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ORIGINAL RESEARCH

Construction of a measure of noncompactness in Sobolev spaces with an application to functional integral-differential equations

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Abstract

In this paper, first we introduce a measure of noncompactness in the Sobolev space $W^{k,1}(\Omega)$ and then, as an application, we study the existence of solutions for a class of the functional integral-differential equations using Darbo's fixed point theorem associated with this new measure of noncompactness. Further, two examples are presented to verify the effectiveness and applicability of our main results.

Keywords Darbo's fixed point theorem · Integral-differential equation · Measure of noncompactness · Sobolev space

Mathematical Subject Classification 45J05 · 47H08 · 47H10

Introduction

Sobolev spaces [11], i.e., the class of functions with derivatives in L^p, play an outstanding role in the modern analysis. In the last decades, there has been increasing attempts to study these spaces. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces. They also highlighted in approximation theory, calculus of variation, differential geometry, spectral theory etc.

On the other hand, integral-differential equations (IDE) have a great deal of applications in some branches of sciences. It arises especially in a variety of models from applied mathematics, biological science, physics and another phenomenon, such as the theory of electrodynamics, electromagnetic, fluid dynamics, heat and oscillating magnetic, etc. [9, 12, 18, 21, 24]. There have appeared recently a number of interesting papers [2, 6, 10, 19, 22, 23, 27] on the solvability of various integral equations with help of measures of noncompactness.

The first such measure was defined by Kuratowski [25]. Next, Banaś et al. [8] proposed a generalization of this notion which is more convenient in the applications. The technique of measures of noncompactness is frequently applicable in several branches of nonlinear analysis, in particular the technique turns out to be a very useful tool in the existence theory for several types of integral and integral-differential equations. Furthermore, they are often used in the functional equations, fractional partial differential equations, ordinary and partial differential equations, operator theory and optimal control theory [1, 3, 7, 13, 15–17, 26, 28, 29]. The most important application of measures of noncompactness in the fixed point theory is contained in the Darbo's fixed point theorem [4, 5].

Now, in this paper, we introduce a new measure of noncompactness in the Sobolev space $W^{k,1}(\Omega)$ as a more effective approach. Then, we study the problem of existence of solutions of the functional integral-differential equation

$$u(x) = p(x) + q(x)u(x) + \int_\Omega k(x,y)g\Big(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)\Big)\mathrm{d}y.$$

We provide some notations, definitions and auxiliary facts which will be needed further on.

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Throughout this paper, $\mathbb{R}_+$ indicates the interval $[0, +\infty)$ and for the Lebesgue measurable subset D of $\mathbb{R}$, $m(D)$ denotes the Lebesgue measure of D. Moreover, let $L^1(D)$ be the space of all Lebesgue integrable functions f on D equipped with the standard norm $\|f\|_{L^1(D)} = \int_D |f(x)|\mathrm{d}x$.

Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. The symbol $\overline{B}(x, r)$ stands for the closed ball centered at x with radius r and put $\overline{B}_r = \overline{B}(0, r)$. Denote by $\mathfrak{M}_E$ the family of nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets of E. For a nonempty subset X of E, the symbols $\overline{X}$ and $ConvX$ will denote the closure and the closed convex hull of X, respectively.

Definition 1.1 [8] A mapping $\mu : \mathfrak{M}_{\mathfrak{E}} \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_{\mathfrak{E}} \;\; \mu(\mathfrak{X}) = \}$ is nonempty and $\ker \mu \subset \mathfrak{N}_{\mathfrak{E}}$.
2° $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(ConvX) = \mu(X)$.
5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_{\mathfrak{E}}$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty. A measure of noncompactness μ is said to be regular if it additionally satisfies the following conditions:
7° $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.
8° $\mu(X + Y) \le \mu(X) + \mu(Y)$.
9° $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.
10° $\ker \mu = \mathfrak{N}_{\mathfrak{E}}$.

In what follows, we recall the well known Darbo's fixed point theorem.

Theorem 1.2 [13] *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $F : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property*

$$\mu(FX) \le k\mu(X), \tag{1}$$

for any nonempty subset X of Ω, *where μ is a measure of noncompactness defined in E. Then, F has a fixed point in the set Ω.*

Construction of a measure of noncompactness in Sobolev spaces

In this section, we introduce a measure of noncompactness in the Sobolev space $W^{k,1}(\Omega)$.

Let Ω be a subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $W^{k,1}(\Omega)$ the space of functions f which, together with all their distributional derivatives $D^\alpha f$ of order $|\alpha| \le k$, belong to $L^1(\Omega)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, *i.e.*, each α_j is a nonnegative integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}.$$

Then, $W^{k,1}(\Omega)$ is equipped with the complete norm

$$\|f\|_{k,1} = \max_{0 \le |\alpha| \le k} \|D^\alpha f\|_{L^1(\Omega)}.$$

We present the following theorem which characterizes the compact subsets of the Sobolev spaces.

