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ORIGINAL RESEARCH

Fuzzy semi-numbers and a distance on them with a case study in medicine

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Abstract

In this paper, we present the novel concept of fuzzy semi-numbers. Then, a method for assigning distance between every pair of fuzzy semi-numbers is given. Moreover, it is shown that this distance is a metric on the set of all trapezoidal fuzzy semi-numbers with the same height and is a pseudo-metric on the set of all fuzzy semi-numbers. Also, by utilizing this distance, we propose an approximation of a fuzzy semi-number with given height and apply this approximation method in a medical case study.

Keywords Fuzzy sets · Fuzzy numbers · Fuzzy semi-numbers · Generalized LR fuzzy semi-numbers · *H*-value · *H*-ambiguity

Introduction

Conventional works on fuzzy logic assume that fuzzy sets satisfy the conditions of convexity and normality (the height is one) which are called fuzzy numbers. Many of the researches in fuzzy logic have contributed to the approximation methods of a fuzzy number. Such an approximation has been done in several ways. Some authors have assigned a crisp number to a fuzzy number as a ranking method. Such methods suffer from a great amount of data loss. Some other methods [10, 14] have defined an interval as an approximation of a fuzzy number. But, in this case, the modal value (the core with height 1) of the fuzzy numbers is lost. In some works such as [1, 3, 13, 15, 16], the authors have tried to solve an optimization problem to obtain a trapezoidal fuzzy number as a nearest approximation. Recently, some works have been done on approximation of a fuzzy number [4, 6–9, 11] by defining distance functions.

All the aforementioned methods destroy much of the useful data associated with the fuzzy number and as a result imprecision of the approximation increases. Such loss is not negligible specially in high critical domains such as medicine where most of the clinical data are intrinsically inexact and ambiguous. To mitigate this, most often applying fuzzy mathematical modeling is favorable approach. However, these data are not necessarily normal fuzzy sets. To overcome these problems, this paper works on fuzzy sets in general form and without regard to their height. At first, we present the novel concept of fuzzy semi-number. Then, we propose a distance to approximate an arbitrary fuzzy semi-number. Depending on some predefined conditions, the result of approximation can be either a fuzzy number or fuzzy semi-number. To this end, in this paper an approximation for a given fuzzy set without considering their height is proposed. Also, for more clarification, a real application of fuzzy semi-number will be studied in terms of a case study.

The structure of the this paper is as follows. In "Preliminaries", the basic concepts of our work are introduced; then in next section, we review fuzzy semi-numbers. Section "Height source distance between fuzzy semi-numbers" introduces a new distance namely Height Source

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Distance for fuzzy semi-numbers. In next section, the nearest trapezoidal fuzzy semi-number to an arbitrary fuzzy semi-number is introduced and a simple method for its computation is presented. Next section contains some numerical examples. Section "A medical case study" presents a case study in medicine and the final section concludes the article.

Preliminaries

Before delving in to our contributions, let us take a glance at some basic fuzzy theory definitions. Let $F(\mathbb{R})$ be the set of all real fuzzy numbers (which are normal, upper semi-continuous, convex and compactly supported fuzzy sets).

The *parametric form* of a fuzzy number is denoted by $\tilde{u} = (\underline{u}, \overline{u})$, where functions $\underline{u}$ and $\overline{u}$ for each $\alpha \in [0, 1]$ satisfy the following requirements [18, 19]:

1. $\underline{u}$ is a monotonically increasing left continuous function.
2. $\overline{u}$ is a monotonically decreasing left continuous function.
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha), \quad 0 \leq \alpha \leq 1$.

Let $\tilde{u} = (\underline{u}, \overline{u})$, $\tilde{v} = (\underline{v}, \overline{v}) \in F(\mathbb{R})$. Some results of applying fuzzy arithmetic on fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are as follows:

- $x > 0 : x\tilde{u} = (x\underline{u}, x\overline{u})$;
- $x < 0 : x\tilde{u} = (x\overline{u}, x\underline{u})$;
- $\tilde{u} + \tilde{v} = (\underline{u} + \underline{v}, \overline{u} + \overline{v})$;
- $\tilde{u} - \tilde{v} = (\underline{u} - \overline{v}, \overline{u} - \underline{v})$.

A fuzzy number $\tilde{v}$ is non-negative (non-positive) if for $x < 0$ $(x > 0)$, we have $\mu_{\tilde{v}}(x) = 0$, equivalently if on [0, 1] we have $\underline{v} \geq 0$ $(\overline{v} \leq 0)$. Also a fuzzy number $\tilde{v}$ is positive (negative) if for $x \leq 0$ $(x \geq 0)$, we have $\mu_{\tilde{v}}(x) = 0$, equivalently if on [0, 1] we have $\underline{v} > 0$ $(\overline{v} < 0)$. For $\alpha \in (0, 1]$, α-cut of a fuzzy set $\tilde{w}$ is defined as $[\tilde{w}]^{\alpha} = \{x | \mu_{\tilde{w}}(x) \geq \alpha\}$ [23].

Definition 1 A function $s \in C[0, 1]$ with the following properties is a *source function* [2, 5] (regular reducing function in [20]) over all fuzzy numbers:

1. $s(\alpha) \geq 0, \quad \alpha \in [0, 1]$
2. $s(0) = 0$,
3. $s(1) = 1$,
4. $\int_0^1 s(\alpha) \mathrm{d}\alpha = \frac{1}{2}$.

Definition 2 The core of a fuzzy number $\tilde{u}$ is defined as follows:

$$\text{core } (\tilde{u}) = \{x | \mu_{\tilde{u}}(x) = 1\} \tag{1}$$

Definition 3 Let s be a *source* function defined on $F(\mathbb{R})$. For $\tilde{u} \in F(\mathbb{R})$, *value* and *ambiguity* associated with s are defined by the following relations [20]:

1. $\mathrm{V}\ (\tilde{u}) = \int_0^1 s(\alpha)[\overline{u}(\alpha) + \underline{u}(\alpha)] \mathrm{d}\alpha$,
2. $\mathrm{A}\ (\tilde{u}) = \int_0^1 s(\alpha)[\overline{u}(\alpha) - \underline{u}(\alpha)] \mathrm{d}\alpha$.

Fuzzy semi-numbers

Here, we present several novel definitions and lemmas which will be used throughout the paper.

Definition 4 A fuzzy set $\tilde{u}$ (with the membership function $\mu_{\tilde{u}}$) is a generalized fuzzy LR semi-number, if there exist a positive number $h \in (0, 1]$ such that

$$\mu_{\tilde{u}}(x) = \begin{cases} l(x), & a \leq x \leq b, \\ h, & b \leq x \leq c, \\ r(x), & c \leq x \leq d, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

where $l(x)$ is nondecreasing on [a, b] and $r(x)$ is nonincreasing on [c, d] such that $l(a) = r(d) = 0$ and $l(b) = r(c) = h$ and we denote fuzzy LR semi-number by $(a, b, c, d; h)_{\mathrm{LR}}$. We show fuzzy semi-number $\tilde{u}$ with height h as $\tilde{u}_h$ and if $h = 1$ then $\tilde{u}$ is an LR fuzzy number [23]. We denote the set of all fuzzy numbers by $F(\mathbb{R})$ and the set of all fuzzy semi-numbers of height h by $F_h(\mathbb{R})$. Also the set of all fuzzy semi-numbers is denoted by $\mathrm{FS}(\mathbb{R})$. i.e.,

$$\mathrm{FS}(\mathbb{R}) := \bigcup_{h \in (0,1]} F_h(\mathbb{R}), \tag{3}$$

it is clear that $F(\mathbb{R})$ is a proper subset of $\mathrm{FS}(\mathbb{R})$. (Actually, $F(\mathbb{R})$ includes the fuzzy sets with height one, however, $\mathrm{FS}(\mathbb{R})$ includes fuzzy sets with arbitrary heights.)

A fuzzy semi-number $\tilde{v}_h$ is non-negative (non-positive) if for $x < 0$ $(x > 0)$, we have $\mu_{\tilde{v}}(x) = 0$, equivalently if on [0, h] we have $\underline{v} \geq 0$ $(\overline{v} \leq 0)$. Also a fuzzy semi-number $\tilde{v}_h$ is positive (negative) if for $x \leq 0$ $(x \geq 0)$, we have $\mu_{\tilde{v}}(x) = 0$, equivalently if on [0, h] we have $\underline{v} > 0$ $(\overline{v} < 0)$.

