [0, 2.5]$ with $\epsilon \in (1, 1.9]$.

Case II. If $x, y \in (2.5, 5]$, then it follows from (29) that

$$\text{L.H.S.} = \alpha(x, y)F(s^\epsilon p_b(fx,fy)) = 0 \le \log\left(\frac{120x^2 + 8}{492}\right) - \frac{1}{100(x^2 + 3)}.$$

Following Fig. 8 validates that R.H.S. with blue line overshadows the L.H.S. with green line. Consequently, we conclude that condition (29) is satisfied for $x, y \in (2.5, 5]$.

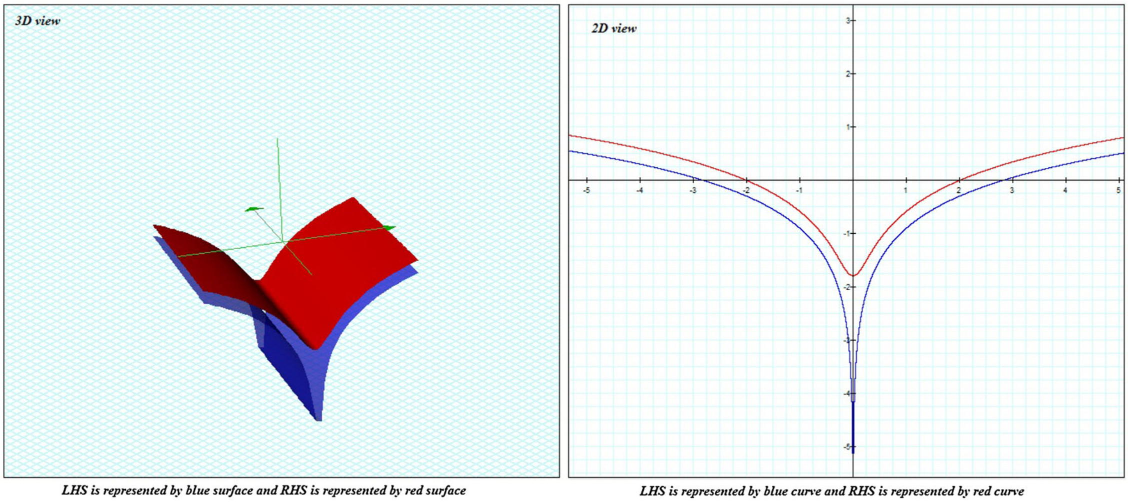


Fig. 6 Plot of inequality for Case I when $\epsilon = 1.1$

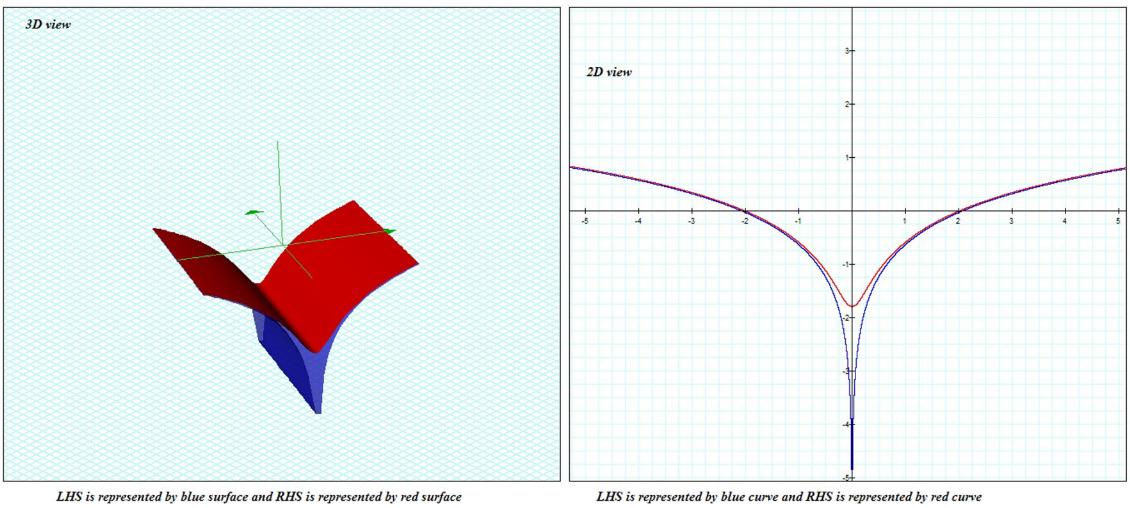
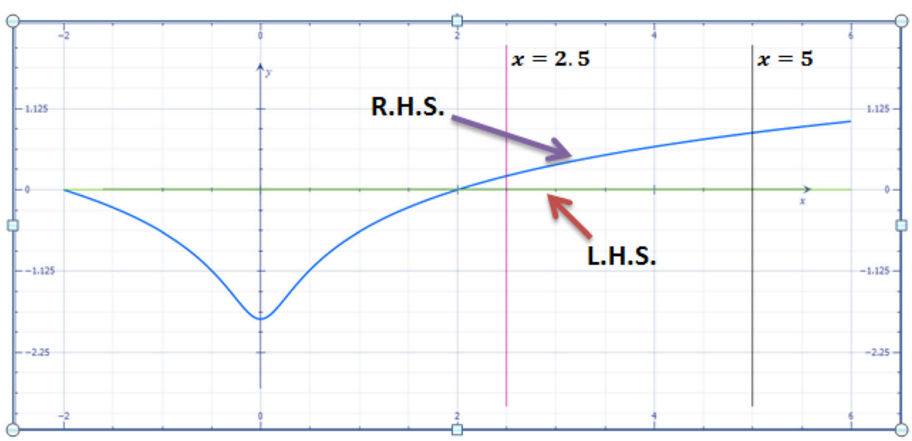


Fig. 7 Plot of inequality for Case I when $\epsilon = 1.9$

Fig. 8 Plot of inequality for Case II



Case III. If $y \in [0, 2.5]$ and $x \in (2.5, 5]$, then Case III is analogous to Case II that is why we omit the details.

Thus, $x = 0$ is the coincidence point of mappings f and g. It is easy to observe that the pair $\{f, g\}$ is weakly compatible and condition (6) holds whenever $f0 = g0$.

Hence, all the conditions of Theorem 2.3 are satisfied and the subsequent figure (see Fig. 9) show that $x = 0$ is a unique common fixed point f and g.

In view of Remark 1.1, the following observations are worth noticing in the perspective of Theorems 2.1, 2.2, and 2.3.

