Discrete septic spline quasi-interpolants for solving generalized Fredholm integral equation of the second kind via three degenerate kernel methods

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Abstract Three main contributions are presented in this paper. First, the septic quasi-interpolants are calculated with all their coefficients. Second, we explore the results to solve a generalized and broad class of Fredholm integral equations of the second kind. Finally, we present three degenerate kernel methods; the latter is a combination of the two previously established methods in the literature. Moreover, we provide a convergence analysis and we give new error bounds. Finally, we exhibit some numerical examples and compare them with previous results in the literature.

Keywords Septic spline · Quasi-interpolation · Integral equations

Introduction

In the last years, many authors have presented different approaches to the solution of Fredholm integral equations using spline kernel approximations, to limit the number of terms in the approximate kernel. The results of this limitation provide high accuracy excluding unnecessary computational costs due to large linear systems.

The authors of [14] exploit Galerkin method to approximate the solution of the time-dependent Dirac equation in prolate spheroidal coordinates for an electron-molecular two-center system. They use balanced basis of kinetic-tomic to evaluate the initial state from a variational

principle. Consequently, they obtain an exact and effective determination of the Dirac spectrum and eigenfunctions.

Using the polar method on parametric cubic spline technique, Prabhakar and Uma get wave resonating quadruplets calculated by the nonlinear source term of the wave model. They decide about the points of the locus for the two spacings, constant spacing and variable spacing. Refer to [27] for details.

The aim of [23] is to numerically solve the time fractional subdiffusion equation with Dirichlet boundary value conditions by use of the collocation method based on quadratic spline. In the paper, the authors explore, in details, the co-efficient matrix of the discretized linear system.

In [10], the authors propose a collocation method based on the quadratic spline in the pricing problem under a finite activity jump diffusion model for the first time.

For the sake of analyzing the nonlinear elastoplastic behavior of prismatic thin-walled members, the authors of [12] found a method of beam finite element working on generalized beam theory.

In another work [21], the authors decide to use quadratic and cubic B-spline quasi-interpolants to achieve higher order numerical methods only for a limited equations of Sobolev type in one dimension. Their aim is to compare the accuracy and convergence rate of these methods' performance.

Barton and Calo derive a new way for generating optimal quadrature rules for splines by making an association of source space and a known optimal quadrature and get the rule from the source space to the target one, but they maintain the point number and the optimality of quadrature. The aim of this process is to produce an optimal quadrature rule in a given spline space (see [7]).

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The purpose of [25] is to develop the BS Hermite spline quasi-interpolation scheme, which is related to the continuous extension of the BS linear multistep methods for solving the ordinary differential equations.

The author of [15] explores the quadratic spline quasi-interpolants on bounded domains, and provide some applications to different areas of the approximation theory. In the same scope, a different work gives some significant approximation formulas for derivatives of approximated functions through the application of univariate and multivariate quadratic spline quasi-interpolants. It uses the superconvergence properties of these operators to achieve very accurate derivatives of approximated functions at certain points (see [16]).

In [3], the authors apply a couple of well-known methods in the literature of the integral equations theory (see for instance [1, 6]) to solve a Fredholm integral equation of the second kind, through the discrete quartic spline quasi-interpolant. The result of their work is two-fold: on the one hand, they achieve an approximation order $O(h^5)$ for the left method, on the other hand, they achieve another approximation of the order of $O(h^6)$ for the right method. Recently, a paper reached the sextic spline, working on two collocation methods, using both, spline interpolants and spline quasi-interpolants to solve the fifth-order boundary value problems (see [22]).

According to what have been stated so far, we assume the importance of the spline functions in the theory of approximation. Therefore, the present paper builds and carries on previously published works by so many authors. The main idea of this paper offers three contributions. The first one is to build the septic quasi-interpolants with all its coefficients. In the second contribution, we explore the achieved results to solve a generalized and broad class of Fredholm integral equations of the second kind. Finally, we present three degenerate kernel methods, where the latter method is original. We also provide a convergence analysis and we give new error bounds. Moreover, we compare the results of our calculations with other results.

In the following, we start by developing B-splines and monomials of degree seven and then we build the discrete septic quasi-interpolant. Next, we use our new results to solve a class of Fredholm integral equations of the second kind on a broader level using degenerate kernel methods. Then, we provide the order of the convergence for each method. In Sect. 5, we illustrate our theory by different examples, comparing the results we achieved with already given results in the field to argue that ours are more efficient than the previous ones. Finally, some conclusions are presented.

Construction of discrete septic quasi-interpolant

B-splines and monomials of degree seven

Let $\mathcal{X} := \mathcal{C}^0([a, b], \mathbb{R})$ be the space of all continuous functions, equipped with the max-norm $\|\cdot\|_\infty$. Let us consider the nodes $x_0, x_1, x_2, \dots, x_n$ in the interval $[a, b]$ with

$$x_k = a + kh_k \quad \text{and} \quad h_k = x_k - x_{k-1} \quad \text{for all} \quad 1 \leq k \leq n.$$

Let $\mathcal{X}_n := \{x_k, 0 \leq k \leq n\}$ denote the partition of the interval $[a, b]$ into n subintervals.

Define $S_7 := S_7([a, b], \mathcal{X}_n)$ to be the space of septic splines of class $\mathcal{C}^6$ on this partition.

Consider the set

$$\Gamma_n := \{1, 2, \dots, n+7\}.$$

Let a canonical basis of S_7 be $\{B_k, k \in \Gamma_n\}$, which is shaped by the $n+7$ normalized B-splines. We add multiple knots at the endpoints to obtain the support of B_k which is the interval $[x_{k-8}, x_k]$ (see [28]). It is well-known that the representation of monomials using symmetric functions $\text{sym}_r(N_k)$ of interior knots

$$N_k := \{x_{k-7}, x_{k-6}, x_{k-5}, x_{k-4}, x_{k-3}, x_{k-2}, x_{k-1}\} \quad \text{in} \quad \text{Supp}(B_k).$$

Denoting by D^j the derivation operator of the order j , we consider the following function

$$\begin{aligned} \phi_k(t) &:= (x_{k-7} - t)(x_{k-6} - t)(x_{k-5} - t)(x_{k-4} - t) \\ &\quad (x_{k-3} - t)(x_{k-2} - t)(x_{k-1} - t). \end{aligned}$$

For $0 \leq r \leq 7$, we have:

$$\begin{aligned} m_r(x) &= x^r \\ &= \sum_{k \in \Gamma_n} (-1)^{7-r} \frac{r!}{7!} D^{7-r} \phi_k(0) B_k(x) \\ &= \sum_{k \in \Gamma_n} \theta_k^r B_k(x), \end{aligned}$$

such that

$$\theta_k^r = \frac{\text{sym}_r(N_k)}{\alpha_r}, \quad 0 \leq r \leq 7,$$

where

$$\alpha_r := \binom{7}{r}.$$

Specifically, if $r = 0$, we have $\theta_k^{(0)} = 1$ for all $k \in \Gamma_n$.