In addition, if $l(x)$ and $r(x)$ are linear, then $\tilde{u}_h$ is a trapezoidal fuzzy semi-number which is denoted by (a, b, c, d; h). In this case if $b = c$, we denote it by (a, b, d; h), which is a triangular fuzzy semi-number. Furthermore, let $TF(\mathbb{R})$ and $TF_h(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers and all trapezoidal fuzzy semi-numbers on $\mathbb{R}$ with height h, respectively:

$$\begin{cases} TF(\mathbb{R}) = \{(a, b, c, d) : a \leq b \leq c \leq d\}, \\ TF_h(\mathbb{R}) = \{(a, b, c, d; h) : a \leq b \leq c \leq d,\ 0 < h \leq 1\}, \end{cases} \tag{4}$$

we denote the set of all trapezoidal fuzzy semi-numbers with arbitrary heights by TFS($\mathbb{R}$)

$$\mathrm{TFS}(\mathbb{R}) := \bigcup_{h\in(0,1]} TF_h(\mathbb{R}). \tag{5}$$

Definition 5 The core of a fuzzy semi-number $\tilde{u}_h$ with height $h \in (0,1]$ is called H-core and is defined as follows:

$$\text{H-core } (\tilde{u}_h) = \{x|\mu_{\tilde{u}}(x) = h\} \tag{6}$$

Definition 6 A function $s_h \in C[0,h]$ with the following properties is a *source function*:

1. $s_h(\alpha) \geq 0, \quad \alpha \in [0,h]$
2. $s_h(0) = 0,$
3. $s_h(h) = h,$
4. $\int_0^h s_h(\alpha)\mathrm{d}\alpha = \frac{1}{2}h^2.$

Definition 7 Let s_{h_1} be a *source function* over $[0,h_1]$. s_{h_2} is equivalent to s_{h_1} if for a given height h_2, s_{h_2} is defined as $s_{h_2}(\alpha) = \frac{h_2}{h_1}s_{h_1}(\frac{h_1}{h_2}\alpha)$. This equivalence is denoted by $s_{h_1} \approx s_{h_2}$.

Lemma 1 *Let s_{h_1} be a source function over $[0,h_1]$. If $s_{h_2} \approx s_{h_1}$, then s_{h_2} is a source function over $[0,h_2]$.*

Proof For s_{h_2} we have the following relations:

1. $s_{h_2}(\alpha) = \frac{h_2}{h_1}s_{h_1}(\frac{h_1}{h_2}\alpha) \geq 0, \quad \alpha \in [0,h_2]$
2. $s_{h_2}(0) = \frac{h_2}{h_1}s_{h_1}(0) = 0,$
3. $s_{h_2}(h_2) = \frac{h_2}{h_1}s_{h_1}(\frac{h_1}{h_2}h_2) = h_2,$
4. $\int_0^{h_2} s_{h_2}(\alpha)\mathrm{d}\alpha = \frac{h_2}{h_1}\int_0^{h_2} s_{h_1}(\frac{h_1}{h_2}\alpha)\mathrm{d}\alpha =$
   $\frac{h_2}{h_1}\frac{h_2}{h_1}\int_0^{h_1} s_{h_1}(\beta)\mathrm{d}\beta = \frac{1}{2}h_2^2,$

where $\beta = \frac{h_1}{h_2}\alpha$ and $\mathrm{d}\beta = \frac{h_1}{h_2}\mathrm{d}\alpha$.

Hence s_{h_2} is a *source function* over $[0,h_2]$. □

For a fuzzy semi-number $\tilde{u}_h \in F_h(\mathbb{R})$ and $\alpha \in [0,h]$ we define $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ as follows:

$$\underline{u}(\alpha) = \begin{cases} \inf\{x \ : \mu_{\tilde{u}_h}(x) \geq \alpha\}, & \alpha \in (0,h], \\ \inf \ \operatorname{supp} \ (\mu_{\tilde{u}_h}(x)), & \alpha = 0. \end{cases} \tag{7}$$

$$\overline{u}(\alpha) = \begin{cases} \sup\{x \ : \mu_{\tilde{u}_h}(x) \geq \alpha\}, & \alpha \in (0,h], \\ \sup \ \operatorname{supp} \ (\mu_{\tilde{u}_h}(x)), & \alpha = 0. \end{cases} \tag{8}$$

So the *parametric form* of a fuzzy semi-number is denoted by $\tilde{u}_h = (\underline{u}, \overline{u}; h)$, where functions $\underline{u}$ and $\overline{u}$ satisfy the following requirements:

1. $\underline{u}$ is a monotonically increasing left continuous function over [0, h].
2. $\overline{u}$ is a monotonically decreasing left continuous function over [0, h].
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha), \quad 0 \leq \alpha \leq h.$
4. $\underline{u}(\alpha) = \overline{u}(\alpha) = 0$, for $\alpha \notin [0,h]$.

For a trapezoidal fuzzy semi-number which is denoted by $\tilde{u} = (a,b,c,d;h)$, we have

$$\underline{u}(\alpha) = a + \frac{b-a}{h}\alpha, \tag{9}$$

$$\overline{u}(\alpha) = d - \frac{d-c}{h}\alpha. \tag{10}$$

Definition 8 Crisp semi-number a_h°, is a fuzzy singleton set with height h, then $\underline{u}(\alpha) = \overline{u}(\alpha) = a$, for $\forall\alpha \in [0,h]$ and we denote $a_h^\circ = (a;h)$. We denote the set of all crisp semi-numbers of height h by $\mathbb{R}_h^\circ$ and we have $\mathbb{R}_1^\circ = \mathbb{R}$. The additive identity on $F_h(\mathbb{R})$ is $0_h^\circ = (0;h)$.

Definition 9 Let s_h be a *source* function defined over [0, h]. For a fuzzy semi-number $\tilde{u}_h \in F_h(\mathbb{R})$, *H-value* and *H-ambiguity* associated with s_h are defined by the following relations:

1. HV $(\tilde{u}_h) = \int_0^h s_h(\alpha)[\overline{u}(\alpha) + \underline{u}(\alpha)]\mathrm{d}\alpha,$
2. HA $(\tilde{u}_h) = \int_0^h s_h(\alpha)[\overline{u}(\alpha) - \underline{u}(\alpha)]\mathrm{d}\alpha.$

Lemma 2 *For $\tilde{u}_h \in F_h(\mathbb{R})$ and $k \in \mathbb{R}$, we have following properties*:

1. HV $(k\tilde{u}_h) = k$ HV $(\tilde{u}_h)$,
2. HA $(k\tilde{u}_h) = |k|$ HA $(\tilde{u}_h)$.

Proof straightforward. □

Lemma 3 *For $\tilde{u}_h \in F_h(\mathbb{R})$ and $k \in \mathbb{R}$, we have following properties*:

1. HV $(\tilde{u}_h + k) =$ HV $(\tilde{u}_h) + h^2k$,
2. HA $(\tilde{u}_h + k) =$ HA $(\tilde{u}_h)$.

Proof By definition of *H-value* and *H-ambiguity* we can write as follows:

$$\begin{aligned} 1.\ \text{HV } (\tilde{u}_h + k) &= \int_0^h s_h(\alpha)[\overline{u}(\alpha) + k + \underline{u}(\alpha) + k]\mathrm{d}\alpha \\ &= \int_0^h s_h(\alpha)[\overline{u}(\alpha) + \underline{u}(\alpha) + 2k]\mathrm{d}\alpha \\ &= \int_0^h s_h(\alpha)[\overline{u}(\alpha) + \underline{u}(\alpha)]\mathrm{d}\alpha + 2k\int_0^h s_h(\alpha)\mathrm{d}\alpha \\ &= \text{HV } (\tilde{u}_h) + h^2k. \end{aligned}$$

$$\begin{aligned} 2.\ \text{HA } (\tilde{u}_h + k) &= \int_0^h s_h(\alpha)[\overline{u}(\alpha) + k - (\underline{u}(\alpha) + k)]\mathrm{d}\alpha \\ &= \int_0^h s_h(\alpha)[\overline{u}(\alpha) - \underline{u}(\alpha)]\mathrm{d}\alpha \\ &= \text{HA } (\tilde{u}_h). \end{aligned}$$

Lemma 4 *In* FS($\mathbb{R}$), *for H-value and H-ambiguity we have following properties*:

1. HV $(\tilde{u}_h \pm \tilde{v}_h) =$ HV $(\tilde{u}_h) \pm$ HV $(\tilde{v}_h)$,
2. HA $(\tilde{u}_h \pm \tilde{v}_h) =$ HA $(\tilde{u}_h) +$ HA $(\tilde{v}_h)$.

Proof straightforward. □

Definition 10 Let s be a *source* function, then the *source number* $I_{s,h}$ is defined on s as follows:

$$I_{s,h} = \int_0^h s(\alpha)\alpha \mathrm{d}\alpha. \tag{11}$$

Lemma 5 *If* s_1 *and* s_2 *are equivalent source functions defined over* $[0, h_1]$ *and* $[0, h_2]$, *respectively, then we have*:

$$I_{s_2,h_2} = \left(\frac{h_2}{h_1}\right)^3 I_{s_1,h_1}$$

Proof Using Definition 10 we have:

$$\begin{aligned}I_{s_2,h_2} &= \int_0^{h_2} s_{h_2}(\alpha)\alpha \mathrm{d}\alpha = \frac{h_2}{h_1}\int_0^{h_2} s_{h_1}\left(\frac{h_1}{h_2}\alpha\right)\alpha \mathrm{d}\alpha \\ &= \left(\frac{h_2}{h_1}\right)^2 \int_0^{h_1} s_{h_1}(\beta)\left(\frac{h_2}{h_1}\beta\right)d\beta \\ &= \left(\frac{h_2}{h_1}\right)^3 \int_0^{h_1} s_{h_1}(\beta)\beta d\beta = \left(\frac{h_2}{h_1}\right)^3 I_{s_1,h_1}\end{aligned}$$

where $\beta = \frac{h_1}{h_2}\alpha$ and $\mathrm{d}\beta = \frac{h_1}{h_2}\mathrm{d}\alpha$. □

Lemma 6 *For an arbitrary fuzzy semi-number* $\tilde{u}$ *with height* h, *we have* $I_{s,h} < \frac{1}{2}h^3$.