Remark 2.1 Theorem 10 and Corollary 13 of Shukla et al.[19] are particular case of Theorem 2.1 by taking $s = 1,\ \psi(p_b(x,y)) = \tau > 0,\ \phi(t) = t,\ \alpha(x,y) = 1$ and $s = 1,\ f = g,\ \psi(p_b(x,y)) = \tau > 0,\ \phi(t) = t,\ \alpha(x,y) = 1$, respectively.

Remark 2.2 If we take $s = 1$, $\alpha(x,y) = 1$ $\psi(p_b(x,y)) = \tau > 0$, $\phi(t) = t$ and $f = g$ in Theorem 2.1, then we obtain Theorem 3.2 [20] of Radenovic and Kadelburg along with Shukla.

Remark 2.3 We generalize the Theorem 17 of Alsulami et al. in [21] for partial b-metric space.

Remark 2.4 Theorem 2.1 of Singh et al. [22] is particular case of Theorem 2.3 by taking $\alpha(x,y) = 1$, $gx = x$, and $s = 1$.

In [7], author stated that the F-contraction is the modified version of Banach contraction principle. Wardowski deduced that the Banach contractions are particular case of F-contractions and the author supported his finding by presenting some F-contractions which are not Banach contractions.

In view of aforesaid, we generalized and extend the following results present in the literature:

Remark 2.5 By introducing Theorems 2.1 and 2.3, we generalized the results of Satish Shukla [5] and obtained the F-contraction version of [5] in partial b-metric spaces.