For $r = 1$, using the formula

$$\sum_{k \in \Gamma_n} B_k(x) = 1,$$

we obtain the Greville abscissae:

Table 1 θ_j^1

j	θ_j^1
1	a
2	$a + \frac{h}{7}$
3	$a + \frac{3h}{7}$
4	$a + \frac{6h}{7}$
5	$a + \frac{10h}{7}$
6	$a + \frac{15h}{7}$
$7 \leq j \leq n+1$	$hj + a - 4h$
$n+2$	$b - \frac{15h}{7}$
$n+3$	$b - \frac{10h}{7}$
$n+4$	$b - \frac{6h}{7}$
$n+5$	$b - \frac{3h}{7}$
$n+6$	$b - \frac{h}{7}$
$n+7$	b

Table 2 θ_j^2

j	θ_j^2
1	a^2
2	$a^2 + \frac{2ha}{7}$
3	$a^2 + \frac{6ha}{7} + \frac{2h^2}{21}$
4	$a^2 + \frac{12ha}{7} + \frac{11h^2}{21}$
5	$a^2 + \frac{20ha}{7} + \frac{5h^2}{3}$
6	$a^2 + \frac{30ha}{7} + \frac{85h^2}{21}$
$7 \leq j \leq n+1$	$h^2j^2 + (2ah - 8h^2)j + a^2 - 8ah + \frac{46h^2}{3}$
$n+2$	$b^2 - \frac{30hb}{7} + \frac{85h^2}{21}$
$n+3$	$b^2 - \frac{20hb}{7} + \frac{5h^2}{3}$
$n+4$	$b^2 - \frac{12hb}{7} + \frac{11h^2}{21}$
$n+5$	$b^2 - \frac{6hb}{7} + \frac{2h^2}{21}$
$n+6$	$b^2 - \frac{2hb}{7}$
$n+7$	b^2

$$\theta_k = \theta_k^{(1)} = \frac{1}{7} \sum_{l=1}^7 x_{k-l},$$

which are the coefficients of

$$m_1(x) = \sum_{k \in \Gamma_n} \theta_k B_k(x).$$

For $r = 2$, we obtain

$$\theta_k = \theta_k^2 = \binom{7}{2}^{-1} \sum_{1 \leq l < l' \leq 7} x_{k-l} x_{k-l'}.$$

For $r = 3$, we obtain

$$\theta_k = \theta_k^3 = \binom{7}{3}^{-1} \sum_{1 \leq l < l' < l'' \leq 7} x_{k-l} x_{k-l'} x_{k-l''}.$$

We have calculated all the coefficients $\theta_j^r, j \in \Gamma_n, 0 \leq r \leq 7$; the results are given in Tables 1, 2, 3, 4, 5, 6 and 7 respectively.

Discrete quasi-interpolant of degree 7

The discrete septic spline quasi-interpolant, abbreviated as dSSQI, is the operator

$$Qf = \sum_{k \in \Gamma_n} \mu_k(f) B_k,$$

whose coefficients are linear combinations of discrete values of f on the set of data points $\mathcal{X}_n$.

The dSSQI is developed to be exact on P_7 , i.e.

$$Qp = p \quad \text{for all } p \in P_7,$$

in other words, $Qm_r = m_r$, where

$$m_r(x) = \sum_{k \in \Gamma_n} \mu_k(m_r) B_k(x) = \sum_{k \in \Gamma_n} \theta_k^r B_k(x), \quad 0 \leq r \leq 7.$$

Therefore, we obtain the following conditions

$$\mu_k(m_r) = \theta_k^r \quad \text{for } k \in \Gamma_n, \quad 0 \leq r \leq 7.$$

For $7 \leq k \leq n+1$, the functionals $\mu_k(f)$ merely use values of f in a neighborhood of the support of B_k , that is why it should be expressed as $\mu_k(f)$ thus

$$\begin{aligned} \mu_k(f) = & \alpha_k f_{k-7} + \beta_k f_{k-6} + \gamma_k f_{k-5} + \delta_k f_{k-4} \\ & + \lambda_k f_{k-3} + \mu_k f_{k-2} + \nu_k f_{k-1}, \end{aligned}$$

where $f_k = f(x_k)$. The conditions above are the same as the systems of linear equations:

$$\begin{aligned} \alpha_k x_{k-7}^r + \beta_k x_{k-6}^r + \gamma_k x_{k-5}^r + \delta_k x_{k-4}^r + \lambda_k x_{k-3}^r \\ + \mu_k x_{k-2}^r + \nu_k x_{k-1}^r = \theta_k^r, \quad 0 \leq r \leq 7. \end{aligned}$$

For $1 \leq k \leq 6$ and $n+2 \leq k \leq n+7$, we get the following equations, respectively

$$\mu_k(f) = \alpha_k f_0 + \beta_k f_1 + \gamma_k f_2 + \delta_k f_3 + \lambda_k f_4 + \mu_k f_5 + \nu_k f_6 + \omega_k f_7$$

$$\begin{aligned} \mu_k(f) = & \alpha_k f_{n-7} + \beta_k f_{n-6} + \gamma_k f_{n-5} + \delta_k f_{n-4} + \lambda_k f_{n-3} \\ & + \mu_k f_{n-2} + \nu_k f_{n-1} + \omega_k f_n. \end{aligned}$$

All these systems have Vandermonde determinants

$$V_8(x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}, x_{k+7}) \neq 0.$$

While the $x_j, j \in J_n := \{0, 1, \dots, n\}$, are different, they have unique solutions. Hence, the existence and uniqueness of the dSSQI. In the case of a uniform partition, the coefficient functionals are, respectively, defined by the following formulæ:

$$\mu_1(f) = f_0.$$

Table 3 θ_j^3

j	θ_j^3
1	a^3
2	$a^3 + \frac{3ha^2}{7}$
3	$a^3 + \frac{9ha^2}{7} + \frac{2h^2}{7}$
4	$a^3 + \frac{18ha^2}{7} + \frac{11h^2a}{7} + \frac{6h^3}{35}$
5	$a^3 + \frac{30ha^2}{7} + 5ah^2 + \frac{10h^3}{7}$
6	$a^3 + \frac{45ha^2}{7} + \frac{85ah^2}{21} + \frac{45h^3}{7}$
$7 \leq j \leq n+1$	$h^3j^3 + (3ah^2 - 12h^3)j^2 + (3ha^2 - 24ah^2 + 46h^3)j + a^3 - 12a^2h + 46ah^2 - 56h^3$
$n+2$	$b^3 - \frac{45hb^2}{7} + \frac{85bh^2}{7} - \frac{45h^3}{7}$
$n+3$	$b^3 - \frac{30hb^2}{7} + 5bh^2 - \frac{10h^3}{7}$
$n+4$	$b^3 - \frac{18hb^2}{7} + \frac{11h^2b}{7} - \frac{6h^3}{35}$
$n+5$	$b^3 - \frac{9hb^2}{7} + \frac{2bh^2}{7}$
$n+6$	$b^3 - \frac{3hb^2}{7}$
$n+7$	b^3