Proof By Mid-point Theorem, the proof is straightforward. □

Height source distance between fuzzy semi-numbers

In this section, we introduce a new distance function namely Height Source Distance to measure the distance between fuzzy semi-numbers which is invariant on translation only in cases where heights are the same.

Definition 11 For $\tilde{u}_{h_u}, \tilde{v}_{h_v} \in \mathrm{FS}(\mathbb{R})$ with heights h_u and h_v, respectively, *Height Source Distance* HSD, is defined as follows:

$$\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = \begin{cases} \frac{1}{2}\{|h_u \text{ HV } (\tilde{u}_{h_u}) - h_v \text{ HV } (\tilde{v}_{h_v})| \\ \quad +|h_u \text{ HA } (\tilde{u}_{h_u}) - h_v \text{ HA } (\tilde{v}_{h_v})| & \text{if } \tilde{u} \neq 0^\circ_{h_u} \textit{ or } \tilde{v} \neq 0^\circ_{h_v} \\ \quad\quad +d_H(h_u^2[\tilde{u}]^{h_u}, h_v^2[\tilde{v}]^{h_v})\}, \\ |h_u - h_v|, & \text{if } \tilde{u} = 0^\circ_{h_u} \textit{ and } \tilde{v} = 0^\circ_{h_v} \end{cases} \tag{12}$$

where d_H is the Hausdorff meter, and $[\tilde{w}]^h$ is the h-cut of fuzzy set $\tilde{w}$.

In the following, we present some theorems of the proposed distance function and investigate its properties from the analytic geometry perspective.

Theorem 7 *For* $\tilde{u}_{h_u}, \tilde{v}_{h_v}, \tilde{w}_{h_w} \in \mathrm{FS}(\mathbb{R})$ *and* $k \in \mathbb{R}$, *HSD satisfies the following properties*:

1. $\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{u}_{h_u}) = 0$,
2. $\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = \mathrm{HSD}(\tilde{v}_{h_v}, \tilde{u}_{h_u})$,
3. $\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{w}_{h_w}) \leq \mathrm{HSD}(\tilde{u}_{h_u}, \tilde{v}_{h_v}) + \mathrm{HSD}(\tilde{v}_{h_v}, \tilde{w}_{h_w})$,
4. $\mathrm{HSD}(k\tilde{u}_{h_u}, k\tilde{v}_{h_v}) = |k|\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{v}_{h_v})$.

Proof If $\tilde{u} = 0^\circ_{h_u}$ and $\tilde{v} = 0^\circ_{h_v}$ then the proof is trivial; otherwise, for each step we can write as follows:

1. Since d_H is a meter we have:

$$\begin{aligned}\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{u}_{h_u}) = \frac{1}{2}\{&|h_u \text{ HV } (\tilde{u}_{h_u}) - h_u \text{ HV } (\tilde{u}_{h_u})| \\ &+ |h_u \text{ HA } (\tilde{u}_{h_u}) - h_u \text{ HA } (\tilde{u}_{h_u})| \\ &+ d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_u^2[\tilde{u}_{h_u}]^{h_u})\Big\} = 0\end{aligned}$$

2. Since d_H is a meter we have:

$$\begin{aligned}\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{u}_{h_u}) = \frac{1}{2}\{&|h_u \text{ HV } (\tilde{u}_{h_u}) - h_u \text{ HV } (\tilde{u}_{h_u})| \\ &+ |h_u \text{ HA } (\tilde{u}_{h_u}) - h_u \text{ HA } (\tilde{u}_{h_u})| \\ &+ d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_u^2[\tilde{u}_{h_u}]^{h_u})\} = 0\end{aligned}$$

3. By d_H is a meter and triangle inequality we have:

$$\begin{aligned}&\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{w}_{h_w}) = \frac{1}{2}\{|h_u \text{ HV } (\tilde{u}_{h_u}) - h_w \text{ HV } (\tilde{w}_{h_w})| \\ &\quad + |h_u \text{ HA } (\tilde{u}_{h_u}) - h_w \text{ HA } (\tilde{w}_{h_w})| + d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_w^2[\tilde{w}_{h_w}]^{h_w})\} \\ &= \frac{1}{2}\{|h_u \text{ HV } (\tilde{u}_{h_u}) - h_v \text{ HV } (\tilde{v}_{h_v}) + h_v \text{ HV } (\tilde{v}_{h_v}) - h_w \text{ HV } (\tilde{w}_{h_w})| \\ &\quad + |h_u \text{ HA } (\tilde{u}_{h_u}) - h_v \text{ HA } (\tilde{v}_{h_v}) + h_v \text{ HA } (\tilde{v}_{h_v}) - h_w \text{ HA } (\tilde{w}_{h_w})| \\ &\quad + d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_w^2[\tilde{w}_{h_w}]^{h_w})\Big\} \\ &\leq \frac{1}{2}\{h_u| \text{ HV } (\tilde{u}_{h_u}) - h_v \text{ HV } (\tilde{v}_{h_v})| + |h_v \text{ HV } (\tilde{v}_{h_v}) - h_w \text{ HV } (\tilde{w}_{h_w})| \\ &\quad + |h_u \text{ HA } (\tilde{u}_{h_u}) - h_v \text{ HA } (\tilde{v}_{h_v})| + |h_v \text{ HA } (\tilde{v}_{h_v}) - h_w \text{ HA } (\tilde{w}_{h_w})| \\ &\quad + d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_v^2[\tilde{v}_{h_v}]^{h_v}) + d_H(h_v^2[\tilde{v}_{h_v}]^{h_v}, h_w^2[\tilde{w}_{h_w}]^{h_w})\Big\} \\ &= \frac{1}{2}\{|h_u \text{ HV } (\tilde{u}_{h_u}) - h_v \text{ HV } (\tilde{v}_{h_v})| + |h_u \text{ HA } (\tilde{u}_{h_u}) - h_v \text{ HA } (\tilde{v}_{h_v})| \\ &\quad + d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_v^2[\tilde{v}_{h_v}]^{h_v})\Big\} + \frac{1}{2}\{|h_v \text{ HV } (\tilde{v}_{h_v}) - h_w \text{ HV } (\tilde{w}_{h_w})| \\ &\quad + |h_v \text{ HA } (\tilde{v}_{h_v}) - h_w \text{ HA } (\tilde{w}_{h_w})| d_H(h_v^2[\tilde{v}_{h_v}]^{h_v}, h_w^2[\tilde{w}_{h_w}]^{h_w})\Big\} \\ &= \mathrm{HSD}(\tilde{u}_{h_u}, \tilde{v}_{h_v}) + \mathrm{HSD}(\tilde{v}_{h_v}, \tilde{w}_{h_w}).\end{aligned}$$

4. By d_H is a meter and Lemma 2 we have:

$$\begin{aligned}
\mathrm{HSD}(k\tilde{u}_{h_u}, k\tilde{v}_{h_v}) &= \frac{1}{2}\{|h_u \text{ HV } (k\tilde{u}_{h_u}) - h_v \text{ HV } (k\tilde{v}_{h_v})| \\
&\quad + |h_u \text{ HA } (k\tilde{u}_{h_u}) - h_v \text{ HA } (k\tilde{v}_{h_v})| + d_H(h_u^2[k\tilde{u}_{h_u}]^{h_u}, h_v^2[k\tilde{v}_{h_v}]^{h_v})\} \\
&= \frac{1}{2}\{|kh_u \text{ HV } (\tilde{u}_{h_u}) - kh_v \text{ HV } (\tilde{v}_{h_v})| + |kh_u \text{ HA } (\tilde{u}_{h_u}) - kh_v \text{ HA } (\tilde{v}_{h_v})| \\
&\quad + d_H(h_u^2[k\tilde{u}_{h_u}]^{h_u}, h_v^2[k\tilde{v}_{h_v}]^{h_v})\Big\} \\
&= \frac{1}{2}\{|k||h_u \text{ HV }_{h_u}(\tilde{u}) - h_v \text{ HV }_{h_v}(\tilde{v})| + |k||h_u \text{ HA }_{h_u}(\tilde{u}) - h_v \text{ HA }_{h_v}(\tilde{v})| \\
&\quad + |k| d_H(h_u^2[\tilde{u}]^{h_u}, h_v^2[\tilde{v}]^{h_v})\Big\} \\
&= \frac{1}{2}|k|\{| \text{ HV } (\tilde{u}_{h_u}) - \text{ HV } (\tilde{v}_{h_v})| + | \text{ HA } (\tilde{u}_{h_u}) - \text{ HA } (\tilde{v}_{h_v})| \\
&\quad + d_H(h_u^2[\tilde{u}_{h_u}]^{h_u}, h_v^2[\tilde{v}_{h_v}]^{h_v})\Big\} = |k|\mathrm{HSD}(\tilde{u}_{h_u}, \tilde{v}_{h_v})
\end{aligned}$$

For $\tilde{u}_h, \tilde{v}_h \in F_h(\mathbb{R})$ with equal height h, HSD simplified as:

$$\mathrm{HSD}(\tilde{u}_h, \tilde{v}_h) = \frac{1}{2}\{h| \text{ HV } (\tilde{u}_h) - \text{ HV } (\tilde{v}_h)| + h| \text{ HA } (\tilde{u}_h) - \text{ HA } (\tilde{v}_h)| + h^2 d_H([\tilde{u}_h]^h, [\tilde{v}_h]^h)\}, \tag{13}$$

Remark 8 For $\tilde{u}, \tilde{v} \in F(\mathbb{R})$, HSD, is the same as the *source distance* defined in [2].