Theorem 2.1 [20] *A subset $\mathcal{F} \subset W^{k,1}(\mathbb{R}^n)$ is totally bounded if, and only if, the following holds:*

(i) *$\mathcal{F}$ is bounded, i.e., there is some M so that*

$$\int |D^\alpha f(x)|\mathrm{d}x < M, \; f \in \mathcal{F}, \; |\alpha| \le k.$$

(ii) For every $\varepsilon > 0$ there is some R so that

$$\int_{\|x\|_{\mathbb{R}^n} > R} |D^\alpha f(x)|\mathrm{d}x < \varepsilon, \; f \in \mathcal{F}, \; |\alpha| \le k.$$

(iii) For every $\varepsilon > 0$ there is some $\rho > 0$ so that

$$\int_{\mathbb{R}^n} |D^\alpha f(x+y) - D^\alpha f(x)|\mathrm{d}x < \varepsilon, \; f \in \mathcal{F}, \; |\alpha| \le k, \; \|y\|_{\mathbb{R}^n} < \rho.$$

Now, we are going to describe a measure of noncompactness in $W^{k,1}(\Omega)$.

Theorem 2.2 *Suppose $1 \le k < \infty$ and U is a bounded subset of $W^{k,1}(\Omega)$. For $u \in U$, $\varepsilon > 0$ and $0 \le |\alpha| \le k$, let*

$$\begin{aligned}
\omega^T(u, \varepsilon) &= \sup\{\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)} : h \in \Omega, \\
&\qquad \|h\|_{\mathbb{R}^n} < \varepsilon, 0 \le |\alpha| \le k\}, \\
\omega^T(U, \varepsilon) &= \sup\{\omega^T(u, \varepsilon) : u \in U\}, \\
\omega^T(U) &= \lim_{\varepsilon\to 0} \omega^T(U, \varepsilon), \\
\omega(U) &= \lim_{T\to\infty} \omega^T(U), \\
d(U) &= \lim_{T\to\infty} \sup\{\|D^\alpha u\|_{L^1(\Omega \setminus B_T)} : u \in U, 0 \le |\alpha| \le k\},
\end{aligned}$$

where $B_T = \{a \in \Omega : \|a\|_{\mathbb{R}^n} \le T\}$ and $\mathcal{T}_h u(t) = u(t+h)$.
Then $\omega_0 : \mathfrak{M}_{\mathfrak{W}^{k,1}(\Omega)} \to \mathbb{R}$ given by

$$\omega_0(U) = \omega(U) + d(U) \tag{2}$$

defines a measure of noncompactness in $W^{k,1}(\Omega)$.

Proof Take $U \in \mathfrak{M}_{\mathfrak{W}^{k,1}(\Omega)}$ such that $\omega_0(U) = 0$. Fix arbitrary α such that $0 \le |\alpha| \le k$. Let $\eta > 0$ be arbitrary, since $\omega_0(U) = 0$,

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\omega^T(U,\varepsilon)=0.$$

Thus, there exists small enough $\delta>0$ and large enough $T>0$ such that $\omega^T(U,\delta)<\eta$. This implies that

$$\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)}<\eta$$

for all $u\in U$ and $h\in\Omega$ such that $\|h\|_{\mathbb{R}^n}<\delta$. Since $\eta>0$ was arbitrary, we obtain

$$\lim_{h\to 0}\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(\Omega)}=\lim_{h\to 0}\lim_{T\to\infty}\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)}=0.$$

Using again the fact that $\omega_0(U)=0$ we have

$$\lim_{T\to\infty}\sup\{\|D^\alpha u\|_{L^1(\Omega\setminus B_T)}: u\in U\}=0,$$

and so for $\varepsilon>0$ there exists large enough $T>0$ such that

$$\|D^\alpha u\|_{L^1(\Omega\setminus B_T)}<\varepsilon \quad \text{for all} \quad u\in U.$$

It follows then from Theorem 2.1 that U is totally bounded. Thus, 1° holds.

2° is obvious by the definition of ω_0.

Now, we check that condition 3° holds. For this purpose, suppose that $U\in\mathfrak{M}_{\mathfrak{W}^{k,1}(\Omega)}$ and $\{u_n\}\subset U$ such that $u_n\to u\in\overline{U}$ in $W^{k,1}(\Omega)$. From the definition of $\omega^T(U,\varepsilon)$, we have

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_T)}\le\omega^T(U,\varepsilon),$$

for any $n\in\mathbb{N}$, $T>0$ and $h\in\Omega$ with $\|h\|_{\mathbb{R}^n}<\varepsilon$. Letting $n\to\infty$, we get

$$\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)}\le\omega^T(U,\varepsilon),$$

for any $T>0$ and $h\in\Omega$ with $\|h\|_{\mathbb{R}^n}<\varepsilon$. Hence

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\omega^T(\overline{U},\varepsilon)\le\lim_{T\to\infty}\lim_{\varepsilon\to 0}\omega^T(U,\varepsilon).$$

This concludes that $\omega(\overline{U})\le\omega(U)$. Similarly, we can show that

$$d(\overline{U})\le d(U),$$

and thus

$$\omega_0(\overline{U})\le\omega_0(U). \tag{3}$$

From (3) and 2° we obtain $\omega_0(\overline{U})=\omega_0(U)$.

4° follows directly from $D^\alpha[Conv(U)]=Conv(D^\alpha U)$ and hence is omitted.

The proof of condition 5° can be obtained by using the equality

$$D^\alpha(\lambda u_1+(1-\lambda)u_2)=\lambda D^\alpha u_1+(1-\lambda)D^\alpha u_2,$$

for all $\lambda\in[0,1]$, $u_1\in X$ and $u_2\in Y$.

It remains only to verify 6°, suppose that $\{U_n\}$ is a sequence of closed and nonempty sets of $\mathfrak{M}_{\mathfrak{W}^{k,1}(\Omega)}$ such that $U_{n+1}\subset U_n$ for $n=1,2,\ldots,$ and $\lim_{n\to\infty}\omega_0(U_n)=0$. Now, for any $n\in\mathbb{N}$, take $u_n\in U_n$ and set $\mathcal{G}=\overline{\{u_n\}}$.