Remark 2.6 Taking $\epsilon = 1, f = g$ and $\phi(t) = kt$, where $k \in [0,1)$ in Theorem 2.2 is akin to Corollary 1 of Mustafa [6] in the sense of F-contraction. On the other hand, to be specific taking $\alpha(x,y) = 1$, $\epsilon = 2$ and $\phi(t) = kt$, where $k \in [0,1)$ in Theorem 2.2 reduces to Corollary 3 of Mustafa [6] for F-contraction.

Remark 2.7 If we take $\epsilon = 1$ and $f = g$ in Theorem 2.1, then Theorem 2.6 in [26] due to Latif et al. is attained.

Remark 2.8 Theorem 2.1 of Huang et al.[23] is particular case of Theorem 2.2 for F-contraction by taking $\alpha(x,y) = 1$ and $\phi(t) = t$

Remark 2.9 In Theorem 2.3, if we put $s = 1$, $\alpha(x,y) = 1$, and $gx = x$, then we obtain Theorem 2.3 for F-contraction by S. Romaguera in [24].

Slip-ups in some recent papers and their remedies

In this section, we point out some slip-ups, which are worth noting in the perspective of some recent papers Arab et al.[25], Latif et al.[26], Shukla et al. [19], and Shahkoohi et al. [27].

1. In [25], we point out that how $\mathbb{R}^+$ can be extended to hold the definition of α-admissible map.
2. In Definition 2.1, authors [25] defined the almost generalized $(\alpha\text{–}\psi\text{–}\phi\text{–}\theta)$-contraction. On using this authors reported that

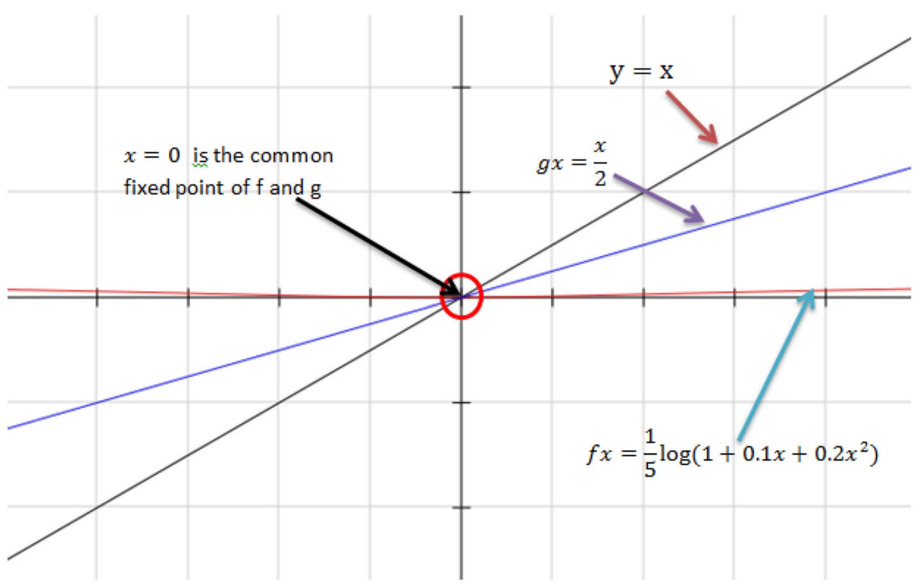


Fig. 9 Common fixed point of f and g

$$\psi(d(gx_n, gx_{n+1})) \le \alpha(gx_{n-1}, gx_n)\psi(s^3 d(Tx_{n-1}, Tx_n))$$
(see inequality (2.4)),

which is worthless, one need to replace $\alpha(gx_{n-1}, gx_n)$ by $\alpha(x_{n-1}, x_n)$. The authors committed the same mistakes on the page no. 7 and 8. On the other hand, authors wrote

$$\psi(s^3 d(Tu, Tv)) \le \alpha(u, v)\psi(s^3 d(Tu, Tv)) \quad \text{(see page no. 9)}.$$

Therefore, there was a dispute regarding to condition (2.1) in the whole paper [25].