Example 9 Let

$$\mu_{\tilde{u}}(x) = \begin{cases} h, & x = a, \\ 0, & \text{otherwise}, \end{cases}$$

,

$$\mu_{\tilde{v}}(x) = \begin{cases} h, & x = b, \\ 0, & \text{otherwise}. \end{cases}$$

$$\begin{aligned}
\mathrm{HSD}(\tilde{u}_h, \tilde{v}_h) &= \frac{1}{2}(h^3|a-b| + |0-0| + h^2|a-b|) \\
&= \frac{1}{2}h^2(h+1)|a-b|.
\end{aligned}$$

In this case if $\tilde{u}$ and $\tilde{v}$ are two crisp real numbers, then

$\mathrm{HSD}(\tilde{u}, \tilde{v}) = |a-b|$.

For the set of all fuzzy semi-numbers of the same height, we prove the following theorem.

Theorem 10 *For $\tilde{u}_h, \tilde{v}_h, \tilde{u}'_h, \tilde{v}'_h \in TF_h(\mathbb{R})$ and $k \in \mathbb{R}$, for a fixed height h and non-negative real number k, HSD satisfies the following properties*:

1. $\mathrm{HSD}(\tilde{u}_h + k, \tilde{v}_h + k) = \mathrm{HSD}(\tilde{u}_h, \tilde{v}_h)$,
2. $\mathrm{HSD}(\tilde{u}_h + \tilde{v}_h, \tilde{u}'_h + \tilde{v}'_h) \leq \mathrm{HSD}(\tilde{u}_h, \tilde{u}'_h) + \mathrm{HSD}(\tilde{v}_h, \tilde{v}'_h)$.

Proof straightforward. The proof is obvious if the both $\tilde{u} = 0^{\circ}_{h_u}$ and $\tilde{v} = 0^{\circ}_{h_v}$ are satisfied.Thus, we investigate the proof in other cases. For the first part by d_H is a meter and Lemma 3 we have:

$$\begin{aligned}
\mathrm{HSD}(\tilde{u}_h + k, \tilde{v}_h + k) &= \frac{1}{2}\{|h \text{ HV } (\tilde{u}_h + k) - h \text{ HV } (\tilde{v}_h + k)| \\
&\quad + |h \text{ HA } (\tilde{u}_h + k) - h \text{ HA } (\tilde{v}_h + k)| \\
&\quad + d_H(h^2[\tilde{u}_h + k]^h, h^2[\tilde{v}_h + k]^h)\Big\}, \\
&= \frac{1}{2}\Big\{\big|h \text{ HV } (\tilde{u}_h) + h^3 k - h \text{ HV } (\tilde{v}_h) - h^3 k\big| \\
&\quad + |h \text{ HA } (\tilde{u}_h) - h \text{ HA } (\tilde{v}_h)| \\
&\quad + d_H(h^2[\tilde{u}_h]^h, h^2[\tilde{v}_h]^h)\Big\}, \\
&= \mathrm{HSD}(\tilde{u}_h, \tilde{v}_h)
\end{aligned}$$

For the second part since d_H is a meter by the triangle inequality and Lemma 4 we have:

$$\begin{aligned}
\mathrm{HSD}(\tilde{u}_h + \tilde{v}_h, \tilde{u}'_h + \tilde{v}'_h) &= \frac{1}{2}\Big\{\big|h \text{ HV } (\tilde{u}_h + \tilde{v}_h) - h \text{ HV } (\tilde{u}'_h + \tilde{v}'_h)\big| \\
&\quad + \big|h \text{ HA } (\tilde{u}_h + \tilde{v}_h) - h \text{ HA } (\tilde{u}'_h + \tilde{v}'_h)\big| \\
&\quad + d_H(h^2[\tilde{u}_h + \tilde{v}_h]^h, h^2[\tilde{u}'_h + \tilde{v}'_h]^h)\Big\}, \\
&= \frac{1}{2}\Big\{\big|h \text{ HV } (\tilde{u}_h) + h \text{ HV } (\tilde{v}_h) - h \text{ HV } (\tilde{u}'_h) - h \text{ HV } (\tilde{v}'_h)\big| \\
&\quad + \big|h \text{ HA } (\tilde{u}_h) + h \text{ HA } (\tilde{v}_h) - h \text{ HA } (\tilde{u}'_h) - h \text{ HA } (\tilde{v}'_h)\big| \\
&\quad + d_H(h^2[\tilde{u}_h + \tilde{v}_h]^h, h^2[\tilde{u}'_h + \tilde{v}'_h]^h)\Big\}, \\
&\leq \frac{1}{2}\Big\{\big|h \text{ HV } (\tilde{u}_h) - h \text{ HV } (\tilde{u}'_h)\big| + \big|h \text{ HV } (\tilde{v}_h) - h \text{ HV } (\tilde{v}'_h)\big| \\
&\quad + \big|h \text{ HA } (\tilde{u}_h) - h \text{ HA } (\tilde{u}'_h)\big| + \big|h \text{ HA } (\tilde{v}_h) - h \text{ HA } (\tilde{v}'_h)\big| \\
&\quad + d_H(h^2[\tilde{u}_h]^h, h^2[\tilde{u}'_h]^h) + d_H(h^2[\tilde{v}_h]^h, h^2[\tilde{v}'_h]^h)\Big\}, \\
&= \frac{1}{2}\Big\{\big|h \text{ HV } (\tilde{u}_h) - h \text{ HV } (\tilde{u}'_h)\big| + \big|h \text{ HA } (\tilde{u}_h) - h \text{ HA } (\tilde{u}'_h)\big| \\
&\quad + d_H(h^2[\tilde{u}_h]^h, h^2[\tilde{u}'_h]^h)\Big\} + \frac{1}{2}\Big\{\big|h \text{ HV } (\tilde{v}_h) - h \text{ HV } (\tilde{v}'_h)\big| \\
&\quad + \big|h \text{ HA } (\tilde{v}_h) - h \text{ HA } (\tilde{v}'_h)\big| + d_H(h^2[\tilde{v}_h]^h, h^2[\tilde{v}'_h]^h)\Big\}, \\
&= \mathrm{HSD}(\tilde{u}_h, \tilde{u}'_h) + \mathrm{HSD}(\tilde{v}_h, \tilde{v}'_h).
\end{aligned}$$

□

In definition of HSD, two source functions s_1 and s_2 are defined over $[0, h_1]$ and $[0, h_2]$, respectively. If s_1 over $[0, h_1]$ is equivalent to s_2 over $[0, h_2]$, $(s_1 \approx s_2)$, then we denote this distance by HSD_s with respect to the source function defined over [0, 1].

Example 11 In Fig. 1, functions $s_1(\alpha) = \frac{1}{2}s(2\alpha)$ over $[0, \frac{1}{2}]$, $s_2(\alpha) = \frac{1}{5}s(5\alpha)$ over $[0, \frac{1}{5}]$ and $s_h(\alpha) = \frac{1}{h}s(h\alpha)$ over $[0, \frac{1}{h}]$ (where $h \geq 1$) are all equivalent to the trivial source function $s(\alpha) = \alpha$ over [0, 1].

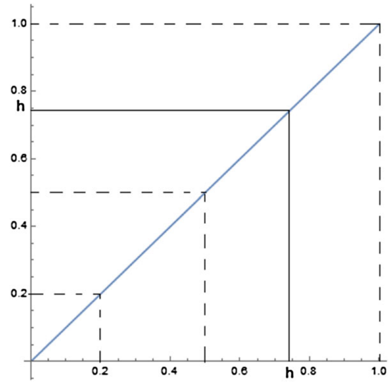

Fig. 1 Some source functions equivalent to the trivial source function

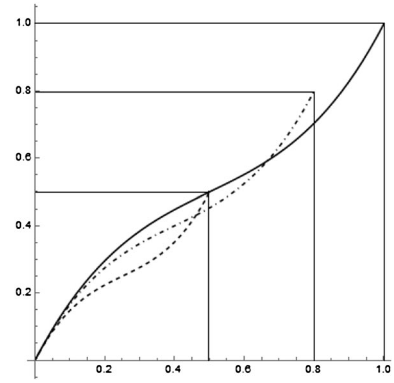

Fig. 2 Some source functions equivalent to a non-trivial source function

Example 12 In Fig. 2, functions $s_1(\alpha) = \frac{1}{2}s(2\alpha)$ over $[0, \frac{1}{2}]$ and $s_2(\alpha) = \frac{1}{5}s(5\alpha)$ over $[0, \frac{1}{5}]$ are both equivalent to a non-trivial source function $s(\alpha) = 2\alpha^3 - 3\alpha^2 + 2\alpha$ over [0, 1].

The nearest approximation of a fuzzy semi-number

In this section, we use HSD to find the nearest approximation of an arbitrary fuzzy semi-number. We start with some definitions and continue with presenting a theorem on the set of all fuzzy semi-numbers with equivalent source functions.