Claim: $\mathcal{G}$ is a compact set in $W^{k,1}(\Omega)$.

Let $\varepsilon>0$ be fixed, since $\lim_{n\to\infty}\omega_0(U_n)=0$, there exists sufficiently large $m_1\in\mathbb{N}$ such that $\omega_0(U_{m_1})<\varepsilon$. Hence, we can find small enough $\delta_1>0$ and large enough $T_1>0$ such that $\omega^{T_1}(U_{m_1},\delta_1)<\varepsilon$ and $d(U_{m_1})<\varepsilon$. Therefore,

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_1})}<\varepsilon,$$

and

$$\|D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})}<\varepsilon,$$

for all $n>m_1$, $0\le|\alpha|\le k$ and $h\in\Omega$ with $\|h\|_{\mathbb{R}^n}<\delta_1$. Thus we have

$$\begin{aligned}&\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega)}\\ &\le\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_1})}+\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})}\\ &\le\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_1})}+\|\mathcal{T}_h D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})}\\ &\quad+\|D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_1})}\\ &<3\varepsilon.\end{aligned}$$

On the other hand, we know that the set $\{u_1,u_2,\ldots,u_{m_1}\}$ is compact, hence there exist $\delta_2>0$ and $T_2>0$ such that

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(B_{T_2})}<\varepsilon, \tag{4}$$

for all $n=1,2,\ldots,m_1$, $0\le|\alpha|\le k$ and $h\in\Omega$ with $\|h\|_{\mathbb{R}^n}<\delta_2$.

Furthermore,

$$\|D^\alpha u_n\|_{L^1(\Omega\setminus B_{T_2})}<\varepsilon, \tag{5}$$

which implies that

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega)}<3\varepsilon,$$

for all $n=1,2,\ldots,m_1$.

Thus,

$$\|\mathcal{T}_h D^\alpha u_n - D^\alpha u_n\|_{L^1(\Omega)}<3\varepsilon,$$

and

$$\|D^\alpha u_n\|_{L^1(\Omega\setminus B_T)}<\varepsilon<3\varepsilon, \tag{6}$$

for all $n\in\mathbb{N}$, $\|h\|_{\mathbb{R}^n}<\min\{\delta_1,\delta_2\}$ and $T=\max\{T_1,T_2\}$. Therefore, all the hypotheses of Theorem 2.1 are satisfied, that completes the proof of the claim.

Using the above claim, there exists a subsequence $\{u_{n_j}\}$ and $u_0\in W^{k,1}(\Omega)$ such that $u_{n_j}\to u_0$. Since $u_n\in U_n$, $U_{n+1}\subset U_n$ and U_n is closed for all $n\in\mathbb{N}$, we yield

$$u_0 \in \bigcap_{n=1}^{\infty} U_n = U_\infty,$$

that finishes the proof of 6°. □

We now investigate the regularity of ω_0.

Theorem 2.3 *The measure of noncompactness ω_0 defined in (2) is regular.*

Proof Suppose that $X, Y \in \mathfrak{M}_{\mathfrak{W}^{k,1}(\Omega)}$. First, notice that for all $\varepsilon > 0$, $\lambda \in \mathbb{R}$ and $T > 0$ we have

$$\begin{aligned}
\omega^T(X \cup Y, \varepsilon) &= \max\{\omega^T(X, \varepsilon), \omega^T(Y, \varepsilon)\},\\
\omega^T(X + Y, \varepsilon) &\leq \omega^T(X, \varepsilon) + \omega^T(Y, \varepsilon),\\
\omega^T(\lambda X, \varepsilon) &= |\lambda|\omega^T(X, \varepsilon),\\
\sup_{u\in X\cup Y} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)} &= \max\{\sup_{u\in X} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)},\\
&\qquad \sup_{u\in Y} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)}\},\\
\sup_{u\in X+Y} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)} &\leq \sup_{u\in X} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)}\\
&\qquad + \sup_{u\in Y} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)},\\
\sup_{u\in \lambda X} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)} &= |\lambda| \sup_{u\in X} \|D^\alpha u\|_{L^1(\Omega\setminus B_T)}.
\end{aligned}$$

Then, the hypotheses 7°–9° hold. Next, we show that 10° holds. Take $U \in \mathfrak{N}_{\mathfrak{W}^{k,1}(\Omega)}$. Thus, the closure of U in $W^{k,1}(\Omega)$ is compact. By Theorem 2.1, for all $|\alpha| \leq k$ and for all $\varepsilon > 0$, there exists $T > 0$ such that $\|D^\alpha u\|_{L^1(\Omega\setminus B_T)} < \varepsilon$ for all $u \in U$, and there exists $\delta > 0$ such that $\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)} < \varepsilon$ for all $h \in \Omega$ with $\|h\|_{\mathbb{R}^n} < \delta$. Then, for all $u \in U$ we have

$$\omega^T(u, \delta) = \sup\{\|\mathcal{T}_h D^\alpha u - D^\alpha u\|_{L^1(B_T)} : h \in \Omega, \\ \|h\|_{\mathbb{R}^n} < \delta\} \leq \varepsilon.$$