3. In [26], authors committed a blunder. Notice that, in the context of Theorem 2.6, authors defined $M_s(x, y)$ in terms of $d(x, y)$, and in whole proof of Theorem 2.6, they used $M_s(x, y)$ in the form of $p_b(x, y)$, which is unsound as for this to hold one needs to supplant $d(x, y)$ by $p_b(x, y)$ in the statement of Theorem 2.6.
4. In [19] from inequality (7), authors got a contradiction and concluded that $F(p(x_{2n}, x_{2n+2})) = 0$ that is $x_{2n} = x_{2n+2}$. Now, by the property of partial metric space $x_{2n} = x_{2n+2}$ when $p(x_{2n}, x_{2n+2}) = 0$, which is incorrect because the function F is not defined at the point 0.
5. Note that, in application section of [27], authors established equivalency between $2^{p-1}|Sx(t) - Sy(t)| \le \sqrt[p]{ln(M(x, y) + 1)}$ and $2^{p-1}|Sx(t) - Sy(t)|^p \le ln(M(x, y) + 1)$, which is not possible. For this, we suggested rectification in our application part.

Applications

Application to solutions of integral equations

In this section, we obtain the solution of the following integral equation for an unknown function u:

$$u(t) = v(t) + \int_a^b G(t, z)f(z, u(z))\mathrm{d}z, t \in [a, b], \tag{50}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, $G : [a, b] \times [a, b] \to [0, \infty)$, $v : [a, b] \to \mathbb{R}$ are given continuous functions.

Let X be the set $C[a, b]$ of real continuous functions on $[a, b]$ and let $p_b : X \times X \to [0, \infty)$ be given by

$$p_b(u, v) = \max_{a \le t \le b} |u(t) - v(t)|^q + r, \tag{51}$$

where $q > 1$ and $r \ge 0$. One can easily see that (X, p_b) is a complete partial b-metric space. Let the mapping $T : X \to X$ is defined by

$$Tu(t) = v(t) + \int_a^b G(t, z)f(z, u(z))\mathrm{d}z, t \in [a, b], \tag{52}$$

then u is a solution of (50) if and only it is a fixed point of T. Now, we prove the subsequent theorem to show the existence of solution of integral equation.

Theorem 4.1 Assume that the following assumptions hold:

(1)
$$\max_{a \le t \le b} \int_a^b |G(t, z)|^q \mathrm{d}z \le \frac{2}{b - a}.$$

(2) For all $x, y \in \mathbb{R}$, the following inequality holds:
$$|f(z, x) - f(z, y)|^q \le \frac{1}{2s^\epsilon}([|x - y|^q + r]e^{-\tau} - s^\epsilon r).$$

Then, the integral equation (50) has a solution.

Proof Utilizing conditions (1)–(2) and in account of inequality (50), we have

$$\begin{aligned}
s^\epsilon p_b(Tu_1, Tu_2) &= s^\epsilon \max_{a \le t \le b} |Tu_1(t) - Tu_2(t)|^q + s^\epsilon r \\
&= s^\epsilon \max_{a \le t \le b} \Big| v(t) + \int_a^b G(t, z)f(z, u_1(z))\mathrm{d}z \\
&\quad - \left(v(t) + \int_a^b G(t, z)f(z, u_2(z))\mathrm{d}z \right)|^q + s^\epsilon r \\
&= s^\epsilon \max_{a \le t \le b} \left\{ \left| \int_a^b \Big(G(t, z)f(z, u_1(z)) - G(t, z)f(z, u_2(z)) \Big) \mathrm{d}z \right|^q \right\} + s^\epsilon r \\
&\le s^\epsilon \max_{a \le t \le b} \left\{ \int_a^b |G(t, z)|^q \mathrm{d}z. \int_a^b |f(z, u_1(z)) - f(z, u_2(z))|^q \mathrm{d}z \right\} + s^\epsilon r \\
&= s^\epsilon \left\{ \max_{a \le t \le b} \int_a^b |G(t, z)|^q \mathrm{d}z \right\}. \left\{ \int_a^b |f(z, u_1(z)) - f(z, u_2(z))|^q \mathrm{d}z \right\} + s^\epsilon r \\
&\le s^\epsilon \left\{ \frac{2}{b - a} \right\}. \left\{ \frac{1}{2s^\epsilon} \int_a^b (|u_1(z) - u_2(z)|^q + r)e^{-\tau} - s^\epsilon r \Big] \mathrm{d}z \right\} + s^\epsilon r \\
&\le \frac{1}{b - a} \int_a^b \max_{a \le t \le b} [(|u_1(t) - u_2(t)|^q + r)e^{-\tau} - s^\epsilon r]\mathrm{d}z + s^\epsilon r \\
&= \max_{a \le t \le b} (|u_1(t) - u_2(t)|^q + r)e^{-\tau} \\
&= p_b(u_1, u_2)e^{-\tau} \\
&\le M_s(u_1, u_2)e^{-\tau},
\end{aligned}$$