Table 4 θ_j^4

j	θ_j^4
1	a^4
2	$a^4 + \frac{4ha^3}{7}$
3	$a^4 + \frac{12ha^3}{7} + \frac{4a^2h^2}{7}$
4	$a^4 + \frac{24ha^3}{7} + \frac{22a^2h^2}{7} + \frac{24ah^3}{35}$
5	$a^4 + \frac{40ha^3}{7} + 10a^2h^2 + \frac{40ah^3}{7} + \frac{24h^4}{35}$
6	$a^4 + \frac{60ha^3}{7} + \frac{170a^2h^2}{7} + \frac{180ah^3}{7} + \frac{274h^4}{35}$
$7 \leq j \leq n+1$	$h^4j^4 + (4ah^3 - 16h^4)j^3 + (6h^2a^2 - 48ah^3 + 92h^4)j^2 + (4a^3h - 48h^2a^2 + 184ah^3 - 224h^4)j$ $+ a^4 - 16a^3h + 92a^2h^2 - 224ah^3 + \frac{967h^4}{5}$
$n+2$	$b^4 - \frac{60hb^3}{7} + \frac{170b^2h^2}{7} - \frac{180bh^3}{7} + \frac{274h^4}{35}$
$n+3$	$b^4 - \frac{40hb^3}{7} + 10b^2h^2 - \frac{40bh^3}{7} + \frac{24h^4}{35}$
$n+4$	$b^4 - \frac{24hb^3}{7} + \frac{22b^2h^2}{7} - \frac{24bh^3}{35}$
$n+5$	$b^4 - \frac{12hb^3}{7} + \frac{4b^2h^2}{7}$
$n+6$	$b^4 - \frac{4hb^3}{7}$
$n+7$	b^4

$$\begin{aligned} \mu_2(f) &= \frac{617}{980}f_0 + f_1 - \frac{3}{2}f_2 - \frac{5}{3}f_3 - \frac{5}{4}f_4 + \frac{3}{5}f_5 - \frac{1}{6}f_6 + \frac{1}{49}f_7. & \mu_4(f) &= -\frac{23,251}{264,600}f_0 + \frac{2887}{2100}f_1 - \frac{207}{700}f_2 - \frac{391}{3780}f_3 \\ & & &+ \frac{5}{24}f_4 - \frac{93}{700}f_5 + \frac{401}{9450}f_6 - \frac{83}{14,700}f_7. \\ \mu_3(f) &= \frac{3623}{26,460}f_0 + \frac{407}{210}f_1 - \frac{337}{140}f_2 + \frac{941}{378}f_3 - \frac{151}{84}f_4 \\ &+ \frac{59}{70}f_5 - \frac{871}{3780}f_6 + \frac{41}{1470}f_7. & \mu_5(f) &= -\frac{713}{52,920}f_0 - \frac{53}{1260}f_1 + \frac{241}{105}f_2 - \frac{8317}{3780}f_3 \\ & & &+ \frac{521}{360}f_4 - \frac{53}{84}f_5 + \frac{617}{3780}f_6 - \frac{167}{8820}f_7. \end{aligned}$$

Table 5 θ_j^5

j	θ_j^5
1	a^5
2	$a^5 + \frac{5ha^4}{7}$
3	$a^5 + \frac{15ha^4}{7} + \frac{20a^3h^2}{7}$
4	$a^5 + \frac{30ha^4}{7} + \frac{110a^3h^2}{21} + \frac{12a^2h^3}{7}$
5	$a^5 + \frac{50ha^4}{7} + \frac{50a^3h^2}{3} + \frac{100a^2h^3}{7} + \frac{24ah^4}{7}$
6	$a^5 + \frac{75ha^4}{7} + \frac{850a^3h^2}{21} + \frac{450a^2h^3}{7} + \frac{274ah^4}{7} + \frac{40h^5}{7}$
.....	
$7 \leq j \leq n+1$	$h^5j^5 + (5ah^4 - 20h^5)j^4 + (10h^3a^2 - 80ah^4 + \frac{460h^5}{3})j^3 + (10h^2a^3 - 120a^2h^3 + 460ah^4 - 560h^5)j^2$ $+ (5a^4h - 80a^3h^2 + 460a^2h^3 - 1120ah^4 + 967h^5)j + a^5 - 20a^4h + \frac{460a^3h^2}{3} - 560a^2h^3 + 967ah^4 - \frac{1876h^5}{3}$
.....	
$n+2$	$b^5 - \frac{75hb^4}{7} + \frac{850b^3h^2}{21} - \frac{450b^2h^3}{7} + \frac{274bh^4}{7} - \frac{40h^5}{7}$
$n+3$	$b^5 - \frac{50hb^4}{7} + \frac{50b^3h^2}{3} - \frac{100b^2h^3}{7} + \frac{24bh^4}{7}$
$n+4$	$b^5 - \frac{30hb^4}{7} + \frac{110b^3h^2}{21} - \frac{12b^2h^3}{7}$
$n+5$	$b^5 - \frac{15hb^4}{7} + \frac{20b^3h^2}{7}$
$n+6$	$b^5 - \frac{5hb^4}{7}$
$n+7$	b^5

Table 6 θ_j^6

j	θ_j^6
1	a^6
2	$a^6 + \frac{6ha^5}{7}$
3	$a^6 + \frac{18ha^5}{7} + \frac{10a^4h^2}{7}$
4	$a^6 + \frac{36ha^5}{7} + \frac{55a^4h^2}{7} + \frac{24a^3h^3}{7}$
5	$a^6 + \frac{60ha^5}{7} + 25a^4h^2 + \frac{200a^3h^3}{7} + \frac{72a^2h^4}{7}$
6	$a^6 + \frac{90ha^5}{7} + \frac{4250a^4h^2}{7} + \frac{900a^3h^3}{7} + \frac{822a^2h^4}{7} + \frac{240h^6}{7}$
.....	
$7 \leq j \leq n+1$	$h^6j^6 + (6ah^5 - 24h^6)j^5 + (15h^4a^2 - 120ah^5 + 230h^6)j^4$ $+ (20h^3a^3 - 240a^2h^4 + 920ah^5 - 1120h^6)j^3$ $+ (15a^4h^2 - 240a^3h^3 + 1380a^2h^4 - 3360ah^5 + 2901h^6)j^2$ $+ (6a^5h - 120a^4h^2 + 920a^3h^3 - 3360a^2h^4 + 5802ah^5 - 3752h^6)j$ $+ a^6 - 24a^5h + 230a^4h^2 - 1120a^3h^3 + 2901a^2h^4 - 3752ah^5 + \frac{13068h^6}{7}$
.....	
$n+2$	$b^6 - \frac{90hb^5}{7} + \frac{425b^4h^2}{7} - \frac{900b^3h^3}{7} + \frac{822b^2h^4}{7} - \frac{240h^6}{7}b$
$n+3$	$b^6 - \frac{60hb^5}{7} + 25b^4h^2 - \frac{200b^3h^3}{7} + \frac{72b^2h^4}{7}$
$n+4$	$b^6 - \frac{36hb^5}{7} + \frac{55b^4h^2}{7} - \frac{24b^3h^3}{7}$
$n+5$	$b^6 - \frac{18hb^5}{7} + \frac{10b^4h^2}{7}$
$n+6$	$b^6 - \frac{6hb^5}{7}$
$n+7$	b^6