Therefore,

$$\omega^T(U, \delta) = \sup\{\|\omega^T(u, \delta)\| : u \in U\} = 0.$$

It yields that

$$\omega(U) = \lim_{T\to\infty}\lim_{\delta\to 0} \omega^T(U, \delta) = 0.$$

Furthermore,

$$d(U) = \lim_{T\to\infty} \sup\{\|D^\alpha u\|_{L^1(\Omega\setminus B_T)} : u \in U\} = 0.$$

Then $\omega_0(U) = 0$ and $\ker(\omega_0) = \mathfrak{N}_{\mathfrak{W}^{k,1}(\Omega)}$. □

Theorem 2.4 *Let $Q = \{u \in W^{k,1}(\mathbb{R}^n) : \|u\|_{1,1} \leq 1\}$. Then $\omega_0(Q) = 3$.*

Proof Applying the same strategy as ([4], Theorem [14]), we observe that $\omega_0(Q) \leq 3$. It remains to verify $\omega_0(Q) \geq 3$. For any $k \in \mathbb{N}$, there exists $E_k \subset \mathbb{R}^n$ such that $m(E_k) = \frac{1}{10k}$, $diam(E_k) \leq \frac{1}{k}$, $E_k \cap B_k = \emptyset$ and $E_k \subset B_{2k}$. Define $f_k : \mathbb{R}^n \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} 10k, & x \in E_k, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, observe that $\|f_k\|_{1,1} = 1$, $\|D^\alpha \mathcal{T}_{\beta_k} f_k - D^\alpha f_k\|_{L^1(B_{2k})} = 2$ and $\|D^\alpha f_k\|_{L^1(\mathbb{R}^n\setminus B_k)} = 1$ for all $k \in \mathbb{N}$, where $\beta_k = (\frac{1}{k}, \ldots, \frac{1}{k}) \in \mathbb{R}^n$. Thus, we conclude that $\omega_0(Q) \geq \omega_0(\{f_k\}) = 3$. □

Application

In this section, we study the existence of solutions for some functional integral-differential equations. We also provide some illustrative examples to verify effectiveness and applicability of our results.

We start with some preliminaries which we need in subsequence.

Lemma 3.1 [14] *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. If $\{f_n\}$ is convergent to f in the L^p-norm, then there is a subsequence $\{f_{n_k}\}$ which converges to f a.e., and there is $g \in L^p(\Omega)$, $g \geq 0$, such that*

$$|f_{n_k}(x)| \leq g(x) \quad \text{for a.e. } x \in \Omega.$$

Definition 3.2 [4] We say that a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the Carathéodory conditions if the function $f(., u)$ is measurable for any $u \in \mathbb{R}^m$ and the function $f(x, .)$ is continuous for almost all $x \in \mathbb{R}^n$.

Let Ω be a subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $BC^k(\Omega)$ the space of functions f which are bounded and k-times continuously differentiable on Ω with the standard norm

$$\|f\|_{BC^k(\Omega)} = \max_{0\leq|\alpha|\leq k} \|D^\alpha f\|_u,$$

where $\|D^\alpha f\|_u = \sup\{|D^\alpha f(x)| : x \in \Omega\}$.

Theorem 3.3 *Let Ω be a subset of $\mathbb{R}^n$ with $m(\Omega) < \infty$. Assume that the following conditions are satisfied:*

(i) *$p \in W^{1,1}(\Omega)$, $q \in BC^1(\Omega)$ and*

$$\lambda := \sup\{\|q\|_u + \|\frac{\partial q}{\partial x_i}\|_u : i = 1, \ldots, n\} < 1. \tag{7}$$

(ii) *$g : \Omega \times \mathbb{R}^{n+2} \to \mathbb{R}$ satisfies the Carath éodory conditions and there exist a bounded continuous function $a : \Omega \to \mathbb{R}_+$ with $|a(x)| \leq M$ for all $x \in \Omega$ and some $M > 0$ and a concave, lower semi-*

continuous and nondecreasing function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|g(x, u_0, u_1, \ldots, u_{n+1})| \le a(x)\zeta(\max_{0 \le i \le n+1} |u_i|). \tag{8}$$

(iii) $k : \Omega \times \Omega \to \mathbb{R}$ *satisfies the Carath éodory conditions and has a derivative of order 1 with respect to the first argument. Moreover, there exist* $g_1, g_3 \in W^{1,1}(\Omega)$ *and* $g_2 \in L^\infty(\Omega)$ *such that*

$$|k(x,y)| \le g_1(x)g_2(y), \ |k(x_1,y) - k(x_2,y)| \le g_2(y)|g_3(x_1) - g_3(x_2)|,$$

and

$$\left|\frac{\partial k}{\partial x_i}(x,y)\right| \le g_1(x)g_2(y), \ \left|\frac{\partial k}{\partial x_i}(x_1,y) - \frac{\partial k}{\partial x_i}(x_2,y)\right| \le g_2(y)|g_3(x_1) - g_3(x_2)|,$$

for almost $x, y, x_1, x_2 \in \Omega$ *and* $1 \le i \le n$.

(iv) There exists a positive solution r_0 of the inequality

$$\|p\|_{1,1} + \lambda r + Mm(\Omega)\|g_1\|_{L^1(\Omega)}\|g_2\|_{L^\infty} \zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right) \le r. \tag{9}$$

(v) $T : W^{1,1}(\Omega) \to L^1(\Omega)$ *is a continuous operator such that for any* $x \in W^{1,1}(\Omega)$ *we have*

$$\|T(x)\|_{L^1(\Omega)} \le \|x\|_{1,1}.$$

Then, the functional integral-differential equation

$$u(x) = p(x) + q(x)u(x) + \int_\Omega k(x,y)g\left(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)\right)\mathrm{d}y \tag{10}$$

has at least one solution in the space $W^{1,1}(\Omega)$.

