Error estimate of the MQ-RBF collocation method for fractional differential equations with Caputo–Fabrizio derivative

B. Fakhr Kazemi¹ · H. Jafari¹

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Abstract A collocation method based on multiquadric radial basis functions is proposed for numerical solution of fractional differential equations. The fractional derivative is sense of Caputo–Fabrizio derivative. An efficient error bound of the method is also introduced in the L_2 norm, using properties of native spaces. We test this approach for two examples. The obtained numerical results confirm theoretical prediction of the convergence of this method.

Keywords Fractional differential equations · Caputo–Fabrizio derivative · Multiquadric radial basis functions · Native spaces

Introduction

Over the last several decades, radial basis functions (RBFs) were known as a powerful tool for the scattered data interpolation problems. The multiquadric (MQ) RBF interpolation method was introduced by Hardy in 1971 [11] and the method of meshless radial basis functions was introduced by Kansa for solving differential equations in 1990 [14, 15]. More recently, the RBF methods have been used to obtain numerical solution of a large type of ordinary and partial differential equations [8, 12, 23].

In the last decades, fractional differential equations have been demonstrated applications in the several fields of

mathematics, physics, and out of them [6, 13, 19] and the several methods have been proposed to solve them. There are various definitions of derivative with fractional order, such as Riemann–Liouville, Caputo, etc. [16, 20]. Caputo and Fabrizio have suggested a new definition of fractional derivative which, contrary to the old definition, has no singularity [5].

In this paper, the collocation method based on MQ-RBFs is applied to solve the Caputo–Fabrizio fractional differential equations. The first author and Ghoreishi have applied this method for fractional differential equations with Caputo derivative [7].

Definition 1 (From [5]) Assume that $u \in H^1(a, b)$, $a < b$, the Caputo–Fabrizio derivative with fractional order $0 \leq \alpha \leq 1$ of the function $u(x)$ is defined by

$${}^{\text{CF}}D^\alpha u(x) = \frac{M(\alpha)}{1-\alpha} \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) u'(t) dt, \quad (1)$$

where $M(\alpha)$ is a normalization function, such that $M(0) = M(1) = 1$.

Losada and Nieto proposed the fractional integral of Caputo–Fabrizio type as follows (see [18]).

Definition 2 Let $0 < \alpha < 1$. The fractional integral of order α of a function u is defined by

$${}^{\text{CF}}I^\alpha u(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(x) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^x u(t) dt. \quad (2)$$

Due to the above definition, the fractional integral of order α is an average between of the function and its

✉ H. Jafari
jafari@umz.ac.ir

B. Fakhr Kazemi
b.fakhrkazemi@umz.ac.ir

¹ Department of Mathematics, University of Mazandaran, Babolsar, Iran

integral of order one. Using this idea, they obtain the normalized function to be

$$M(\alpha) = \frac{2}{2-\alpha}, \quad 0 \leq \alpha \leq 1.$$

Therefore, the Caputo–Fabrizio fractional derivative of order $0 < \alpha < 1$ was reformulated by Losada and Nieto as

$${}^{\text{CF}}D^\alpha u(x) = \frac{1}{1-\alpha} \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) u'(t) dt. \quad (3)$$

The remainder this paper is organized as follows. In Sect. 2, we present the necessary definitions and some necessary facts about RBFs and some other concepts. In Sect. 3, we introduce MQ-RBF collocation method for numerical solution of the Caputo–Fabrizio fractional differential equations. Error estimate of the proposed method was devoted in Sect. 4. Finally, numerical examples are carried out to verify the theoretical results in Sect. 5.

Basic definitions

Let φ be a continuous function on $\mathbb{R}^d$ which is conditionally positive definite of order m . Given a set of N distinct data points $\mathcal{X} = \{x_j\}_{j=1}^N$ corresponding data value $\{g_j\}_{j=1}^N$, the RBF interpolant is defined by

$$\mathcal{P}g(x) = \sum_{j=1}^N \lambda_j \varphi(\|x - x_j\|) + \sum_{i=1}^\eta \lambda'_i \psi_i(x), \quad (4)$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$, $\{\psi_i\}_{i=1}^\eta$ is a basis of the polynomial space $\mathbb{P}_d^{m-1}$, that is, all polynomials of total degree less than m in d variables and $\eta = \dim \Pi_m^d = \binom{d+m-1}{m-1}$.

The expansion coefficients $\{\lambda_j\}_{j=1}^N$ and $\{\lambda'_i\}_{i=1}^\eta$ are then determined from the interpolation conditions and the further conditions:

$$\sum_{j=1}^N \lambda_j \psi_i(x_j) = 0, \quad 1 \leq i \leq \eta, \quad (5)$$

which leads to the following linear system:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda' \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}, \quad (6)$$

where $A_{N \times N}$ and $B_{N \times \eta}$ are matrices with the elements $A_{ji} = \varphi(x_j - x_i)$, $i, j = 1, \dots, N$ and $B_{ji} = \psi_i(x_j)$, $j = 1, \dots, N$, $i = 1, \dots, \eta$, respectively. Furthermore, λ and λ' are the vectors of coefficients of $\mathcal{P}g(x)$ and the components of g are the data $g(x_j)$, $j = 1, \dots, N$.