Definition 12 Let $\tilde{u}_h \in \mathrm{FS}(\mathbb{R})$ and $h^* \in (0, 1]$. $\tilde{u}^*$ is *the nearest approximation* to $\tilde{u}_h$ out of $F_{h^*}(\mathbb{R})$, if and only if

$$\mathrm{HSD}(\tilde{u}_h, \tilde{u}^*_{h^*}) = \min_{\tilde{v}_{h^*} \in F_{h^*}(\mathbb{R})} \mathrm{HSD}(\tilde{u}_h, \tilde{v}_{h^*}). \tag{14}$$

Definition 13 Let $\tilde{u}_h$ be an arbitrary fuzzy semi-number. For $h^* \in (0, 1]$, $\tilde{u}^*_{h^*}$ is *the consistent nearest approximation* to $\tilde{u}_h$ out of $\mathrm{TFS}(\mathbb{R})$, if and only if

$$\mathrm{HSD}(\tilde{u}_h, \tilde{u}^*_{h^*}) = 0. \tag{15}$$

Theorem 13 *Let* $\tilde{u}_{h_u}, \tilde{v}_{h_v} \in TF_h(\mathbb{R})$, *then* $\mathrm{HSD}_s(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = 0$, *if and only if* $\tilde{u}_{h_u} = \tilde{v}_{h_v}$.

Proof If $\tilde{u}_{h_u} = \tilde{v}_{h_v}$, from Theorem 7 we have $\mathrm{HSD}_s(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = 0$.

Conversely, assume $\mathrm{HSD}_s(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = 0$. If $\tilde{u} = 0^\circ_{h_u}$ and $\tilde{v} = 0^\circ_{h_v}$, we have $|h_u - h_v| = 0$. Thus, $h_u = h_v$ and $\tilde{u}_{h_u} = \tilde{v}_{h_v}$. Without any loss of generality let $\tilde{u}_{h_u} = (a_u, b_u, c_u, d_u; h_u)$ and $\tilde{v}_{h_v} = (a_v, b_v, c_v, d_v; h_v)$ be two trapezoidal fuzzy semi-numbers where $\tilde{u}_{h_u} \neq 0^\circ_{h_u}$ or $\tilde{v}_{h_v} \neq 0^\circ_{h_v}$. If $\mathrm{HSD}_s(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = 0$ then

$$\begin{cases} \text{(i)} & d_H(h_u^2[c_u, b_u], h_v^2[c_v, b_v]) = 0, \\ \text{(ii)} & h_u HV(\tilde{u}_{h_u}) = h_v HV(\tilde{v}_{h_v}), \\ \text{(iii)} & h_u HA(\tilde{u}_{h_u}) = h_v HA(\tilde{v}_{h_v}). \end{cases} \tag{16}$$

From (i), we have $\max\{|h_u^2 c_u - h_v^2 c_v|, |h_u^2 b_u - h_v^2 b_v|\} = 0$ and hence $b_v = (\frac{h_u}{h_v})^2 b_u$ and $c_v = (\frac{h_u}{h_v})^2 c_u$. From (ii) and (iii)

$$\begin{aligned} h_u HV(\tilde{u}_{h_u}) + h_u HA(\tilde{u}_{h_u}) &= 2h_u \int_0^{h_u} s(\alpha)\overline{u}(\alpha)\mathrm{d}\alpha \\ &= 2h_v \int_0^{h_v} s(\alpha)\overline{v}(\alpha)\mathrm{d}\alpha \\ &= h_v HV(\tilde{v}_{h_v}) \\ &\quad + h_v HA(\tilde{v}_{h_v}), h_u HV(\tilde{u}_{h_u}) \\ &\quad - h_u HA(\tilde{u}_{h_u}) \\ &= 2h_u \int_0^{h_u} s(\alpha)\underline{u}(\alpha)\mathrm{d}\alpha \\ &= 2h_v \int_0^{h_v} s(\alpha)\underline{v}(\alpha)\mathrm{d}\alpha \\ &= h_v HV(\tilde{v}_{h_v}) - h_v HA(\tilde{v}_{h_v}), \end{aligned}$$

and hence

$$\begin{cases} \dfrac{d_u\ h_u^3}{2} - (d_u - c_u)I_{s,h_u} = \dfrac{d_v\ h_v^3}{2} - (d_v - c_v)I_{s,h_v}, \\ \dfrac{a_u\ h_u^3}{2} + (b_u - a_u)I_{s,h_u} = \dfrac{a_v\ h_v^3}{2} + (b_v - a_v)I_{s,h_v}. \end{cases}$$

which are equivalent to the following relations:

$$\begin{cases} d_u(\ h_u^3 - 2I_{s,h_u}) + 2c_u I_{s,h_u} = d_v(\ h_v^3 - 2I_{s,h_v}) + 2c_v I_{s,h_v}, \\ a_u(\ h_u^3 - 2I_{s,h_u}) + 2b_u I_{s,h_u} = a_v(\ h_v^3 - 2I_{s,h_v}) + 2b_v I_{s,h_v}. \end{cases} \tag{17}$$

Since $b_v = (\frac{h_u}{h_v})^2 b_u$ and $c_v = (\frac{h_u}{h_v})^2 c_u$, we have:

$$\begin{cases} a_v = \dfrac{h_u^3 - 2I_{s,h_u}}{h_v^3 - 2I_{s,h_v}} a_u + \dfrac{2h_v^2 I_{s,h_u} - 2h_u^2 I_{s,h_v}}{h_v^5 - 2h_v^2 I_{s,h_v}} b_u, \\ d_v = \dfrac{h_u^3 - 2I_{s,h_u}}{h_v^3 - 2I_{s,h_v}} d_u + \dfrac{2h_v^2 I_{s,h_u} - 2h_u^2 I_{s,h_v}}{h_v^5 - 2h_v^2 I_{s,h_v}} c_u. \end{cases} \tag{18}$$

By assumption, we have $h_u = h_v$, it implies that $d_v = d_u$ and $a_v = a_u$; therefore, we have $\tilde{u}_{h_u} = \tilde{v}_{h_v}$. □

From Theorem 7, Corollaries 4.2 and 4.3 can be immediately derived.

Corollary 14 HSD$_s$, *is a metric on* $TF_h(\mathbb{R})$, *for a fixed height* h.

Proof By Theorems 7 and 13 the proof is clear.

Corollary 15 *Let* $\tilde{u}_{h_u} \in TF_{h_u}(\mathbb{R})$, $\tilde{v}_{h_v} \in TF_{h_v}(\mathbb{R})$ *and* s_{h_u} *be equivalent to* s_{h_v}. *If* $\mathrm{HSD}_s(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = 0$, *then*

$$\begin{cases} a_v = \left(\dfrac{h_u}{h_v}\right)^3 a_u + \dfrac{2h_u^2(h_u - h_v)I_{s,h_u}}{h_v^3(h_u^3 - 2I_{s,h_u})} b_u, \\ b_v = \left(\dfrac{h_u}{h_v}\right)^2 b_u, \\ c_v = \left(\dfrac{h_u}{h_v}\right)^2 c_u, \\ d_v = \left(\dfrac{h_u}{h_v}\right)^3 d_u + \dfrac{2h_u^2(h_u - h_v)I_{s,h_u}}{h_v^3(h_u^3 - 2I_{s,h_u})} c_u. \end{cases}$$

Proof By Theorem 7 and Lemma 5 the proof is clear. □

For an arbitrary fuzzy semi-number $\tilde{u}_{h_u} \in TF_{h_u}(\mathbb{R})$ with height h_u, the consistent nearest approximation $\tilde{v}_{h_v} = (a_v, b_v, c_v, d_v; h_v)$ can be found by applying equation (18) for an arbitrary height h_v

$$a_v = \frac{h_u^3 - 2I_{s,h_u}}{h_v^3 - 2I_{s,h_v}} a_u + \frac{2h_v^2 I_{s,h_u} - 2h_u^2 I_{s,h_v}}{h_v^5 - 2h_v^2 I_{s,h_v}} b_u, \tag{19}$$

$$b_v = \left(\frac{h_u}{h_v}\right)^2 b_u, \tag{20}$$

$$c_v = \left(\frac{h_u}{h_v}\right)^2 c_u, \tag{21}$$

$$d_v = \frac{h_u^3 - 2I_{s,h_u}}{h_v^3 - 2I_{s,h_v}} d_u + \frac{2h_v^2 I_{s,h_u} - 2h_u^2 I_{s,h_v}}{h_v^5 - 2h_v^2 I_{s,h_v}} c_u. \tag{22}$$

Since the approximation depends on the height of known trapezoidal fuzzy semi-number, the trapezoidal approximation may not exist for a specific height. Example 16 explains this case.

Example 16 Let $\tilde{u} = (1, 2, 3, 4; \frac{1}{2})$ and $s(r) = r$. The consistent nearest approximation of $\tilde{u}$ for height $\frac{1}{3}$, is $\tilde{v} = (\frac{63}{8}, \frac{9}{2}, \frac{27}{4}, \frac{81}{4}; \frac{1}{3})$. Since $a_v \geq b_v$, $\tilde{v}$ is not a trapezoidal fuzzy semi-number.

Therefore, in the following subsections, a range is specified for h associated with a non-negative or non-positive trapezoidal fuzzy semi-number. This range must satisfy some constraints which will be explained in the following subsections.

A constraint on the nearest approximation' height of a non-negative fuzzy semi-number

Let $\tilde{u} = (a_u, b_u, c_u, d_u; h_u)$ be an arbitrary non-negative trapezoidal fuzzy semi-number, for the consistent nearest approximation via HSD$_s$, with height h_v, where $\tilde{v} = (a_v, b_v, c_v, d_v; h_v) \in TF_{h_v}(\mathbb{R})$, the Lemmas 17 and 18 can be derived.