Proof We define the operator $F : W^{1,1}(\Omega) \to W^{1,1}(\Omega)$ by

$$Fu(x) = p(x) + q(x)u(x) + \int_\Omega k(x,y)g\left(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)\right)\mathrm{d}y.$$

Obviously, Fu is measurable for any $u \in W^{1,1}(\Omega)$. Also, for any $x \in \Omega$ we have

$$\frac{\partial(Fu)}{\partial x_i}(x) = \frac{\partial p}{\partial x_i}(x) + \frac{\partial q}{\partial x_i}(x)u(x) + q(x)\frac{\partial u}{\partial x_i}(x) + \int_\Omega \frac{\partial k}{\partial x_i}(x,y)g\left(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)\right)\mathrm{d}y,$$

and Fu has measurable derivatives. We show that, $Fu \in W^{1,1}(\Omega)$. Using our assumptions, for arbitrarily fixed $x \in \Omega$, we have

$$|Fu(x)| \le |p(x)| + |q(x)||u(x)| + \left|\int_\Omega k(x,y)g\left(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), (y)\right)\mathrm{d}y\right|.$$

According to the Jensen's inequality, we deduce

$$\|Fu\|_{L^1(\Omega)} \le \|p\|_{L^1(\Omega)} + \|q\|_u\|u\|_{L^1(\Omega)} + Mm(\Omega)\|g_1\|_{L^1(\Omega)}\|g_2\|_{L^\infty}\zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right).$$

By the same argument as above,

$$\left|\frac{\partial(Fu)}{\partial x_i}(x)\right| \le \left|\frac{\partial p}{\partial x_i}(x)\right| + \left|\frac{\partial q}{\partial x_i}(x)\right||u(x)| + |q(x)|\left|\frac{\partial u}{\partial x_i}(x)\right| + \left|\int_\Omega \frac{\partial k}{\partial x_i}(x,y)g\left(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)\right)\mathrm{d}y\right|$$

and

$$\left\|\frac{\partial(Fu)}{\partial x_i}\right\|_{L^1(\Omega)} \le \left\|\frac{\partial p}{\partial x_i}(x)\right\|_{L^1(\Omega)} + \left\|\frac{\partial q}{\partial x_i}\right\|_u\|u\|_{L^1(\Omega)} + \|q\|_u \left\|\frac{\partial u}{\partial x_i}\right\|_{L^1(\Omega)} + Mm(\Omega)\|g_1\|_{L^1(\Omega)}\|g_2\|_{L^\infty}\zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right).$$

Thus, we obtain

$$\|Fu\|_{1,1} \le \|p\|_{1,1} + \lambda\|u\|_{1,1} + Mm(\Omega)\|g_1\|_{L^1(\Omega)}\|g_2\|_{L^\infty}\zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right). \tag{11}$$

Due to (11) and using condition (iv), we derive that F is a mapping from $\bar{B}_{r_0}$ into $\bar{B}_{r_0}$. Now, we show that the map F is continuous. Let $\{u_m\}$ be an arbitrary sequence in $W^{1,1}(\Omega)$ which converges to $u \in W^{1,1}(\Omega)$. By Lemma 3.1 there is a subsequence $\{u_{m_k}\}$ which converges to u a.e., $\{\frac{\partial u_{m_k}}{\partial x_i}\}$ converges to $\{\frac{\partial u}{\partial x_i}\}$ a.e., $\{Tu_{m_k}\}$ converges to Tu a.e. and there is $h \in L^1(\Omega)$, $h \ge 0$, such that

$$\max\{|u_{m_k}(y)|, \left|\frac{\partial u_{m_k}}{\partial x_1}(y)\right|, \left|\frac{\partial u_{m_k}}{\partial x_2}(y)\right|, \ldots, |Tu_{m_k}(y)|\} \le h(y) \quad \text{for a.e.} \quad y \in \Omega.$$

Since $u_{m_k} \to u$ almost everywhere and g satisfies the Carathéodory conditions, it follows that

$$g(y, u_{m_k}(y), \frac{\partial u_{m_k}}{\partial x_1}(y), \ldots, Tu_{m_k}(y)) \to g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, Tu(y)),$$

for almost all $y \in \Omega$.

From condition (ii) we have

$$g(y, u_{m_k}(y), \frac{\partial u_{m_k}}{\partial x_1}(y), \ldots, Tu_{m_k}(y)) \leq a(y)\zeta(h(y)) \quad \text{for} \quad \text{a.e.} \quad y \in \Omega.$$

As a consequence of the Lebesgue's Dominated Convergence Theorem, it yields that

$$\int g(y, u_{m_k}(y), \frac{\partial u_{m_k}}{\partial x_1}(y), \ldots, Tu_{m_k}(y))\mathrm{d}y \to \int g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, Tu(y))\mathrm{d}y, \tag{12}$$

for almost all $y \in \Omega$. Inequality (12) and condition (iii) imply that

$$\|Fu_{m_k} - Fu\|_{1,1} \to 0 \quad \text{and} \quad \left\|\frac{\partial \mathrm{Fu_{m_k}}}{\partial \mathrm{x_i}} - \frac{\partial \mathrm{Fu}}{\partial \mathrm{x_i}}\right\|_{1,1} \to 0 \quad \text{as} \quad k \to \infty \quad (1 \leq i \leq n).$$

Therefore, $F : W^{1,1}(\Omega) \longrightarrow W^{1,1}(\Omega)$ is continuous.

