On weighted Iyengar-type inequalities for conformable fractional integrals

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Abstract In this study, we establish some new weighted Iyengar-type integral inequalities using fractional Steffensen's inequality for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works.

Keywords Trapezoid inequality · Iyengar inequality · Weighted quadrature rule · Conformable fractional integral

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Introduction

The presence of a differentiable mappings was introduced in 1938 for the first time and the results were announced by Iyengar. In recent years, Iyengar-type integral inequalities have been studied by several authors such as [5, 6, 9, 10, 12]. In this paper, we concentrate on Iyengar-type integral inequalities and present some generalizations. Section 2 consists of conformable integrals, derivatives, and some properties of them. Steffensen's inequality for

conformable fractional integral is also given. Finally, in Sect. 3, by means of Steffensen's inequality, we obtain some main interesting results. We first establish two new identities for conformable integral involving some moments. Generalizing this, using these equalities and considering fractional Steffensen's inequality for conformable fractional integrals, we get a new version of weighted Iyengar-type integral inequalities for conformable integrals.

The results presented here will provide generalizations of those given in earlier works. Iyengar proved a useful inequality which gives an upper bound for the approximation of the integral average by the mean of the values of mapping at end points of the interval that is given below (see [7] or [11, p.471]).

Theorem 1.1 *Let f be a differentiable function on (a, b) and assume that there is a constant $M > 0$, such that $|f'(x)| \leq M$, $\forall x \in (a, b)$. Then, we have*

$$\left| \int_a^b f(x) dx - (b-a) \frac{f(a)+f(b)}{2} \right| \leq \frac{M(b-a)^2}{4} - \frac{1}{4M} (f(b) - f(a))^2. \quad (1.1)$$

In [3], Agarwal and Dragomir proved the following generalization of Theorem 1.1.

Theorem 1.2 *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° (the interior of I), $[a, b] \subset I^\circ$ with $a < b$. Assume that $M = \sup_{x \in [a, b]} f'(x) < \infty$ and $m = \inf_{x \in [a, b]} f'(x) > -\infty$. If $M > m$ and f' is integrable on $[a, b]$, then the following inequality holds:*

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$$\left| \int_a^b f(x) dx - (b-a) \frac{f(a) + f(b)}{2} \right| \leq \frac{[M(b-a) - f(b) + f(a)][f(b) - f(a) - m(b-a)]}{2(M-m)}.$$

Using a classical Steffensen's inequality, Liu proved a generalization of the above result involving weighted integrals in terms of bounds involving the first derivative of the function.

The following Lemmas and Theorem was given by him in [9] (or [5]).

Lemma 1.1 Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $[a, b] \subset I^\circ$ with $M = \sup_{x \in [a, b]} f'(x) < \infty$, $m = \inf_{x \in [a, b]} f'(x) > -\infty$, and $M > m$. Assume that $w(x) \geq 0$ for all $x \in [a, b]$ $\mu = \int_a^b w(x) dx < \infty$ and $v = \int_a^b xw(x) dx < \infty$ be the zeroth and first moments of $w(\cdot)$ on $[a, b]$. If f' is integrable on $[a, b]$, then the following inequality holds:

$$[M - m][Q_b - (b - \lambda)P_b] \leq \int_a^b w(x)f(x)dx - \mu(f(a) - ma) - mv \leq [M - m][Q_a - (\lambda + a)P_a + \lambda\mu]. \quad (1.2)$$

where $P_a = \int_a^{a+\lambda} w(x)dx$, $Q_b = \int_{b-\lambda}^b w(x)dx$ and $\lambda = \frac{b-a}{M-m} \left(\frac{f(b)-f(a)}{b-a} - m \right)$.

Lemma 1.2 Let the conditions be as in Lemma 1.1, then the following inequality holds:

$$[M - m][Q_b - (\lambda - b)P_b - \lambda\mu] \leq \int_a^b w(x)f(x)dx - \mu(f(b) - mb) - mv \leq [M - m][Q_a - (\lambda + a)P_a]. \quad (1.3)$$

Let the condition of Lemma 1.1 and 1.2 be maintained. Then, the following inequality holds:

$$[M - m] \left[Q_b - (b - \lambda)P_b - \frac{\lambda}{2}\mu \right] \leq \int_a^b w(x)f(x)dx - \frac{\mu}{2}(f(a) + f(b) - m(a + b)) - mv \leq [M - m] \left[Q_a - (\lambda + a)P_a + \frac{\lambda}{2}\mu \right]. \quad (1.4)$$

Now, we will introduce the conformable integral and derivative:

Definition and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in (see, [1, 2, 4, 8]).