Lemma 17 *If* $h_v \geq h_u$, *the H-core of the consistent nearest approximated non-negative trapezoidal fuzzy semi-number,* $\tilde{v}_{h_v}$, *tends to the left side and similarly if* $h_v \leq h_u$, *the H-core of the consistent nearest approximated non-negative trapezoidal fuzzy semi-number,* $\tilde{v}_{h_v}$, *tends to the right side.*

Proof When $h_v \geq h_u$, from (20) and (21) we will have $b_v \leq b_u$ and $c_v \leq c_u$, when $h_v \leq h_u$, from (20) and (21) we will have $b_v \geq b_u$ and $c_v \geq c_u$, since $a_u \geq b_u \geq c_u \geq d_u \geq 0$, the proof of the lemma is complete. □

Lemma 18 *Let* $\tilde{u}$ *be a non-negative trapezoidal fuzzy semi-number with fixed height* h_u. *For each* $h_v \leq h_u$ *and the source function* s_{h_v} *which is equivalent to* s_{h_u}, *we have* $c_v \leq d_v$, *similarly for each* $h_v \geq h_u$ *we have* $a_v \leq b_v$.

Proof Let $I_u = I_{s_{h_u}, h_u}$ and $I_v = I_{s_{h_v}, h_v}$. Since $s_{h_u} \approx s_{h_v}$, with respect to Lemma 5 we have

$$I_v = \left(\frac{h_v}{h_u}\right)^3 I_u. \tag{23}$$

From Eqs. (21) and (22) we can write as follows:

$$d_v - c_v = \frac{h_u^3 - 2I_u}{h_v^3 - 2I_v} d_u + \frac{2h_v^2 I_u - 2h_u^2 I_v}{h_v^5 - 2h_v^2 I_v} c_u - \left(\frac{h_u}{h_v}\right)^2 c_u,$$

then by Eq. (23),

$$\begin{aligned} d_v - c_v &= \left(\frac{h_u}{h_v}\right)^3 \left(\frac{d_u h_u^3 - c_u h_u^2 h_v + 2(c_u - d_u)I_u}{h_u^3 - 2I_u}\right), \\ &= \left(\frac{h_u}{h_v}\right)^3 (d_u - c_u) \left(\frac{h_u^2 (\frac{d_u h_u - c_u h_v}{d_u - c_u}) - 2I_u}{h_u^3 - 2I_u}\right), \end{aligned}$$

since $d_u \geq c_u \geq 0$ and $h_u \geq h_v$, we have $d_u h_u - c_u h_v \geq d_u h_u - c_u h_u$ and from that we have $(\frac{d_u h_u - c_u h_v}{d_u - c_u}) \geq h_u$. By multiply h_u^2 to both sides and subtract $2I_u$ from both side we'll get $h_u^2(\frac{d_u h_v - c_u h_u}{d_u - c_u}) - 2I_u \geq h_u^3 - 2I_u$. By Lemma 6 and $d_u - c_u \geq 0$,

$$h_u^2\frac{(d_uh_v-c_uh_u)}{(d_u-c_u)}-2I_u\geq h_u^3-2I_u>0,$$

hence $d_v-c_v\geq 0$.

Similarly for $h_v\geq h_u$ from Eqs. (20), (19), (23) and $b_u-a_u\geq 0$, we have:

$$b_v-a_v=\left(\frac{h_u}{h_v}\right)^3(b_u-a_u)\left(\frac{h_u^2(\frac{b_uh_v-a_uh_u}{b_u-a_u})-2I_u}{h_u^3-2I_u}\right)\geq 0.$$

□

Theorem 19 *Let s be a source function, for an arbitrary known non-negative trapezoidal fuzzy semi-number* $\tilde{u}=(a_u,b_u,c_u,d_u;h_u)$ *with height* h_u*. The consistent nearest approximation via* HSD_s *is a trapezoidal fuzzy semi-number with height* h_v*, if and only if* h_v *satisfies one of the following conditions*:

(i) If $h_v\leq h_u$, *then*

$$h_v\geq\frac{a_uh_u^3-2a_uI_{s,h_u}+2b_uI_{s,h_u}}{b_uh_u^2},\tag{24}$$

(ii) If $h_v\geq h_u$, *then*

$$h_v\leq\frac{d_uh_u^3-2d_uI_{s,h_u}+2c_uI_{s,h_u}}{c_uh_u^2}.\tag{25}$$

Proof Let $I_u=I_{s_{h_u},h_u}$, $I_v=I_{s_{h_v},h_v}$ and $\tilde{v}=(a_v,b_v,c_v,d_v;h_v)$ be the consistent nearest approximation of $\tilde{u}_{h_u}$. $\tilde{v}_{h_v}$ is a trapezoidal fuzzy semi-number if $a_v\leq b_v\leq c_v\leq d_v$. From Eqs. (20) and (21) it is clear that $b_v\leq c_v$.

(i) From Lemma 17 and 18, whenever $h_v\leq h_u$, it is sufficient that $a_v\leq b_v$ holds, therefore, by Eqs. (20) and (19) we write as follows:

$$\begin{aligned}b_v-a_v&=\left(\frac{h_u}{h_v}\right)^2b_u-\frac{h_u^3-2I_u}{h_v^3-2I_v}a_u\\&\quad-\frac{2h_v^2I_u-2h_u^2I_v}{h_v^5-2h_v^2I_v}b_u,\\&=\left(\frac{h_u}{h_v}\right)^3\left(\frac{-a_uh_u^3+b_uh_u^2h_v+2a_uI_u-2b_uI_u}{h_u^3-2I_u}\right),\end{aligned}$$

From Lemma 6, $b_v\geq a_v$ implies that

$$-a_uh_u^3+b_uh_u^2h_v+2a_uI_u-2b_uI_u\geq 0,$$

since $b_u\geq a_u\geq 0$ hence

$$h_v\geq\frac{a_uh_u^3-2a_uI_u+2b_uI_u}{b_uh_u^2}.$$

(ii) From Lemmas 17 and 18 ,whenever $h_v\geq h_u$ it is sufficient that $c_v\leq d_v$ holds; therefore, by Eqs. (21) and (22) we write as follows:

$$d_v-c_v=\left(\frac{h_u}{h_v}\right)^3\frac{d_uh_u^3-c_uh_u^2h_v-2d_uI_u+2c_uI_u}{h_u^3-2I_u},$$

From Lemma 6, $d_v\geq c_v$ implies that

$$d_uh_u^3-c_uh_u^2h_v-2d_uI_u+2c_uI_u\geq 0,$$

since $d_u\geq c_u\geq 0$, then

$$h_v\leq\frac{d_uh_u^3-2d_uI_u+2c_uI_u}{c_uh_u^2}.$$

□

Now, the Corollaries 20 and 21 can be derived from Theorem 19.

Corollary 20 *Let* $\tilde{u}=(a_u,b_u,c_u,d_u;h)$ *be an arbitrary non-negative fuzzy semi-number with height h. When* $1\leq\frac{d_uh^3-2d_uI_{s,h}+2c_uI_{s,h}}{c_uh^2}$, *the consistent nearest trapezoidal fuzzy number of* $\tilde{u}_h$ *is* $\tilde{v}=(a_v,b_v,c_v,d_v)$, *where*

$$\begin{aligned}a_v&=\frac{h^3-2I_{s,h}}{1-2I_{s,1}}a_u+\frac{2I_{s,h}-2h^2I_{s,1}}{1-2I_{s,1}}b_u,\\b_v&=h^2b_u,\\c_v&=h^2c_u,\\d_v&=\frac{h^3-2I_{s,h}}{1-2I_{s,1}}d_u+\frac{2I_{s,h}-2h^2I_{s,1}}{1-2I_{s,1}}c_u.\end{aligned}$$

Proof Straightforward. □

Corollary 21 *Let* $\tilde{u}=(a_u,b_u,c_u,d_u)$ *be an arbitrary non-negative trapezoidal fuzzy number. The consistent nearest fuzzy semi-number of* $\tilde{u}_h$ *with height h, for* $h\geq\frac{a_u-2a_uI_{s,1}+2b_uI_{s,1}}{b_u}$ *is* $\tilde{v}=(a_v,b_v,c_v,d_v;h)$, *where*

$$\begin{aligned}a_v&=\frac{1-2I_{s,1}}{h^3-2I_{s,h}}a_u+\frac{2h^2I_{s1}-2I_{s,h}}{h^5-2h^2I_{s,h}}b_u,\\b_v&=\left(\frac{1}{h}\right)^2b_u,\\c_v&=\left(\frac{1}{h}\right)^2c_u,\\d_v&=\frac{1-2I_{s,1}}{h^3-2I_{s,h}}d_u+\frac{2h^2I_{s,1}-2I_{s,h}}{h^5-2h^2I_{s,h}}c_u.\end{aligned}$$

Proof Straightforward. □

A constraint on the nearest approximation' height of a non-positive fuzzy semi-number

Let $\tilde{u}=(a_u,b_u,c_u,d_u;h_u)$ be an arbitrary known non-positive trapezoidal fuzzy semi-number, for the consistent nearest approximation via HSD_s, with height h_v, where $\tilde{v}=(a_v,b_v,c_v,d_v;h_v)\in F_{h_v}(\mathbb{R})$, Lemmas 22 and 23 are obtained.