It is worth mentioning that system (6) has a unique solution when $\mathcal{X}$ contains a $\mathbb{P}_d^{m-1}$ -unisolvant subset. We call a set of points $\mathcal{X} \subset \mathbb{R}^d$ $\mathbb{P}_d^{m-1}$ -unisolvant if the only polynomial of total degree at most $m-1$ interpolating zero data on $\mathcal{X}$ is the zero polynomial. A complete treatment of this topic can be seen in [3, 9].

We will use multiquadric radial basis functions for the numerical scheme introduced in Sect. 3. The multiquadric RBFs can be defined as

$$\varphi(x) = (-1)^{\lceil m-d/2 \rceil} (x^2 + a^2)^{m-d/2}, \quad d \text{ odd}, \quad (7)$$

where $\lceil m-d/2 \rceil$ indicates the smallest integer greater than $m-d/2$. The MQ-RBFs in (7) were named one of the shifted surface spline functions by Yoon [22].

We need the native and Sobolev spaces to discuss about error estimates of RBF interpolation.

Each radial basis function $\varphi(x)$ induces a native space denoted by $\mathcal{F}_\varphi$ which is a semi-Hilbert space for $m > 0$.

Definition 3 Consider the Fourier transform of $f \in L_1(\mathbb{R}^d)$ as

$$\hat{f}(\theta) = \int_{\mathbb{R}^d} f(t) \exp(-i\theta t) dt.$$

For the given basis function φ , there arises a function space

$$\mathcal{F}_\varphi = \left\{ g : |g|_\varphi^2 = \int_{\mathbb{R}^d} \frac{|\hat{g}(\theta)|^2}{\hat{\varphi}(\theta)} d\theta < \infty \right\},$$

which is called the “native” space for φ . This function space $\mathcal{F}_\varphi$ is equipped with the semi-inner product

$$(f, g)_\varphi = \int_{\mathbb{R}^d} \hat{f}(\theta) \bar{\hat{g}}(\theta) \hat{\varphi}^{-1} d\theta.$$

The definition and relevant properties of the native space can be found in [3, 9].

Now, we recall some basic definitions about the Sobolev spaces. Assume that Ω is an open domain in $\mathbb{R}^d$, $\rho = (\rho_1, \dots, \rho_d)$ is a d -tuple of non-negative integers and $|\rho| = \sum_{k=1}^d \rho_k$. We set

$$D^\rho v = \frac{\partial^{|\rho|} v}{\partial x_1^{\rho_1} \dots \partial x_d^{\rho_d}},$$

and consider the following definition from [1].

Definition 4 Assume that $1 \leq p \leq \infty$ and k be a non-negative integer, the Sobolev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) = \{f \in L_p(\Omega) : \text{for each non-negative multi-index } \rho \text{ with } |\rho| \leq k, D^\rho f \in L_p(\Omega)\}.$$

It is equipped with the following norm:

$$\|f\|_{k,L_p(\Omega)} = \begin{cases} \left(\sum_{|\rho| \leq k} \|D^\rho f\|_{L_p(\Omega)}^p \right)^{1/p}, & 1 \leq p < \infty, \\ \sum_{|\rho| \leq k} \|D^\rho f\|_{L_\infty(\Omega)}, & p = \infty. \end{cases}$$

The Sobolev spaces W_p^k form a hierarchy of spaces for $d = 1$, in the sense that $W_p^k(\Omega) \subset W_p^{k-1}(\Omega) \subset \dots \subset W_p^0(\Omega) = L_p(\Omega)$.

In the special case $p = 2$, $W_2^k(\Omega)$ is denoted by $H^k(\Omega)$. This space is a Hilbert space for the inner product:

$$\langle f, g \rangle_{k,\Omega} = \sum_{|\rho| \leq k} \langle D^\rho f, D^\rho g \rangle_{L_2(\Omega)} = \sum_{|\rho| \leq k} \int_\Omega D^\rho f D^\rho g d\mu.$$

The k th order L_p -Sobolev semi-norm $|f|_{k,L_p(\Omega)}$ is defined by

$$|f|_{k,L_p(\Omega)} = \left(\sum_{|\rho|=k} \|D^\rho f\|_{L_p(\Omega)}^p \right)^{1/p}.$$