Definition 2.1 (Conformable fractional derivative) Given a function $f : [0, \infty) \rightarrow \mathbb{R}$. Then, the “conformable fractional derivative” of f of order α is defined by

$$D_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad (2.1)$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, $\alpha > 0$, $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t). \quad (2.2)$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α . In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α -differentiable.

Theorem 2.1 Let $\alpha \in (0, 1]$ and f and g be α -differentiable at a point $t > 0$. Then

- (i) $D_\alpha(af + bg) = aD_\alpha(f) + bD_\alpha(g)$, for all $a, b \in \mathbb{R}$,
- (ii) $D_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$,
- (iii) $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$,
- (iv) $D_\alpha\left(\frac{f}{g}\right) = \frac{fD_\alpha(g) - gD_\alpha(f)}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \quad (2.3)$$

Definition 2.2 (Conformable fractional integral) Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \rightarrow \mathbb{R}$ is α -fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(x) d_\alpha x := \int_a^b f(x) x^{\alpha-1} dx \quad (2.4)$$

exists and is finite. All α -fractional integrable on $[a, b]$ is indicated by $L_\alpha^1([a, b])$.

Remark 2.1

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Theorem 2.2 Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$, we have

$$I_\alpha^\alpha D_\alpha^\alpha f(t) = f(t) - f(a). \quad (2.5)$$

Theorem 2.3 (Integration by parts) *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two functions, such that fg is differentiable. Then*

$$\int_a^b f(x) D_\alpha^\alpha (g)(x) d_\alpha x = fg|_a^b - \int_a^b g(x) D_\alpha^\alpha (f)(x) d_\alpha x. \quad (2.6)$$

The following theorem gave by Anderson [4].

Theorem 2.4 (Fractional Steffensen's inequality) *Let $\alpha \in (0, 1]$ and a and b be real numbers, such that $0 \leq a < b$. Let $h : [a, b] \rightarrow [0, \infty)$ and $g : [a, b] \rightarrow [0, 1]$ be α -fractional integrable functions on $[a, b]$ with h is decreasing. Then*

$$\int_{b-\ell}^b h(x) d_\alpha x \leq \int_a^b h(x) g(x) d_\alpha x \leq \int_a^{a+\ell} h(x) d_\alpha x, \quad (2.7)$$

where $\ell := \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(x) d_\alpha x$.

Weighted trapezoidal inequality for conformable fractional integrals

Some definitions are required to simplify the subsequent work.

Definition 3.1 Let $w(x)$ be a positive conformable integrable function on $[a, b]$ and $\alpha \in (0, 1]$. Let μ and ν be its zeroth and first moments about zero, so that

$$\mu_\alpha = \frac{1}{\alpha} \int_a^b w(x) d_\alpha x < \infty, \quad (3.1)$$

and

$$\nu_\alpha = \frac{1}{\alpha} \int_a^b x^\alpha w(x) d_\alpha x < \infty. \quad (3.2)$$

Definition 3.2 P and Q will be used to denote the zeroth and first moments of $w(x)$ over a subinterval $[a, b]$. In particular, for $\ell > 0$, the subscript a and b will be used to indicate the intervals $[a, a + \ell]$ and $[b - \ell, b]$, respectively. Thus, for example

$$P_\alpha(a, \ell) := \frac{1}{\alpha} \int_a^{a+\ell} w(x) d_\alpha x, \quad (3.3)$$

$$Q_\alpha(a, \ell) := \frac{1}{\alpha} \int_a^{a+\ell} x^\alpha w(x) d_\alpha x, \quad (3.4)$$

$$P_\alpha(\ell, b) := \frac{1}{\alpha} \int_{b-\ell}^b w(x) d_\alpha x, \quad (3.5)$$

and

$$Q_\alpha(\ell, b) := \frac{1}{\alpha} \int_{b-\ell}^b x^\alpha w(x) d_\alpha x. \quad (3.6)$$

Fractional Steffensen's inequality (2.7) will now be used to obtain inequalities for conformable fractional integrals to give weighted trapezoidal-type quadrature rules. First, the following lemmas will need to be proved by the use of the Fractional Steffensen's inequality.