Lemma 22 *If $h_v \geq h_u$ the H-core of the consistent nearest approximated non-positive trapezoidal fuzzy semi-number, $\tilde{v}_{h_v}$, tends to the right side and similarly if $h_v \leq h_u$ the H-core of the consistent nearest approximated non-positive trapezoidal fuzzy semi-number, $\tilde{v}_{h_v}$, tends to the left side.*

Lemma 23 *Let $\tilde{u}_{h_u} \in TF_{h_u}(\mathbb{R})$ and $\tilde{u}_{h_u}$ be non-positive for a fixed h_u. For each $h_v \leq h_u$ and the source function s_{h_v} which is equivalent to s_{h_u}, we have $a_v \leq b_v$, similarly for each $h_v \geq h_u$ we have $c_v \leq d_v$.*

Theorem 24 *Let s be a source function, for an arbitrary known non-positive trapezoidal fuzzy semi-number $\tilde{u} = (a_u, b_u, c_u, d_u; h_u)$ with height h_u. The consistent nearest approximation via HSD_s is a trapezoidal fuzzy semi-number with height h_v, if and only if h_v satisfies one of the following conditions*:

(i) If $h_v \leq h_u$, then

$$h_v \geq \frac{d_u h_u^3 - 2d_u I_u + 2c_u I_u}{c_u h_u^2}, \tag{26}$$

(ii) If $h_v \geq h_u$, then

$$h_v \leq \frac{a_u h_u^3 - 2a_u I_u + 2b_u I_u}{b_u h_u^2}. \tag{27}$$

Corollary 25 *Let $\tilde{u} = (a_u, b_u, c_u, d_u; h)$ be an arbitrary non-positive fuzzy semi-number with height h. When $1 \leq \frac{a_u h^3 - 2a_u I_u + 2b_u I_u}{b_u h^2}$, the consistent nearest trapezoidal fuzzy number of $\tilde{u}_h$ is $\tilde{v} = (a_v, b_v, c_v, d_v)$, where*

$$a_v = \frac{h^3 - 2I_{s,h}}{1 - 2I_{s,1}} a_u + \frac{2I_{s,h} - 2h^2 I_{s,1}}{1 - 2I_{s,1}} b_u,$$
$$b_v = h^2 b_u,$$
$$c_v = h^2 c_u,$$
$$d_v = \frac{h^3 - 2I_{s,h}}{1 - 2I_{s,1}} d_u + \frac{2I_{s,h} - 2h^2 I_{s,1}}{1 - 2I_{s,1}} c_u.$$

Corollary 26 *Let $\tilde{u} = (a_u, b_u, c_u, d_u)$* be an arbitrary non-*positive trapezoidal fuzzy number. The consistent nearest fuzzy semi-number of $\tilde{u}$ with height h for $h \geq \frac{a_u - 2a_u I_{s,1} + 2b_u I_{s,1}}{b_u}$, is $\tilde{v} = (a_v, b_v, c_v, d_v; h)$, where*

$$a_v = \frac{1 - 2I_{s,1}}{h^3 - 2I_{s,h}} a_u + \frac{2h^2 I_{s1} - 2I_{s,h}}{h^5 - 2h^2 I_{s,h}} b_u,$$
$$b_v = \left(\frac{1}{h}\right)^2 b_u,$$
$$c_v = \left(\frac{1}{h}\right)^2 c_u,$$
$$d_v = \frac{1 - 2I_{s,1}}{h^3 - 2I_{s,h}} d_u + \frac{2h^2 I_{s,1} - 2I_{s,h}}{h^5 - 2h^2 I_{s,h}} c_u.$$

The proofs of the lemmas, theorem and corollaries in this subsection are the same as the previous subsection ("A constraint on the nearest approximation' height ofa non-negative fuzzy semi-number"). Alternatively, these proofs can be derived, assuming $\tilde{u}^+ = -\tilde{u}$ and $\tilde{v}^+ = -\tilde{v}$.

Numerical examples

In this section, we present some numerical examples using the proposed method during paper. The simplest source function is identity function and because of computational complexity we use $s(r) = r$ in our examples.

Example 27 Let $\tilde{u} = (1, 2, 3, 4; \frac{1}{2})$ and $s(r) = r$. The consistent nearest trapezoidal fuzzy semi-number of $\tilde{u}$ with height $\frac{7}{13}$, is $\tilde{v} = (\frac{1521}{2744}, \frac{169}{98}, \frac{507}{196}, \frac{3887}{1372}; \frac{7}{13})$. In Fig. 3, $\tilde{u}$ and $\tilde{v}$ are shown by solid and dashed lines, respectively.

The nearest trapezoidal fuzzy semi-number of $\tilde{u}$ with height $\frac{6}{13}$, is $\tilde{w} = (\frac{2873}{1728}, \frac{169}{72}, \frac{169}{48}, \frac{4901}{864}; \frac{6}{13})$. In Fig. 4, $\tilde{u}$ and $\tilde{w}$ are shown by solid and dashed lines, respectively.

If $h_v \geq \frac{1}{2}$, then from Theorem 19 we should have $h_v \leq \frac{5}{9}$ and if $h_v \leq \frac{1}{2}$, then from Theorem 19 we should have $h_v \geq \frac{5}{12}$. Since $\tilde{u}$ is positive and $h_v \geq h_u$, $\tilde{v}$ tend to left side as it can be seen in Fig. 3 regarding Lemma 17, it was predictable and because $h_w \leq h_u$, $\tilde{w}$ tend to right side as it can be seen in Fig. 4 according to Lemma 17 it could be foretold.

Example 28 Let $\tilde{u} = (-5, -3, -2, -1, \frac{3}{4})$ and $s(r) = r$. The consistent nearest fuzzy semi-number of $\tilde{u}$ with height $\frac{7}{8}$, is $\tilde{v} = (-\frac{864}{343}, -\frac{108}{49}, -\frac{72}{49}, -\frac{72}{343}, \frac{7}{8})$. In Fig. 4, $\tilde{u}$ and $\tilde{v}$ are shown by solid and dashed lines, respectively.

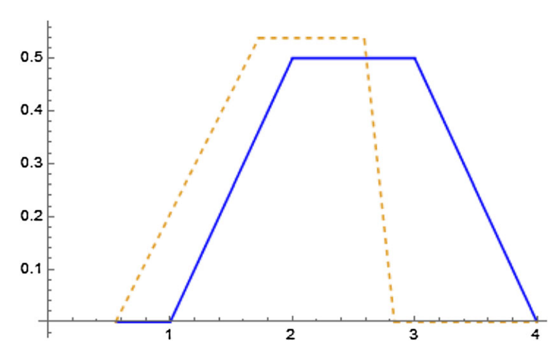

Fig. 3 The nearest trapezoidal fuzzy semi-number with height $\frac{7}{13}$

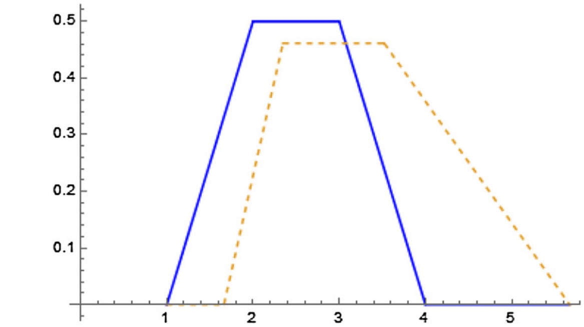

Fig. 4 The nearest trapezoidal fuzzy semi-number with height $\frac{6}{13}$

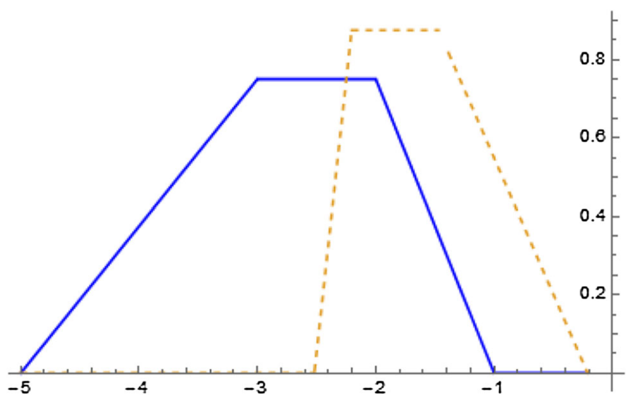

Fig. 5 The nearest fuzzy semi-number with height $\frac{7}{8}$

The nearest fuzzy semi-number of $\tilde{u}$ with height $\frac{2}{3}$, is $\tilde{w} = (-\frac{4131}{512}, -\frac{243}{64}, -\frac{81}{32}, -\frac{1053}{512}, \frac{2}{3})$. In Fig. 5, $\tilde{u}$ and $\tilde{w}$ are shown by solid and dashed lines, respectively.

Example 29 Let $\tilde{u} = (1, 2, 3, 5; \frac{1}{2})$ and $s(r) = r$. The consistent nearest trapezoidal fuzzy semi-number of $\tilde{u}$ with height $\frac{11}{21}$, $\frac{4}{7}$, $\frac{10}{22}$ and $\frac{21}{44}$ are as follows:

$$\tilde{m} = \left(\frac{7497}{10648}, \frac{441}{242}, \frac{1323}{484}, \frac{3969}{968}; \frac{11}{21}\right), \tilde{n}$$
$$= \left(\frac{147}{512}, \frac{49}{32}, \frac{147}{64}, \frac{1421}{512}; \frac{4}{7}\right), \tilde{v}$$
$$= \left(\frac{363}{200}, \frac{121}{50}, \frac{363}{100}, \frac{7381}{1000}; \frac{10}{22}\right), \tilde{w}$$
$$= \left(\frac{12584}{9261}, \frac{968}{441}, \frac{484}{147}, \frac{56144}{9261}; \frac{21}{44}\right).$$

In Fig. 5, $\tilde{u}$, $\tilde{m}$, $\tilde{n}$, $\tilde{v}$ and $\tilde{w}$ are shown .