Table 7 θ_j^7

j	θ_j^7
1	a^7
2	$a^7 + ha^6$
3	$a^7 + 3ha^6 + 2a^5h^2$
4	$a^7 + 6ha^6 + 11a^5h^2 + 6a^4h^3$
5	$a^7 + 10ha^6 + 35a^5h^2 + 50a^4h^3 + 24a^3h^4$
6	$a^7 + 15ha^6 + 85a^5h^2 + 225a^4h^3 + 274a^3h^4 + 120a^2h^5$
.....	
$7 \leq j \leq n+1$	$h^7j^7 + (7ah^6 - 28h^7)j^6 + (21h^5a^2 - 168ah^6 + 322h^7)j^5$ $+ (35h^4a^3 - 420a^2h^5 + 1610ah^6 - 1960h^7)j^4$ $+ (35a^4h^3 - 560a^3h^4 + 3220a^2h^5 - 7840ah^6 + 6769h^7)j^3$ $+ (21a^5h^2 - 420a^4h^3 + 3220a^3h^4 - 11760a^2h^5 + 20307ah^6 - 13132h^7)j^2$ $+ (7a^6h - 168a^5h^2 + 1610a^4h^3 - 7840a^3h^4 + 20307a^2h^5 - 26264ah^6 + 13068h^7)j$ $+ a^7 - 28a^6h + 322a^5h^2 - 1960a^4h^3 + 6769a^3h^4 - 13132a^2h^5 + 13068ah^6 - 5040h^7$
.....	
$n+2$	$b^7 - 15hb^6 + 85b^5h^2 - 225b^4h^3 + 274b^3h^4 - 120b^2h^5$
$n+3$	$b^7 - 10hb^6 + 35b^5h^2 - 50b^4h^3 + 24b^3h^4$
$n+4$	$b^7 - 6hb^6 + 11b^5h^2 - 6b^4h^3$
$n+5$	$b^7 - hb^6 + 2b^5h^2$
$n+6$	$b^7 - 6hb^6$
$n+7$	b^7

$$\mu_6(f) = \frac{3769}{105,840}f_0 - \frac{811}{2520}f_1 + \frac{263}{240}f_2 + \frac{5741}{7560}f_3$$

$$- \frac{683}{720}f_4 + \frac{85}{168}f_5 + \frac{311}{2160}f_6 + \frac{311}{7640}f_7.$$

For $7 \leq j \leq n+1$,

$$\mu_j(f) = -\frac{311}{15,120}(f_{j-7} + f_{j-1}) + \frac{22}{105}(f_{j-6} + f_{j-2})$$

$$- \frac{1657}{1680}(f_{j-5} + f_{j-3}) + \frac{2452}{945}f_{j-4}.$$

$$\mu_{n+2}(f) = \frac{3769}{105,840}f_n - \frac{811}{2520}f_{n-1} + \frac{263}{240}f_{n-2}$$

$$+ \frac{5741}{7560}f_{n-3} - \frac{683}{720}f_{n-4}$$

$$- \frac{85}{168}f_{n-5} - \frac{311}{2160}f_{n-6} + \frac{311}{7640}f_{n-7}.$$

$$\mu_{n+3}(f) = -\frac{713}{52,920}f_n - \frac{53}{1260}f_{n-1} + \frac{241}{105}f_{n-2}$$

$$- \frac{8317}{3780}f_{n-3} + \frac{521}{360}f_{n-4} - \frac{53}{84}f_{n-5}$$

$$+ \frac{617}{3780}f_{n-6} - \frac{167}{8820}f_{n-7}.$$

$$\mu_{n+4}(f) = -\frac{23,251}{264,600}f_n + \frac{2887}{2100}f_{n-1} - \frac{207}{700}f_{n-2}$$

$$- \frac{391}{3780}f_{n-3} + \frac{5}{24}f_{n-4} - \frac{93}{700}f_{n-5}$$

$$+ \frac{401}{9450}f_{n-6} - \frac{83}{14,700}f_{n-7}.$$

$$\mu_{n+5}(f) = \frac{3623}{26,460}f_n + \frac{407}{210}f_{n-1} - \frac{337}{140}f_{n-2} + \frac{941}{378}f_{n-3}$$

$$- \frac{151}{84}f_{n-4} + \frac{59}{70}f_{n-5}$$

$$- \frac{871}{3780}f_{n-6} + \frac{41}{1470}f_{n-7}.$$

$$\mu_{n+6}(f) = \frac{617}{980}f_n + f_{n-1} - \frac{3}{2}f_{n-2} + \frac{5}{3}f_{n-3} - \frac{5}{4}f_{n-4} + \frac{3}{5}f_{n-5}$$

$$- \frac{1}{6}f_{n-6} + \frac{1}{49}f_{n-7}.$$

$$\mu_{n+7}(f) = f_n.$$

It is noticeable that

$$|\mu_2|_\infty = |\mu_{n+6}|_\infty \approx 6.8333,$$

$$|\mu_3|_\infty = |\mu_{n+5}|_\infty \approx 9.8704,$$

$$|\mu_4|_\infty = |\mu_{n+4}|_\infty \approx 2.2511,$$

$$|\mu_5|_\infty = |\mu_{n+3}|_\infty \approx 6.8114,$$

$$|\mu_6|_\infty = |\mu_{n+2}|_\infty \approx 3.8288.$$

For $7 \leq j \leq n+1$,

$$|\mu_j|_\infty \approx 5.0275,$$

hence,

$$\|Q\| \leq 9.8704.$$

Furthermore, for $f \in C^8(I)$, we also have

$$\|f - Qf\|_{\infty, I_k} \leq (1 + 9.87)d_{\infty, I_k}(f, \mathbb{P}_7) \leq 11d_{\infty, I_k}(f, \mathbb{P}_7),$$

where

$$I_k = [x_{k-1}, x_k], \quad 1 \leq k \leq n,$$

and

$$d_{\infty, I_k}(f, \mathbb{P}_7) = \max\{\|f - p\|_{\infty, I_k}, p \in \mathbb{P}_7\}.$$