To finish, the proof we have to verify that condition (1) is satisfied. We fix arbitrary $T > 0$ and $\varepsilon > 0$. Let U be a nonempty and bounded subset of $\bar{B}_{r_0}$. Choose $u \in U$ and $x, h \in B_T$ with $\|h\|_{\mathbb{R}^n} \leq \varepsilon$, then we have

$$\begin{aligned}
&\int_{B_T} |Fu(x) - Fu(x+h)|\mathrm{d}x \\
&\leq \int_{B_T} |p(x) - p(x+h)|\mathrm{d}x \\
&+ \int_{B_T} |q(x) - q(x+h)||u(x)|\mathrm{d}x \\
&+ \int_{B_T} |q(x+h)||u(x) - u(x+h)|\mathrm{d}x \\
&+ \int_{B_T} \int_{\Omega} |k(x,y) - k(x+h,y)||g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))|\mathrm{d}y\mathrm{d}x \\
&\leq \omega^T(p,\varepsilon) + \int_{B_T} |q(x) - q(x+h)||u(x)|\mathrm{d}x + \lambda\omega^T(U,\varepsilon) \\
&+ Mm(\Omega)\|g_2\|_{L^\infty}\zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right)\omega^T(g_3,\varepsilon).
\end{aligned} \tag{13}$$

Obviously, $\omega^T(p,\varepsilon) \to 0$, $\omega^T(g_3,\varepsilon) \to 0$ and by continuity of q,

$$\int_{B_T} |q(x) - q(x+h)||u(x)|\mathrm{d}x \to 0,$$

as $\varepsilon \to 0$. Then the right hand side of (13) tends to $\lambda\omega^T(U)$ as $\varepsilon \to 0$.

By a similar argument and using condition (i), for each $i = 1, \ldots, n$, we get

$$\begin{aligned}
&\int_{B_T} \left|\frac{\partial(Fu)}{\partial x_i}(x) - \frac{\partial(Fu)}{\partial x_i}(x+h)\right|\mathrm{d}x \\
&\leq \int_{B_T} \left|\frac{\partial p}{\partial x_i}(x) - \frac{\partial p}{\partial x_i}(x+h)\right|\mathrm{d}x + \int_{B_T} \left|\frac{\partial q}{\partial x_i}(x) - \frac{\partial q}{\partial x_i}(x+h)\right||u(x)|\mathrm{d}x \\
&+ \int_{B_T} \left|\frac{\partial q}{\partial x_i}(x+h)||u(x) - u(x+h)\right|\mathrm{d}x \\
&+ \int_{B_T} \left|q(x)||\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x+h)\right|\mathrm{d}x \\
&+ \int_{B_T} \left|\frac{\partial u}{\partial x_i}(x+h)||q(x) - q(x+h)\right|\mathrm{d}x \\
&+ \int_{B_T} \int_{\Omega} \left|\frac{\partial k}{\partial x_i}(x,y) - \frac{\partial k}{\partial x_i}(x+h,y)\right| \\
&\times \left|g\left(y, u(y), \frac{\partial u}{\partial x}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)\right)\right|\mathrm{d}y\mathrm{d}x \\
&\leq \omega^T(p,\varepsilon) + \int_{B_T} \left|\frac{\partial q}{\partial x_i}(x) - \frac{\partial q}{\partial x_i}(x+h)||u(x)\right|\mathrm{d}x \\
&+ \lambda\omega^T(U,\varepsilon) \\
&+ \int_{B_T} \left|\frac{\partial u}{\partial x_i}(x+h)||q(x) - q(x+h)\right|\mathrm{d}x \\
&+ Mm(\Omega)\|g_2\|_{L^\infty}\zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right)\omega^T(g_3,\varepsilon).
\end{aligned} \tag{14}$$

Applying the same reasoning as above, the right hand side of (14) tends to $\lambda\omega^T(U)$ as $\varepsilon \to 0$, too. Regarding to (13) and (14), and since u is an arbitrary element of U, then $\omega^T(FU) \leq \lambda\omega^T(U)$. Letting $T \to \infty$, we deduce

$$\omega(FU) \leq \lambda\omega(U). \tag{15}$$

Next, let us fix an arbitrary number $T > 0$. Then, taking into account our hypotheses, for an arbitrary function $u \in U$ we derive

$$\begin{aligned}
\|Fu\|_{L^1(\Omega\setminus B_T)} \leq{}& \|p\|_{L^1(\Omega\setminus B_T)} + \|q\|_u\|u\|_{L^1(\Omega\setminus B_T)} \\
&+ Mm(\Omega)\|g_1\|_{L^1(\Omega\setminus B_T)}\|g_2\|_{L^\infty(\Omega\setminus B_T))} \\
&\zeta\left(\frac{1}{m(\Omega)}\|u\|_{1,1}\right).
\end{aligned}$$

Now, since

$$\|p\|_{L^1(\Omega\setminus B_T)} \to 0, \quad \|g_1\|_{L^1(\Omega\setminus B_T)} \to 0 \quad \text{as} \quad T \to \infty,$$

then

$$\lim_{T\to\infty} \|Fu\|_{L^1(\Omega\setminus B_T)} \leq \lambda d(U).$$

Similarly,

$$\lim_{T\to\infty}\left\{\left\|\frac{\partial(Fu)}{\partial x_i}\right\|_{L^1(\Omega\setminus B_T)} : i=1,\ldots,n\right\}\le \lambda d(U).$$

These relations imply that

$$d(FU)\le \lambda d(U). \tag{16}$$

Finally, from (15) and (16) we conclude that $\omega_0(FU)\le \lambda\omega_0(U)$.