Lemma 3.1 *Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{R}$ with $0 \leq a < b$, $f : [a, b] \rightarrow \mathbb{R}$ be a conformable differentiable with $M = \sup_{x \in [a, b]} D_\alpha f(x) < \infty$, $m = \inf_{x \in [a, b]} D_\alpha f(x) > -\infty$ and $M > m$. Assume that $w(x) \geq 0$ for all $x \in [a, b]$ and $\nu = \frac{1}{\alpha} \int_a^b x^\alpha w(x) d_\alpha x < \infty$ be the first moment of $w(\cdot)$ on $[a, b]$. If $D_\alpha f$ is α -fractional integrable on $[a, b]$, then the following inequalities hold:*

$$\begin{aligned} & [M - m][Q_\alpha(\ell, b) - (b - \ell)^\alpha P_\alpha(\ell, b)] \\ & \leq \int_a^b w(x) f(x) d_\alpha x - \mu_\alpha (\alpha f(a) - m a^\alpha) - m \nu_\alpha \\ & \leq [M - m][Q_\alpha(a, \ell) - (a + \ell)^\alpha P_\alpha(a, \ell) + [(a + \ell)^\alpha - a^\alpha] \mu_\alpha], \end{aligned} \quad (3.7)$$

where P_α and Q_α are as described in Definition 3.2.

Proof Let $h_b(x) = \int_x^b w(u) d_\alpha u$ and $g(x) = \frac{D_\alpha f(x) - m}{M - m}$. Then, from inequality (2.7), we get

$$L_b \leq I_b \leq U_b, \quad (3.8)$$

where

$$L_b = (M - m) \int_{b-\ell}^b h_b(x) d_\alpha x, \quad (3.9)$$

$$I_b = \int_a^b h_b(x) (D_\alpha f(x) - m) d_\alpha x, \quad (3.10)$$

and

$$U_b = (M - m) \int_a^{a+\ell} h_b(x) d_\alpha x. \quad (3.11)$$

Now, an integration by parts and change of order formula gives

$$\begin{aligned}
I_b &= \int_a^b \left(\int_x^b w(u) d_\alpha u \right) D_\alpha f(x) d_\alpha x \\
&\quad - m \int_a^b \left(\int_x^b w(u) d_\alpha u \right) d_\alpha x \\
&= \left(\int_x^b w(u) d_\alpha u \right) f(x) \Big|_a^b + \int_a^b w(x) f(x) d_\alpha x \\
&\quad - m \int_a^b w(u) \left(\int_a^u d_\alpha u \right) \\
&= -\mu_\alpha(\alpha f(a) - ma^\alpha) - mv_\alpha + \int_a^b w(x) f(x) d_\alpha x. \quad (3.12)
\end{aligned}$$

In addition

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)} \int_a^b (D_\alpha f(x) - m) d_\alpha x. \quad (3.13)$$

It should be noted that $0 < \ell \leq b - a$. $\square$

For the lower bound L_b , a change of order of integration gives

$$\begin{aligned}
\frac{L_b}{M-m} &= \int_{b-\ell}^b \left(\int_x^b w(u) d_\alpha u \right) d_\alpha x \\
&= Q_\alpha(\ell, b) - (b-\ell)^\alpha P_\alpha(\ell, b), \quad (3.14)
\end{aligned}$$

where $P_\alpha(\ell, b)$ and $Q_\alpha(\ell, b)$ are described in Definition 3.2.

Similarly, the upper bound U_b may be obtained through a change of order of integration to give

$$\begin{aligned}
\frac{U_b}{M-m} &= \int_a^{a+\ell} \left(\int_x^b w(u) d_\alpha u \right) d_\alpha x \\
&= \int_a^b \int_x^b w(u) d_\alpha u d_\alpha x - \int_{a+\ell}^b \int_x^b w(u) d_\alpha u d_\alpha x \\
&= \int_a^b w(u) \left(\int_a^u d_\alpha x \right) d_\alpha u \\
&\quad - \int_{a+\ell}^b w(u) \left(\int_{a+\ell}^u d_\alpha x \right) d_\alpha u \\
&= Q_\alpha(a, \ell) - (a+\ell)^\alpha P_\alpha(a, \ell) + [(a+\ell)^\alpha - a^\alpha] \frac{\mu_\alpha}{2}, \quad (3.15)
\end{aligned}$$

where $P_\alpha(a, \ell)$ and $Q_\alpha(a, \ell)$ are described in Definition 3.2, and v_α is the zeroth moment of $w(x)$ on $[a, b]$.

Using (3.8)–(3.15), the lemma is thus proved. $\square$

Lemma 3.2 Let the conditions be as in Lemma 3.1, then the following inequalities hold:

$$\begin{aligned}
&[M-m][Q_\alpha(\ell, b) - (b-\ell)^\alpha P_\alpha(\ell, b) + [(b-\ell)^\alpha - b^\alpha] \mu_\alpha] \\
&\leq \int_a^b w(x) f(x) d_\alpha x - \mu_\alpha(\alpha f(b) - mb^\alpha) - mv_\alpha \\
&\leq [M-m][Q_\alpha(a, \ell) - (a+\ell)^\alpha P_\alpha(a, \ell)]. \quad (3.16)
\end{aligned}$$

where P_α and Q_α are as described in Definition 3.2.