Then

$$\|f - Qf\|_\infty = O(h^8).$$

We can write the quasi-interpolant Q under the quasi-Lagrange form:

$$Qf = \sum_{j \in J_n} f_j L_j,$$

where

$$L_0 = B_1 + \frac{617}{980}B_2 + \frac{3623}{26,460}B_3 - \frac{23,251}{264,600}B_4 - \frac{713}{52,920}B_5 \\ + \frac{3769}{105,840}B_6 - \frac{311}{15,120}B_7.$$

$$L_1 = B_2 + \frac{407}{210}B_3 + \frac{2887}{2100}B_4 - \frac{53}{1260}B_5 - \frac{811}{2520}B_6 \\ + \frac{22}{105}B_7 - \frac{311}{15,120}B_8.$$

$$L_2 = -\frac{3}{2}B_2 - \frac{337}{140}B_3 - \frac{207}{700}B_4 + \frac{241}{105}B_5 + \frac{263}{240}B_6 \\ - \frac{1657}{1680}B_7 + \frac{22}{105}B_8 - \frac{311}{15,120}B_9.$$

$$L_3 = \frac{5}{3}B_2 + \frac{941}{378}B_3 - \frac{391}{3780}B_4 - \frac{8317}{3780}B_5 + \frac{5741}{7560}B_6 \\ + \frac{2452}{945}B_7 - \frac{1657}{1680}B_8 + \frac{22}{105}B_9 \\ - \frac{311}{15,120}B_{10}.$$

$$L_4 = -\frac{5}{4}B_2 - \frac{151}{84}B_3 + \frac{5}{24}B_4 + \frac{521}{360}B_5 - \frac{683}{720}B_6 \\ - \frac{1657}{1680}B_7 + \frac{2452}{945}B_8 - \frac{1657}{1680}B_9$$

$$+ \frac{22}{105}B_{10} - \frac{311}{15,120}B_{11}.$$

$$L_5 = \frac{3}{5}B_2 + \frac{59}{70}B_3 - \frac{93}{700}B_4 - \frac{53}{84}B_5 + \frac{85}{168}B_6 + \frac{22}{105}B_7 \\ - \frac{1657}{1680}B_8 + \frac{2452}{945}B_9 - \frac{1657}{1680}B_{10}$$

$$+ \frac{22}{105}B_{11} - \frac{311}{15,120}B_{12}.$$

$$L_6 = -\frac{1}{6}B_2 - \frac{871}{3780}B_3 + \frac{401}{9450}B_4 + \frac{617}{3780}B_5 - \frac{311}{2160}B_6 \\ - \frac{311}{15,120}B_7 + \frac{22}{105}B_8 - \frac{1657}{1680}B_9 + \frac{2452}{945}B_{10}$$

$$- \frac{1657}{1680}B_{11} + \frac{22}{105}B_{12} - \frac{311}{15,120}B_{13}.$$

$$L_7 = \frac{1}{49}B_2 + \frac{41}{1470}B_3 - \frac{83}{14,700}B_4 - \frac{167}{8820}B_5 + \frac{311}{17,640}B_6$$

$$- \frac{311}{15,120}B_8 + \frac{22}{105}B_9 - \frac{1657}{1680}B_{10} + \frac{2452}{945}B_{11}$$

$$- \frac{1657}{1680}B_{12} + \frac{22}{105}B_{13} - \frac{311}{15,120}B_{14}.$$

$$L_j = -\frac{311}{15,120}(B_{j+1} + B_{j+7}) + \frac{22}{105}(B_{j+2} + B_{j+6})$$

$$- \frac{1657}{1680}(B_{j+3} + B_{j+5}) + \frac{2452}{945}B_{j+4},$$

$$8 \leq j \leq n-8.$$

$$L_{n-7} = \frac{1}{49}B_{n+6} + \frac{41}{1470}B_{n+5} - \frac{83}{14,700}B_{n+4} - \frac{167}{8820}B_{n+3}$$

$$+ \frac{311}{17,640}B_{n+2} - \frac{311}{15,120}B_n + \frac{22}{105}B_{n-1} - \frac{1657}{1680}B_{n-2}$$

$$+ \frac{2452}{945}B_{n-3} - \frac{1657}{1680}B_{n-4} + \frac{22}{105}B_{n-5} - \frac{311}{15,120}B_{n-6}.$$

$$L_{n-6} = -\frac{1}{6}B_{n+6} + \frac{871}{3780}B_{n+5} + \frac{401}{9450}B_{n+4} + \frac{617}{3780}B_{n+3}$$

$$- \frac{311}{2160}B_{n+2} - \frac{311}{15,120}B_{n+1} + \frac{22}{105}B_n - \frac{1657}{1680}B_{n-1}$$

$$+ \frac{2452}{945}B_{n-2} - \frac{1657}{1680}B_{n-3} + \frac{22}{105}B_{n-4} - \frac{311}{15,120}B_{n-5}.$$

$$L_{n-5} = \frac{3}{5}B_{n+6} + \frac{59}{70}B_{n+5} - \frac{93}{700}B_{n+4} - \frac{53}{84}B_{n+3} - \frac{85}{168}B_{n+2} + \frac{22}{105}B_{n+1}$$

$$- \frac{1657}{1680}B_n + \frac{2452}{945}B_{n-1} - \frac{1657}{1680}B_{n-2} + \frac{22}{105}B_{n-3} - \frac{311}{15,120}B_{n-4}.$$

$$\begin{aligned}
L_{n-4} &= -\frac{5}{4}B_{n+6} - \frac{151}{84}B_{n+5} + \frac{5}{24}B_{n+4} + \frac{521}{360}B_{n+3} \\
&\quad - \frac{683}{720}B_{n+2} - \frac{1657}{1680}B_{n+1} + \frac{2452}{945}B_n - \frac{1657}{1680}B_{n-1} \\
&\quad + \frac{22}{105}B_{n-2} - \frac{311}{15,120}B_{n-3}, \\
L_{n-3} &= \frac{5}{3}B_{n+6} - \frac{941}{378}B_{n+5} - \frac{391}{3780}B_{n+4} - \frac{8317}{3780}B_{n+3} \\
&\quad + \frac{5741}{7560}B_{n+2} + \frac{2452}{945}B_{n+1} \\
&\quad - \frac{1657}{1680}B_n + \frac{22}{105}B_{n-1} - \frac{311}{15,120}B_{n-2}, \\
L_{n-2} &= -\frac{3}{2}B_{n+6} - \frac{337}{140}B_{n+5} - \frac{207}{700}B_{n+4} + \frac{241}{105}B_{n+3} \\
&\quad + \frac{263}{240}B_{n+2} - \frac{1657}{1680}B_{n+1} + \frac{22}{105}B_n - \frac{311}{15,120}B_{n-1}, \\
L_{n-1} &= B_{n+6} + \frac{407}{210}B_{n+5} + \frac{2887}{2100}B_{n+4} \\
&\quad - \frac{53}{1260}B_{n+3} - \frac{811}{2520}B_{n+2} + \frac{22}{105}B_{n+1} \\
&\quad - \frac{311}{15,120}B_n, \\
L_n &= B_{n+7} + \frac{617}{980}B_{n+6} + \frac{3623}{26,460}B_{n+5} \\
&\quad - \frac{23,251}{264,600}B_{n+4} - \frac{713}{52,920}B_{n+3} \\
&\quad + \frac{3769}{105,840}B_{n+2} - \frac{311}{15,120}B_{n+1}.
\end{aligned}$$