According to Theorem 1.2, we obtain that the operator F has a fixed point x in $\bar{B}_{r_0}$, and thus functional integral-differential equation (10) has at least one solution in the space $W^{1,1}(\Omega)$. □

Now, we present two examples which verify the effectiveness and applicability of Theorem 3.3.

Example 3.4 Consider the following functional integral-differential equation

$$\begin{aligned}u(x_1,x_2,x_3) =& \sqrt[4]{x_1^5}+e^{-(x_1+x_2+x_3+1)}u(x_1,x_2,x_3)\\ &+\int_0^1\int_0^1\int_0^1 \frac{e^{-(x_1+x_2+x_3)}}{(y_1+1)^3(y_2+2)^2(y_3+5)}\\ &\times\cos\left(y_1u(x_1,x_2,x_3)\frac{\partial u}{\partial x_1}(y_1,y_2,y_3)+y_2\frac{\partial u}{\partial x_2}(y_1,y_2,y_3)\right.\\ &\left.+y_3\frac{\partial u}{\partial x_3}(y_1,y_2,y_3)+\frac{1}{2}u(y_1,y_2,y_3)\right)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.\end{aligned} \tag{17}$$

Eq. (17) is a special case of Eq. (10) with

$\Omega=[0,1]\times[0,1]\times[0,1], p(x_1,x_2,x_3)=\sqrt[4]{x_1^5}$,
$q(x_1,x_2,x_3)=e^{-(x_1+x_2+x_3+1)}$, $g(y_1,y_2,y_3,u_0,u_1,u_2,u_3,u_4) = \cos(y_1u_0u_1+y_2u_2+y_3u_3+u_4)$, $Tu=\frac{1}{2}u$,

$$k(x_1,x_2,x_3,y_1,y_2,y_3)=\frac{e^{-(x_2+y_2+y_3+1)}}{(y_1+1)^3(y_2+2)^2(y_3+5)},$$

$$g_1(x_1,x_2,x_3)=g_3(x_1,x_2,x_3)=e^{-(x_1+x_2+x_3+1)},$$

and

$$g_2(x_1,x_2,x_3)=\frac{1}{(y_1+1)^2(y_2+2)(y_3+5)}.$$

It is easy to see that $p\in W^{1,1}(\Omega)$, $q\in BC^1(\Omega)$ and $\lambda=2e^{-1}$. Also, g satisfies Carath$\acute{e}$odory conditions and if we define $a(x_1,x_2,x_3)=\zeta(x)=1$, then condition (ii) of Theorem 3.3 holds. We observe that $g_1,g_3\in L^1(\Omega), g_2\in L^\infty(\Omega)$ and k satisfies condition (iii). Moreover, it can be easily shown that each number $r\ge 4$ satisfies the inequality in condition (iv), i.e.,

$$\|p\|_{1,1}+\lambda r+M\|g_1\|_{L^1(\Omega)}\|g_2\|_{L^\infty}\zeta(r)\le 1+2e^{-1}r+\frac{(1-e^{-1})^3}{10}\le r.$$

Thus, as the number r_0 we can take $r_0=4$. Consequently, all the conditions of Theorem 3.3 are satisfied. Hence the functional integral-differential equation (17) has at least one solution in the space $W^{1,1}(\Omega)$.

Example 3.5 Consider the following functional integral-differential equation

$$u(x)=\frac{u(x)}{x+2}+\int_0^1\frac{\sqrt[5]{y^3u(y)+3u^{(2)}(y)+u^{(3)}(y)}}{1+(u')^2(y)e^{\sin(u^{(2)}(y)+1)}}\mathrm{d}y. \tag{18}$$

Eq. (18) is a special case of Eq. (10) with

$$p(x)=0, q(x)=\frac{1}{x+2}, k(x,y)=e^{x-y}, T(u)=0, \Omega=[0,1]$$

and

$$g(y,u_0,u_1,u_2,u_3,u_4)=\frac{\sqrt[5]{y^3u_0+3u_2+u_4}}{1+u_1^2e^{\sin(u_2+1)}}.T(u)=0$$

It is easy to see that $q\in BC^1(\Omega)$ and $\lambda=\frac{3}{4}$. Also, g satisfies Carath$\acute{e}$odory conditions and if we define $a(x)=\sqrt[5]{5}$ and $\zeta(x)=\sqrt[5]{x}$, then condition (ii) of Theorem 3.3 holds. Moreover, k is continuous and has a continuous derivative of order 1 with respect to the first argument. On the other hand, $g_1(x)=g_3(x)=e^x$ and $g_2(x)=e^{-x}$. It can be easily shown that each number $r\ge 10$ satisfies the inequality in condition (iv), i.e.,

$$\|p\|_{1,1}+\lambda r+M\|g_1\|_{L^1(\Omega)}\|g_2\|_{L^\infty}\zeta(r)\le\frac{3}{4}r+\sqrt[5]{5}(e-1)(1-e^{-1})\sqrt[5]{r}\le r.$$

Hence, as the number r_0 we can take $r_0=10$. Consequently, all the conditions of Theorem 3.3 are satisfied. It implies that the functional integral-differential equation (18) has at least one solution in the space $W^{1,1}(\Omega)$.