that is

$$s^\epsilon p_b(Tu_1, Tu_2) \le M_s(u_1, u_2)e^{-\tau}.$$

Consequently, by passing to logarithms, we get

$$\tau + \log(s^\epsilon p_b(Tu_1, Tu_2)) \le \log M_s(u_1, u_2),$$

this turns into

$$\tau + F(s^\epsilon p_b(Tu_1, Tu_2)) \le F(M_s(u_1, u_2)),$$

for $F(t) = \log t$, $t > 0$. Thus, all the conditions of Corollary 2.1 are satisfied. Hence, we conclude that T has a unique fixed point u^* in X, which yields that integral equation (50) has a unique solution which belongs to $X = C[a, b]$. □

Application to solutions of ordinary differential equations:

Consider the following first-order periodic boundary value problem:

$$\begin{cases} u'(t) = k(t, u(t)), & t \in I = [0, T] \\ u(0) = u(T), \end{cases} \tag{53}$$

where $T > 0$ and $k : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Consider the space $X = C(I, \mathbb{R})$ of all real continuous functions on $I = [0, T]$. Let $p_b : X \times X \to \mathbb{R}^+$ be given by

$$p_b(u, v) = \max_{t \in I} |u(t) - v(t)|^q + r,$$

for all $u, v \in X$, where $q > 1$ and $r \geq 0$. Obviously, the space (X, p_b) is a complete partial b-metric space with parameter $s = 2^{q-1}$. The space $X = C(I, \mathbb{R})$ can also equipped with the following order relation:

$$x, y \in C(I, \mathbb{R}) \quad x \leq y \iff x(t) \leq y(t) \quad \text{for all} \quad t \in I.$$

Definition 4.1 An element $\alpha \in X$ is called a lower solution for the problem (53) if

$$\begin{cases} \alpha'(t) = k(t, \alpha(t)), & t \in I = [0, T] \\ \alpha(0) = \alpha(T) \end{cases}$$

Theorem 4.2 *Consider the problem (53) and assume that there exists $\lambda > 0$, such that for any $x, y \in \mathbb{R}$, with $x \geq y$ and $\tau > 0$*

$$\begin{aligned} &|k(t, x(t)) + \lambda x(t) - k(t, y(t)) - \lambda y(t)| \\ &\leq \frac{\lambda}{2^{\frac{\epsilon(q-1)}{q}}} \sqrt[q]{[|x(t) - y(t)|^q + r]e^{-\tau} - 2^{\epsilon(q-1)} r}. \end{aligned} \tag{54}$$

Then, the existence of a lower solution for the periodic boundary value problem (53) provides a unique solution for (53).

Proof The problem (53) is equivalent to the integral equation:

$$u(t) = \int_0^T G(t, s)[k(s, u(s)) + \lambda u(s)]\mathrm{d}s, \tag{55}$$

in which

$$G(t, s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} & 0 \leq s \leq t \leq T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} & 0 \leq t \leq s \leq T. \end{cases} \tag{56}$$

Define $f : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$fu(t) = \int_0^T G(t, s)[k(s, u(s)) + \lambda u(s)]\mathrm{d}s. \tag{57}$$

It is easy to note that if $u \in C(I, \mathbb{R})$ is a fixed point of f, then $u \in C^1(I, \mathbb{R})$ is a solution of the problem (53).