Solving Fredholm integral equations by degenerate kernel approximations

Consider the following generalized Fredholm integral equation of the second kind

$$u(s) - \sum_{k=1}^m \int_a^b H_k(s, t)u(t)dt = f(s), \quad m \in \mathbb{N}^*, \quad a \leq s \leq b, \quad (1)$$

where f is a continuous function. We assume that $H_k(\cdot, \cdot) \in C([a, b] \times [a, b], \mathbb{C})$, $k = 1, \dots, m$. Then the integral operator

$$Tu(s) := \sum_{k=1}^m \int_a^b H_k(s, t)u(t)dt, \quad a \leq s \leq b,$$

is compact from $\mathcal{X}$ into $\mathcal{X}$, we assume that the Eq. (1) has a unique solution. Equation (1) can be rewritten in an operator form as:

$$(I - T)u = f.$$

Degenerate kernel methods are crucial in approximation theory and in scientific computing. They have many interesting applications, particularly to solve integral and integro-differential equations. In [18], the authors explore the classical version of the degenerate kernel method to numerically solve the Hammerstein equations. Later, they extend the degenerate kernel method for single-variable Hammerstein equations to include multi-variable Hammerstein equations in [19]. The authors of [26] treat a degenerate approximation of the kernel using Taylor series and Lagrange interpolation for solving the general non-linear Fredholm integro-differential equations under mixed conditions. The degenerate kernel in the polar coordinates for two subdomains is adopted in [9] for the closed-form fundamental solution of null-field boundary integral equation method. Majidiana and Babolian [24] apply a degenerate kernel method with piecewise constant interpolation with respect to the second variable to approximate isolated eigenvalues of a class of noncompact linear operators. In [11], a new approach to the theory of kernel approximations is developed for the numerical solution of Fredholm integral equations of the second kind using a degenerate-kernel operator of fixed rank. Kalaba and Scott [17] use an initial-value method for integral equations with generalized degenerate kernels. Recently, the authors of [3] rely on two degenerate methods for solving the classical Fredholm integral equation of the second kind, based on (left and right) partial approximations of the kernel through a discrete quartic spline quasi-interpolant. From reviewing the literature, it is noticeable that the degenerate kernel method is commonly used in the development in the theory of approximation on a wide scale, mainly the resolution of integral equations. In a more modern sense, we intend to rely on the same degenerate kernel methods for solving (1) through our newly obtained results of discrete septic spline quasi-interpolants, unlike the previously mentioned work [3].

First septic spline degenerate kernel method

We first approximate the given continuous functions $s \mapsto H_k(s, t)$ by the septic spline quasi-interpolant using quasi-Lagrange form:

$$H_k^F(s, t) = \sum_{j \in J_n} H_k(s_j, t)L_j(s).$$

The left degenerate kernel operator is defined by

$$T_n^F u(s) := \sum_{j \in J_n} L_j(s) \sum_{k=1}^m \int_a^b H_k(s_j, t)u(t)dt.$$

Approximating the Eq. (1) by

$$u_n^F(s) - \sum_{j \in J_n} L_j(s) \sum_{k=1}^m \int_a^b H_k(s_j, t) u_n^F(t) dt = f(s), \quad (2)$$

the approximate solution u_n^F of Eq. (2) is given by

$$u_n^F(s) = f(s) + \sum_{j \in J_n} c_j L_j(s),$$

for some scalars c_j .

As a result, we obtain the linear system

$$\sum_{i \in J_n} c_i L_i(s) - \sum_{i \in J_n} L_i(s) \sum_{k=1}^m \int_a^b H_k(s_i, t) [f(t) + \sum_{j \in J_n} c_j L_j(t)] dt = 0,$$

hence

$$c_i - \sum_{j \in J_n} c_j \sum_{k=1}^m \int_a^b H_k(s_i, t) L_j(t) dt = \sum_{k=1}^m \int_a^b H_k(s_i, t) f(t) dt,$$

that is to say, the coefficients c_j are obtained by solving the following linear system

$$(I - F_n)X_n^F = b_n^F,$$

where, for $i \in J_n$ and $j \in J_n$,

$$F_n(i, j) := \sum_{k=1}^m \int_a^b H_k(s_i, t) L_j(t) dt, \\ b_n^F(i) := \sum_{k=1}^m \int_a^b H_k(s_i, t) f(t) dt.$$

Second septic spline degenerate kernel method

Next, we consider the following right degenerate kernel operator

$$T_n^R u(s) := \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_j) \int_a^b L_j(t) u(t) dt.$$

We approximate the given continuous functions $t \mapsto H_k(s, t)$ by a septic spline quasi-interpolant:

$$H_{k_n}^R(s, t) = \sum_{j \in J_n} H_k(s, t_j) L_j(t),$$

and the approximate solution satisfies

$$u_n^R(s) - \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_j) \int_a^b L_j(t) u_n^R(t) dt = f(s).$$

Therefore, u_n^R is of the form:

$$u_n^R(s) = f(s) + \sum_{i \in J_n} r_i \sum_{k=1}^m H_{k_i}^R(s) \quad \text{with} \\ H_{k_i}^R(s) := H_k(s, t_i),$$

for some scalars r_j . Hence

$$r_i - \sum_{j \in J_n} r_j \sum_{k=1}^m \int_a^b L_j(t) H_{k_i}^R(t) dt = \int_a^b L_i(t) f(t) dt, \quad i \in J_n.$$

The coefficients r_j are obtained by solving the following linear system

$$(I - R_n)X_n^R = b_n^R,$$

where, for $i \in J_n$ and $j \in J_n$,

$$R_n(i, j) := \sum_{k=1}^m \int_a^b L_j(t) H_{k_i}^R(t) dt, \\ b_n^R(i) := \int_a^b L_i(t) f(t) dt.$$

Third septic spline degenerate kernel method

Finally, we approximate the given continuous functions $s \mapsto H_k(s, t)$ by the septic spline quasi-interpolant using quasi-Lagrange form:

$$H_k^{RF}(s, t) = \sum_{j \in J_n} \sum_{i \in J_n} H_k(s_j, t_i) L_j(s) L_i(t).$$

The third degenerate kernel operator is defined by

$$T_n^{RF} u(s) := H_k^{RF}(s, t) = \sum_{j \in J_n} \sum_{i \in J_n} L_j(s) \sum_{k=1}^m H_k(s_j, t_i) \int_a^b L_i(t) u(t) dt.$$

Approximating the Eq. (1) by

$$u_n^{RF}(s) - \sum_{j \in J_n} \sum_{i \in J_n} L_j(s) \sum_{k=1}^m H_k(s_j, t_i) \int_a^b L_i(t) u_n^{RF}(t) dt = f(s), \quad (3)$$

the approximate solution u_n^{RF} of Eq. (3) is given by

$$u_n^{RF}(s) = f(s) + \sum_{j \in J_n} c_j L_j(s),$$

for some scalars c_j .