References

1. Agarwal, R.P., Benchohra, M., Seba, D.: On the application of measure of noncompactness to the existence of solutions for fractional differential equations. Results Math. **55**, 221–230 (2009)
2. Aghajani, A., Allahyari, A., Mursaleen, M.: A generalization of Darbo's theorem with application to the solvability of systems of integral equations. J. Comput. Appl. Math. **260**, 68–77 (2014)

3. Aghajani, A., Banaś, J., Jalilian, Y.: Existence of solutions for a class of nonlinear Volterra singular integral equations. Comput. Math. Appl. **62**, 1215–1227 (2011)
4. Aghajani, A., O'Regan, D., Shole Haghighi, A.: Measure of noncompactness on $L^p(\mathbb{R}^n)$ and applications. Cubo A Math. J. **17**, 85–97 (2015)
5. Arab, R., Allahyari, R., Shole Haghighi, A.: Construction of measures of noncompactness of $C^k(\Omega)$ and C_0^k and their application to functional integral-differential equations. Bull. Iran. Math. Soc. **43**(1), 53–67 (2017)
6. Ayad, A.: Spline approximation for first order Fredholm delay integro-differential equations. Int. J. Comput. Math. **70**, 467–476 (1999)
7. Banaś, J.: Measures of noncompactness in the study of solutions of nonlinear differential and integral equations. Cent. Eur. J. Math. **10**(6), 2003–2011 (2012)
8. Banaś, J., Goebel, K.: Measure of noncompactness in Banach spaces. Lecture notes in pure and applied mathematics, vol. 60. Dekker, New York (1980)
9. Banaś, J., O'Regan, D., Sadarangani, K.: On solutions of a quadratic hammerstein integral equation on an unbounded interval. Dynam. Syst. Appl. **18**, 251–264 (2009)
10. Behiry, S.H., Hashish, H.: Wavelet methods for the numerical solution of Fredholm integro-differential equations. Int. J. Appl. Math. **11**, 27–35 (2002)
11. Brezis, H.: Functional analysis. Sobolev spaces and partial differential equations. Springer Science, Business Media, LLC, New York (2011)
12. Bloom, F.: Asymptotic bounds for solutions to a system of damped integro-differential equations of electromagnetic theory. J. Math. Anal. Appl. **73**, 524–542 (1980)
13. Darbo, G.: Punti uniti in trasformazioni a codominio non compatto. Rend. Sem. Mat. Univ. Padova. **24**, 84–92 (1955)
14. Drabek, P., Milota, J.: Methods of nonlinear analysis. Birkhauser Velgar AG, Basel (2007)
15. Darwish, M.A.: On monotonic solutions of a quadratic integral equation with supremum. Dynam. Syst. Appl. **17**, 539–550 (2008)
16. Darwish, M.A., Henderson, J., O'Regan, D.: Existence and asymptotic stability of solutions of a perturbed fractional functional-integral equation with linear modification of the argument. Bull. Korean Math. Soc. **48**, 539–553 (2011)
17. Dhage, B.C., Bellale, S.S.: Local asymptotic stability for nonlinear quadratic functional integral equations. Electron. J. Qual. Theory Differ. Equ. **10**, 1–13 (2008)
18. Forbes, L.K., Crozier, S., Doddrell, D.M.: Calculating current densities and fields produced by shielded magnetic resonance imaging probes. SIAM J. Appl. Math. **57**, 401–425 (1997)
19. Garey, L.A., Gladwin, C.J.: Direct numerical methods for first order Fredholm integro-differential equations. Int. J. Comput. Math. **34**, 237–246 (1990)
20. Hanche-Olsen, H., Holden, H.: The Kolmogorov–Riesz compactness theorem. Expo. Math. **28**, 385–394 (2010)
21. Holmaker, K.: Global asymptotic stability for a stationary solution of a system of integro-differential equations describing the formation of liver zones. SIAM J. Math. Anal. **24**, 116–128 (1993)
22. Hosseini, S.M., Shahmorad, S.: Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial base. Appl. Math. Model. **27**, 145–154 (2003)
23. Jleli, M., Mursaleen, M., Sadarangani, K., Samet, B.: A cone measure of noncompactness and some generalizations of Darbo's theorem with applications to functional integral equations. J. Funct. Spaces **9896502**, 11 (2016)
24. Kanwal, R.P.: Linear integral differential equations theory and technique. Academic Press, New York (1971)
25. Kuratowski, K.: Sur les espaces complets. Fund. Math. **15**, 301–309 (1930)
26. Liu, L., Guo, F., Wu, C., Wu, Y.: Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. J. Math. Anal. Appl. **309**, 638–649 (2005)
27. Mursaleen, M., Rizvi, S.M.H.: Solvability of infinite system of second order differential equations in c_0 and l_1 by Meir–Keeler condensing operator. Proc. Am. Math. Soc. **144**(10), 4279–4289 (2016)
28. Mursaleen, M., Mohiuddine, S.A.: Applications of noncompactness to the infinite system of differential equations in l_p spaces. Nonlinear Anal. **75**(4), 2111–2115 (2012)
29. Olszowy, L.: Solvability of infinite systems of singular integral equations in Fréchet space of coninuous functions. Comp. Math. Appl. **59**, 2794–2801 (2010)

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