Proof The proof follows along similar lines to that of Lemma 3.1. Let $h_a(x) = -\int_a^x w(u) d_\alpha u$ and $g(x) = \frac{D_\alpha f(x) - m}{M-m}$. Then, from inequality (2.7)

$$L_a \leq I_a \leq U_a, \quad (3.17)$$

where

$$I_a = \int_a^b h_a(x) (D_\alpha f(x) - m) d_\alpha x,$$

$$L_a = (M-m) \int_{b-\ell}^b h_a(x) d_\alpha x$$

and

$$U_a = (M-m) \int_a^{a+\ell} h_a(x) d_\alpha x.$$

Now, a straight forward integration by parts yields

$$\begin{aligned}
I_a &= \int_a^b h_a(x) (D_\alpha f(x) - m) d_\alpha x \\
&= \int_a^b w(x) f(x) d_\alpha x - \mu_\alpha(\alpha f(b) - mb^\alpha) - mv_\alpha. \quad (3.18)
\end{aligned}$$

Furthermore, an interchange of the order of integration and simplification of results yields

$$\begin{aligned}
L_a &= (M-m) \int_{b-\ell}^b h_a(x) d_\alpha x \\
&= (M-m)[Q_\alpha(\ell, b) - (b-\ell)^\alpha P_\alpha(\ell, b) + [(b-\ell)^\alpha - b^\alpha] \mu_\alpha], \quad (3.19)
\end{aligned}$$

and

$$U_a = (M-m) \int_a^{a+\ell} h_a(x) d_\alpha x = Q_\alpha(a, \ell) - (a+\ell)^\alpha P_\alpha(a, \ell). \quad (3.20)$$

Hence, using (3.17)–(3.20), the lemma is thus proved. $\square$

Theorem 3.1 *Let the condition of Lemmas 3.1 and 3.2 be maintained. Then, the following inequality holds:*

$$\begin{aligned} & [M - m] \left[Q_\alpha(\ell, b) - (b - \ell)^\alpha P_\alpha(\ell, b) + [(b - \ell)^\alpha - b^\alpha] \frac{\mu_\alpha}{2} \right] \\ & \leq \int_a^b w(x) f(x) d_\alpha x - \mu_\alpha \left[\alpha \frac{f(a) + f(b)}{2} - m \frac{a^\alpha + b^\alpha}{2} \right] - mv_\alpha \\ & \leq [M - m] \left[Q_\alpha(a, \ell) - (a + \ell)^\alpha P_\alpha(a, \ell) + [(a + \ell)^\alpha - a^\alpha] \frac{\mu_\alpha}{2} \right], \end{aligned} \quad (3.21)$$

where the P_α s and Q_α s are as defined in Definition 3.2.

Proof Adding (3.7) and (3.16), and resulting inequalities divided by 2. This completes the proof. $\square$

Corollary 3.1 *Let the conditions be as in the previous lemmas and theorem of this section. Then*

$$\begin{aligned} & \left| \int_a^b w(x) f(x) d_\alpha x - \mu_\alpha \left[\alpha \frac{f(a) + f(b)}{2} - m \frac{a^\alpha + b^\alpha}{2} \right] - mv_\alpha \right| \\ & \leq \frac{b^\alpha - a^\alpha}{2} \mu_\alpha. \end{aligned}$$

Proof The corollary follows readily from (3.21) on noting that:

$$\begin{aligned} Q_\alpha(\ell, b) &= \frac{1}{\alpha} \int_{b-\ell}^b x^\alpha w(x) d_\alpha x \geq \frac{(b - \ell)^\alpha}{\alpha} \int_{b-\ell}^b w(x) d_\alpha x, \\ Q_\alpha(a, \ell) &= \frac{1}{\alpha} \int_a^{a+\ell} x^\alpha w(x) d_\alpha x \leq \frac{(a + \ell)^\alpha}{\alpha} \int_a^{a+\ell} w(x) d_\alpha x, \end{aligned}$$

and

$$\begin{aligned} (b - \ell)^\alpha - b^\alpha &\geq a^\alpha - b^\alpha, \quad a < b - \ell < b, \\ (a + \ell)^\alpha - a^\alpha &\leq b^\alpha - a^\alpha, \quad a < a + \ell < b. \end{aligned}$$

Hence, this completes the proof. $\square$

Remark 3.1 If we take $\alpha = 1$ in Lemma 3.1, then inequality (3.7) reduces inequality (1.2).

Remark 3.2 If we take $\alpha = 1$ in Lemma 3.2, then inequality (3.16) reduces inequality (1.3).

Remark 3.3 If we take $\alpha = 1$ in Theorem 3.1, then inequality (3.21) reduces inequality (1.4).

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