As a result, we obtain the linear system

$$\sum_{j \in J_n} c_j L_j(s) - \sum_{j \in J_n} \sum_{i \in J_n} L_j(s) \sum_{k=1}^m H_k(s_j, t_i) \int_a^b L_i(t) [f(t) + \sum_{l \in J_n} c_l L_l(t)] dt = 0.$$

Hence

$$c_j - \sum_{i \in J_n} \sum_{l \in J_n} \sum_{k=1}^m c_l H_k(s_j, t_i) \int_a^b L_i(t) L_l(t) dt \\ = \sum_{i \in J_n} \sum_{k=1}^m H_k(s_j, t_i) \int_a^b L_i(t) f(t) dt,$$

that is to say, the coefficients c_j are obtained by solving the following linear system

$$(I - A_n^{\text{RF}})X_n^{\text{RF}} = b_n^{\text{RF}},$$

where, for $j \in J_n$ and $i \in J_n$,

$$A_n^{\text{RF}}(j, i) := \sum_{l \in J_n} \sum_{k=1}^m H_k(s_j, t_i) \int_a^b L_i(t) L_l(t) dt, \\ b_n^{\text{RF}}(j) := \sum_{i \in J_n} \sum_{k=1}^m H_k(s_j, t_i) \int_a^b L_i(t) f(t) dt.$$

Error analysis

We will use the following notation:

$$H_k^s(t) = H_k(s, t) = H_k^t(s).$$

The following theorems hold.

Theorem 1 Let $H_k(\cdot, \cdot) \in C^8([a, b] \times [a, b])$, $k = 1 \dots m$. The following estimate holds:

$$\|u - u_n^{\text{F}}\|_{\infty} \leq \alpha h^8 \sum_{k=1}^m \left\| \frac{\partial^8 H_k}{\partial s^8} \right\|_{\infty} \|u\|_{\infty},$$

for some constant α independent of n .

Proof Following [6],

$$\|T - T_n^{\text{F}}\| = \max_{a \leq s \leq b} \sum_{k=1}^m \int_a^b |H_k(s, t) - H_{k_n}^{\text{F}}(s, t)| dt,$$

and

$$\|(I - T_n^{\text{F}})^{-1}\|_{\infty} \leq \frac{\|(I - T)^{-1}\|}{1 - \|(I - T)^{-1}\| \|T - T_n^{\text{F}}\|}.$$

Since T is compact, according to [6], the operator $I - T_n^{\text{F}}$ is invertible for n is large enough, and its inverse is uniformly bounded with respect to n . Then there exists $c_1 > 0$, such as

$$\sup_n \|(I - T_n^{\text{F}})^{-1}\|_{\infty} \leq c_1.$$

Since

$$u - u_n^{\text{F}} = [(I - T)^{-1} - (I - T_n^{\text{F}})^{-1}]f \\ = (I - T_n^{\text{F}})^{-1}[T - T_n^{\text{F}}](I - T)^{-1}f \\ = (I - T_n^{\text{F}})^{-1}[Tu - T_n^{\text{F}}u],$$

and

$$H_{k_n}^{\text{F}}(s, t) = QH_k^t(s).$$

We obtain

$$\|T^{\text{F}}u - T_n^{\text{F}}u\|_{\infty} \leq c_0 h^8 \sum_{k=1}^m \left\| \frac{\partial^8 H_k}{\partial s^8} \right\|_{\infty} \|u\|_{\infty}$$

for some constant c_0 independent of n ,

hence

$$\|u - u_n^{\text{F}}\|_{\infty} \leq c_1 c_0 h^8 \sum_{k=1}^m \left\| \frac{\partial^8 H_k}{\partial s^8} \right\|_{\infty} \|u\|_{\infty}.$$

Letting $\alpha := c_1 c_0$, the desired result is achieved. $\square$

Theorem 2 Let $H_k(\cdot, \cdot) \in C^8([a, b] \times [a, b])$, $k = 1 \dots m$. The following estimate holds:

$$\|u - u_n^{\text{R}}\|_{\infty} \leq \beta h^8 \sum_{k=1}^m \left\| \frac{\partial^8 H_k}{\partial t^8} \right\|_{\infty} \|u\|_{\infty},$$

where β is a constant independent of n .

Proof We have

$$\|T - T_n^{\text{R}}\| = \max_{a \leq s \leq b} \sum_{k=1}^m \int_a^b |H_k(s, t) - H_{k_n}^{\text{R}}(s, t)| dt.$$

The operator $I - T_n^{\text{R}}$ is invertible (see [1]), and its inverse is uniformly bounded with respect to n , that is, there exists a constant c_2 such that

$$\sup_n \|(I - T_n^{\text{R}})^{-1}\|_{\infty} \leq c_2.$$

Note that

$$u - u_n^{\text{R}} = (I - T_n^{\text{R}})^{-1}[Tu - T_n^{\text{R}}u],$$

and

$$H_{k_n}^{\text{R}}(s, t) = QH_k^s(t).$$

It follows that

$$\|T - T_n^{\text{R}}\| \leq h^8 c_3 \sum_{k=1}^m \left\| \frac{\partial^8 H_k}{\partial t^8} \right\|_{\infty} \|u\|_{\infty}$$

for some constant c_3 independent of n ,

hence

$$\|u - u_n^{\text{R}}\|_{\infty} \leq h^8 c_2 c_3 \sum_{k=1}^m \left\| \frac{\partial^8 H_k}{\partial t^8} \right\|_{\infty} \|u\|_{\infty}.$$

Letting $\beta := c_2 c_3$, the desired result is obtained. $\square$

Theorem 3 Let $H_k(\cdot, \cdot) \in C^8([a, b] \times [a, b])$, $k = 1 \dots m$. The following estimate holds:

$$\|u - u_n^{\text{RF}}\|_{\infty} \leq \gamma h^8 \sum_{k=1}^m \left[\left\| \frac{\partial^8 H_k^t}{\partial s^8} \right\|_{\infty} + 10 \left\| \frac{\partial^8 H_k^s}{\partial t^8} \right\|_{\infty} \right] \|u\|_{\infty},$$

where γ is a constant independent of n .