For any $u, v \in X$ with $u \geq v$, we have

$$\begin{aligned} &2^{\frac{\epsilon(q-1)}{q}}\left|fu(t) - fv(t)\right| = 2^{\frac{\epsilon(q-1)}{q}}\left| \int_0^T G(t, s)[k(s, u(s)) + \lambda u(s)]\mathrm{d}s \right. \\ &\left. - \int_0^T G(t, s)[k(s, v(s)) + \lambda v(s)]\ \mathrm{d}s\right| \\ &\leq 2^{\frac{\epsilon(q-1)}{q}} \int_0^T \left|G(t, s)\right|\left|k(s, u(s)) + \lambda u(s) - k(s, v(s)) - \lambda v(s)\right|\ \mathrm{d}s \\ &\leq 2^{\frac{\epsilon(q-1)}{q}} \int_0^T \left|G(t, s)\right| \frac{\lambda}{2^{\frac{\epsilon(q-1)}{q}}} \sqrt[q]{[|u(s) - v(s)|^q + r]e^{-\tau} - 2^{\epsilon(q-1)}\ r}\ \mathrm{d}s \\ &\leq \int_0^T \left|G(t, s)\right|\lambda \sqrt[q]{[|u(s) - v(s)|^q + r]e^{-\tau} - 2^{\epsilon(q-1)}\ r}\ \mathrm{d}s \\ &\leq \max_{t \in I} \int_0^T \left|G(t, s)\right|\lambda \sqrt[q]{[|u(s) - v(s)|^q + r]e^{-\tau} - 2^{\epsilon(q-1)}\ r}\ \mathrm{d}s \\ &\leq \lambda \sqrt[q]{p_b(u, v)e^{-\tau} - 2^{\epsilon(q-1)}\ r}\ \max_{t \in I} \int_0^T G(t, s)\,\mathrm{d}s \\ &\leq \lambda \sqrt[q]{p_b(u, v)e^{-\tau} - 2^{\epsilon(q-1)}\ r}\ \max_{t \in I}\left[\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}\mathrm{d}s + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}\mathrm{d}s\right] \\ &\leq \lambda \sqrt[q]{p_b(u, v)e^{-\tau} - 2^{\epsilon(q-1)}\ r}\ \left[\frac{1}{\lambda(e^{\lambda T} - 1)}(e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1)\right] \\ &\leq \sqrt[q]{p_b(u, v)e^{-\tau} - 2^{\epsilon(q-1)}\ r} \end{aligned}$$

which yields

$$\begin{aligned} 2^{\epsilon(q-1)}\left|fu(t) - fv(t)\right|^q &\leq p_b(u, v)e^{-\tau} - 2^{\epsilon(q-1)}\ r \\ 2^{\epsilon(q-1)}\left[\left|fu(t) - fv(t)\right|^q + r\right] &\leq p_b(u, v)e^{-\tau} \\ 2^{\epsilon(q-1)} p_b(fu, fv) &\leq M_s(u, v)e^{-\tau}, \end{aligned}$$

where

$$M_s = \max\{p_b(x, y), p_b(x, fx), p_b(y, fy), \frac{1}{2s}[p_b(x, fy) + p_b(y, fx)]\}.$$

Consequently, by passing to logarithms, above inequality deduce to

$$\log(2^{\epsilon(q-1)} p_b(fu, fv)) \leq \log(M_s(u, v)) + \log e^{-\tau}.$$

This turns into

$$\tau + F(2^{\epsilon(q-1)} p_b(fu, fv)) \leq F(M_s(u, v)).$$

for $F(t) = \log t, t > 0$. Finally, let α be a lower solution for (53). Then from [28] we have $\alpha \leq f(\alpha)$. Thus, the hypotheses of Corollary 2.1 are satisfied and f has a unique fixed point in X, i.e., the system of first-order periodic boundary value problem has a unique solution. □

Open problems

- In Theorems 2.1 and 2.3 can Boyd–Wong **type A** and **type B** F-contraction be improved by cyclic contraction.
- Can Theorem 2.1 be extended and generalized replacing α-admissible by twisted (α, β)-admissible.
- Can Theorem 2.3 be extended and generalized replacing g-α-admissible by generalized-α-admissible.

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