If $h_v \geq \frac{1}{2}$, then from Theorem 19 we should have $h_v \leq \frac{3}{5}$ and if $h_v \leq \frac{1}{2}$, then from Theorem 19 we should have $h_v \geq \frac{9}{22}$. Hence, when $h_u = \frac{1}{2}$, given height should belong to $[\frac{9}{22}, \frac{3}{5}]$; otherwise, the nearest approximation of $\tilde{u}$ will not be a trapezoidal fuzzy semi-number. Since $\tilde{u}$ is a positive semi-number by Lemma 17 as it can be seen in Fig. 5

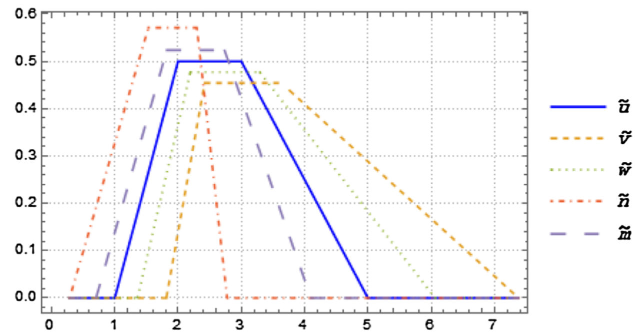

Fig. 6 The nearest trapezoidal fuzzy semi-number with height $\frac{11}{21}$, $\frac{4}{7}$, $\frac{10}{22}$ and $\frac{21}{44}$

whenever the given height increases the nearest approximation of $\tilde{u}$ tend to left and whenever the given height decreases the nearest approximation of $\tilde{u}$ tend to right. As it can be seen in Fig. 5, according to Lemma 18 when the given height increases we do not have to worry about left spread and when the given height decreases we do not have to worry about right spread.

Example 30 Let $\tilde{u} = (0, 2, 4, 5; \frac{1}{2})$ and $\tilde{v} = (1, 2, 3, 4; \frac{3}{4})$ as shown in Fig. 6. These can be written as $\tilde{u}(r) = (4r, 5 - 2r)$ and $\tilde{v}(r) = (1 + \frac{4}{3}r, 4 - \frac{4}{3}r)$. By using $s(r) = r$, *H-value* and *H-ambiguity* of them can be found as follows:

$$\text{HV}\ (\tilde{u}_{h_u}) = \frac{17}{24},$$
$$\text{HA}\ (\tilde{u}_{h_u}) = \frac{3}{8}, \quad \text{HV}\ (\tilde{v}_{h_v}) = \frac{45}{32},$$
$$\text{HA}\ (\tilde{v}_{h_v}) = \frac{15}{32}.$$

The Height Source Distance between these fuzzy semi-numbers is

$$\text{HSD}_s(\tilde{u}_{h_u}, \tilde{v}_{h_v}) = \frac{149}{192}.$$

A medical case study

Without a doubt, controlling the depth of anesthesia is the main task of an anesthetist during surgery. According to the report in [17], annually, anesthesia is the direct responsible for the death of approximately 34 patients. Also it plays an indirect role in another 281 deaths especially in the elderly. Measuring the depth of anesthesia is inherently not straightforward. Thus, anesthetists solve the matter via measuring some other available factors such as blood pressure, heart rate and so on.

In [12], Cullen showed that there is a plausible correlation between blood pressure and anesthetic dose. In other words, gaining a sufficient depth of anesthesia is feasible through controlling the blood pressure related factors such as the MAP (Mean Arterial Pressure which is measured in

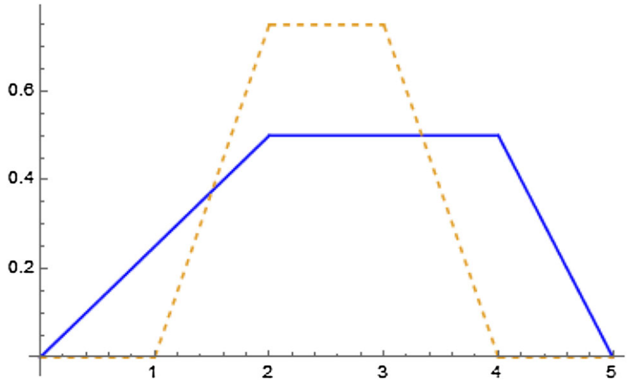

Fig. 7 Two fuzzy semi-numbers $\tilde{u} = (0, 2, 4, 5; \frac{1}{2})$ and $\tilde{v} = (1, 2, 3, 4; \frac{3}{4})$

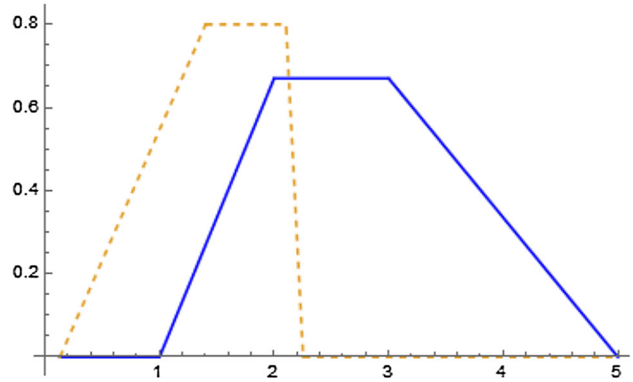

Fig. 8 The solid line shows $\tilde{B}_{h_B}$ and the dashed line shows $\tilde{B}^*_{h_{B^*}}$

mmHg) so that it falls within a predetermined range. This approach has been one of the most prevalent ones in determining the necessary anesthetic dose for decades.

Surgical operations are so risky that most often a small oversight may lead to serious losses. Thus, it is reasonable to automate such tasks as far as possible. In reality, controlling of depth of anesthesia automatically is of highly importance among these tasks, and it releases the surgery staff to pay much more attention to the other activities in the operating room.

There are actually two ways to anesthetize a patient. This is performed either by intravenous injection of drugs or inhaling gasses which are most often a mixture of Isoflurane in oxygen and potentially nitrous oxide. The concentration of Isoflurane used in the process is determined and tuned regarding the type of surgery and the patients psychological condition.

As a case study, in this paper, a fuzzy logic controller is deployed to measure the MAP [21, 22]. This controller simulates the relationship between the inflow concentration of Isoflurane, and the blood pressure. Assume that we are given a fuzzy semi-number which is the average of patient's blood pressure. Due to some disturbances, the blood pressure measurement is an ambiguous process; thus, the given data should be a fuzzy set. Actually, there are different sorts of disturbances such as surgical disturbances to the patient and measurement noise which cause the ambiguity of blood pressure measurement to increase. For this given fuzzy semi-number we find the range of heights which indicates blood pressure variation. Then, we estimate a new fuzzy semi-number with given height related to the new blood pressure to help the anesthetist determine the dose of drugs through the anesthetic process.

The given fuzzy semi-number $\tilde{B}_{h_B} = (1, 2, 3, 5; 0.67)$, shown in Fig. 7, represents the uncertainty in each blood pressure measurement. Here, the height of given semi-number, $h_B = 0.67$, is proportional to blood pressure and its support [1, 5] shows the anesthesia duration.

To this end, we can estimate the range of height associated with the given fuzzy semi-number with (24) and (25) as follows:

$$0.548182 \leqslant h_{\tilde{B}} \leqslant 0.804$$

This range shows how the blood pressure varies with the inflow concentration of Isoflurane by applying related defuzzification methods in medicine (Fig. 8).

Moreover, if we want to change blood pressure we approximate a new fuzzy semi-number, $\tilde{B}^*_{h_{B^*}}$, where $h_{B^*} = 0.8$ is related to this new blood pressure by using (19)–(22) as follows:

$$\tilde{B}^*_{h_{B^*}} = (0.131514, 1.40281, 2.10422, 2.25327; 0.8)$$

With this approximated fuzzy semi-number we will allow the anesthetist to determine the next drug application for the patient.

Conclusion

In most of the works on fuzzy sets, the fuzzy sets are convex and normal (the height is one) which are called fuzzy numbers. Approximation of a fuzzy set has been done in several ways. Assigning a single crisp number to a fuzzy set, defining an interval as an approximation of a fuzzy set, defining distance function and solving an optimization problem in order to obtain a trapezoidal fuzzy set as a nearest approximation are among the most well-known ones. All of these methods suffer from some deficiencies such as precision loss. Moreover, none of them have addressed fuzzy sets with heights less than one. Since too many subjects work on non-normal fuzzy sets, in this paper we worked on fuzzy sets in general, without regard to their height. At first, we reviewed the novel concept of fuzzy semi-numbers. Then, we proposed a distance to approximate an arbitrary fuzzy semi-number. Also, as we observed, depending on some predefined conditions, the result of approximation can be either a fuzzy number or a fuzzy semi-number. We finished the paper with some numerical examples and presented a critical medicine case study which improved anesthetist decision for prescribing dose of drugs and controlling the depth of anesthesia with our approximation method.

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