Proof Since

$$\|T - T_n^{\text{RF}}\|_{\infty} \leq \sum_{k=1}^m [\|(I - Q)H_k^t(\cdot)\|_{\infty} + \|Q\| \|(I - Q)H_k^s(\cdot)\|_{\infty}],$$

we obtain

$$\|T - T_n^{\text{RF}}\|_{\infty} \leq h^8 \sum_{k=1}^m [c_4 \left\| \frac{\partial^8 H_k}{\partial s^8} \right\|_{\infty} + 10c_5 \left\| \frac{\partial^8 H_k}{\partial t^8} \right\|_{\infty}].$$

On the other hand,

$$u - u_n^{\text{RF}} = (I - T_n^{\text{RF}})^{-1} [Tu - T_n^{\text{RF}}u],$$

and

$$\|(I - T_n^{\text{RF}})^{-1}\| \leq c_6 \quad \text{for } n \text{ sufficiently large.}$$

Letting $\gamma := c_6 \max c_4 c_5$, the desired result is obtained. $\square$

Numerical examples

In Examples 1 and 2, we compare the results we obtained with previous results presented in [3]. For a high accuracy, we raised the degree to seven, while the mentioned paper worked on degree four. Examples 1 and 2 show the accuracy of our results vis-a-vis of [3]. Denote by R_n^F , R_n^R and R_n^{RF} error terms for the above three septic spline degenerate kernel method, respectively. We compare our methods with other methods such as discrete Galerkin methods and discrete collocation methods given in [8], Nyström methods given in [5], Iteration methods given in [4] and Petrov–

Galerkin elements via Chebyshev polynomials described in [2].

Example 1

We consider the following Fredholm integral equation

$$u(s) - \frac{1}{2} \int_0^1 (s+1)e^{-st}u(t)dt = e^{-s} - \frac{1}{2} + \frac{1}{2}e^{-(s+1)}, \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = e^{-s}$. We present in Table 8 the corresponding absolute errors R_n^F , R_n^R and R_n^{RF} respectively for this example. We compare our results with the results given in [3].

Example 2

Consider the following Fredholm integral equation

$$u(s) - \int_0^{\frac{\pi}{2}} \sin(s) \cos(t)u(t)dt = \sin s, \quad 0 \leq s \leq \frac{\pi}{2}.$$

The exact solution $u(s) = 2 \sin(s)$. We present in Table 9 the numerical results for this example.

Example 3

Consider the following Fredholm integral equation

$$u(s) + 2 \int_0^1 e^{s-t}u(t)dt = 2se^s, \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = 2e^s(s - \frac{1}{3})$. We present in Table 10 the numerical results for this example.

Table 10 Example 3

n	R_n^F	R_n^R	R_n^{RF}
8	1.0587e − 9	1.2708e − 10	1.1810e − 9
16	5.9771e − 12	1.6609e − 13	5.8168e − 12
32	2.7534e − 14	1.7764e − 15	2.2871e − 14

Table 8 Example 1

n	R_n^F	R_n^F in [3]	R_n^R	R_n^R in [3]	R_n^{RF}
8	1.2978e − 10	4.5–8	4.2236e − 11	1.2e − 8	1.2533e − 10
16	5.0865e − 13	1.6–9	1.4638e − 13	2.6e − 10	5.2422e − 13
32	9.8810e − 15	4.8e − 11	5.2736e − 15	4.6e − 12	2.0817e − 15

Table 9 Example 2

n	R_n^F	R_n^F in [3]	R_n^R	R_n^R in [3]	R_n^{RF}
8	3.0468e − 008	2.7–6	4.2236e − 11	9.5e − 7	3.4793e − 8
16	1.0291e − 10	8.7e − 8	5.4732e − 11	9.5e − 8	1.1094e − 10
32	3.8120e − 13	2.7e − 9	2.9407e − 13	2.8e − 10	5.8328e − 13

Table 11 Example 4

n	R_n^F	R_n^R	R_n^{RF}	Galerkin method in [8]	Collocation method in [8]
8	$2.8380e-11$	$5.2729e-11$	$7.1493e-11$	$7.893595e-05$	$6.855828e-05$
16	$8.5876e-14$	$2.4126e-13$	$3.2456e-13$	$9.778065e-06$	$8.364251e-06$

Table 12 Example 5

n	R_n^F	R_n^R	R_n^{RF}	E_1 in [4]	E_2 in [4]
8	$1.8865e-8$	$7.0747e-10$	$1.9520e-08$	$3.05e-02$	$7.02e-05$
16	$5.5321e-11$	$1.9194e-12$	$5.3475e-11$	$7.99e-03$	$4.38e-06$
32	$1.9158e-13$	$1.4211e-14$	$2.0393e-13$	$2.05e-03$	$2.74e-07$

Table 13 Example 6

n	R_n^F	R_n^R	R_n^{RF}	E_1 in [2]
10	$4.8528e-10$	$2.0806e-07$	$2.0806e-07$	$6.09910e-03$

Example 4 (cf.[8])

We consider the following Fredholm integral equation

$$u(s) - \frac{1}{2} \int_0^1 e^{st} u(t) dt = f(s), \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = e^{-s} \cos s$. We present in Table 11 the corresponding absolute errors R_n^F , R_n^R and R_n^{RF} respectively for this example. We compare our results with the results of discrete Galerkin methods and discrete collocation methods, respectively, given in [8].

Example 5 (cf.[4])

We consider the following Fredholm integral equation

$$u(s) - \int_0^1 s e^{s-t} u(t) dt = s e^s, \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = 2s e^s$. We present in Table 12 the corresponding absolute errors R_n^F , R_n^R and R_n^{RF} respectively for this example. We compare our results with the results given in [4] using iteration methods based on the classical continuous piecewise linear and quadratic Lagrange interpolants. We give the corresponding absolute errors E_1 , E_2 , respectively (Table 12).

Example 6 (cf.[2])

We consider the following Fredholm integral equation

$$u(s) - \int_0^1 \frac{e^s \sin s}{1+t^2} u(t) dt = f(s), \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = s^3$. We present in Table 13 the corresponding absolute errors R_n^F , R_n^R and R_n^{RF} respectively

for this example. We compare our results with the results of Petrov–Galerkin elements via Chebyshev polynomials described in [2], for $k = 1$ and $n = 10$.

Conclusions

In this paper, we present three degenerate kernel methods to numerically solve generalized Fredholm integral equation of the second kind, working on the septic spline quasi-interpolants. These methods are constructed to approach the kernel of the correspondent integral operator. While the first method is an approximation on the left, the second is on right. The last method, nevertheless, combines the former two methods. The strength of this combination lies in the reduction of integrals and calculations in the linear system.

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