Clearly, for $d = 1$

$$|f|_{k,L_p(\Omega)} = \|D^k f\|_{L_p(\Omega)}.$$

MQ-RBF collocation method

Consider the fractional differential equation:

$$\begin{aligned} {}^{\text{CF}}D^\alpha u(x) &= \gamma u(x) + f(x), \quad x \geq 0, \quad \gamma \in \mathbb{R}, \\ u(0) &= u_0. \end{aligned} \quad (8)$$

According to the RBF collocation method, the approximation of the unknown function $u(x)$ may be written as a linear combination of MQ-RBF radial basis functions and the monomial basis for the vector space $\mathbb{P}_1^{m-1}$. Thus

$$\hat{u}(x) = \sum_{j=1}^N \lambda_j (-1)^{\lceil m-1/2 \rceil} ((x - x_j)^2 + a^2)^{m-1/2} + \sum_{i=0}^{m-1} \lambda'_i x^i. \quad (9)$$

Using (9) and collocation points $\{x_k\}_{k=0}^N \subset (0, 1)$, Eq. (3) takes the following form:

$${}^{\text{CF}}D^\alpha(\hat{u})(x_k) = \frac{1}{1-\alpha} \int_0^{x_k} \mathcal{Q}(x_k, t) dt, \quad (10)$$

where

$$\mathcal{Q}(x, t) = \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) \hat{u}'(t). \quad (11)$$

To obtain the integral term in (10), we will transfer the integral interval $[0, x_k]$ to a fixed interval $[-1, 1]$ and then

make use of some appropriate quadrature rules. Therefore, we make a simple linear transformation:

$$t(x, \xi) = \frac{x}{2} \xi + \frac{x}{2}, \quad -1 \leq \xi \leq 1. \quad (12)$$

Then

$$\int_0^{x_k} \mathcal{Q}(x_k, t) dt = \int_{-1}^1 \tilde{\mathcal{Q}}(x_k, \xi) d\xi, \quad 1 \leq k \leq N, \quad (13)$$

where

$$\tilde{\mathcal{Q}}(x, \xi) = \frac{x}{2} \mathcal{Q}(x, t(x, \xi)). \quad (14)$$

Therefore, (10) becomes

$${}^{\text{CF}}D^\alpha \hat{u}(x_k) = \frac{1}{1-\alpha} \int_{-1}^1 \tilde{\mathcal{Q}}(x_k, \xi) d\xi. \quad (15)$$

Using a $(M+1)$ -point Gauss quadrature formula relative to the Legendre weights $\{w_\ell\}_{\ell=1}^M$ gives

$${}^{\text{CF}}D^\alpha \hat{u}(x_k) = \frac{1}{1-\alpha} \sum_{\ell=0}^M \tilde{\mathcal{Q}}(x_k, \xi_\ell) w_\ell, \quad (16)$$

where the set $\{\xi_\ell\}_{\ell=0}^M$ coincides with the Gauss–Legendre nodal points on $[-1, 1]$.

To obtain approximate solution of (8) using the MQ-RBF collocation method, it is necessary that (8) be satisfied exactly by (9) at a set of collocation points $\{x_k\}_{k=1}^N$:

$${}^{\text{CF}}D^\alpha \hat{u}(x_k) = \gamma \hat{u}(x_k) + f(x_k), \quad 1 \leq k \leq N. \quad (17)$$

By inserting (16) into the above equation, the numerical scheme (17) leads to a system of N linear equations in $N + m$ unknowns. Therefore

$$\frac{1}{1-\alpha} \sum_{\ell=0}^M \tilde{\mathcal{Q}}(x_k, \xi_\ell) w_\ell = \gamma \hat{u}(x_k) + f(x_k), \quad 1 \leq k \leq N. \quad (18)$$

To determine the unknown coefficients $\{\lambda_j\}_{j=1}^N$ and $\{\lambda'_i\}_{i=0}^{m-1}$ to the N equations resulting from (18), an extra m equations are required. This is ensured by the m conditions (5) as

$$\sum_{j=1}^N \lambda_j (x_j)^i = 0, \quad 0 \leq i \leq m-1.$$

Error estimate

The aim of this section is to provide an error estimate of the numerical scheme (18), where the MQ-RBF is conditionally positive definite of order $m = 1$, in the other words:

$$\hat{u}(x) = \mathcal{P}u(x) = \sum_{j=1}^N -\lambda_j ((x - x_j)^2 + a^2)^{1/2} + \lambda'_i. \quad (19)$$

First, we state some lemmas and theorems, and then, we prove our main theorem. In this position for subsequent error analysis, we have to assume that $\Omega \subseteq \mathbb{R}^d$ has Lipschitz boundary and the interior cone property. For simplicity, in the sequel, we will assume that $\Omega = (-1, 1)$.

To prove the error estimate for the proposed method, we need the generalized Hardy's inequality and estimates for the interpolation error (see [4, 10]).

Lemma 1 (Generalized Hardy's inequality) *For all measurable functions $f \geq 0$, the following generalized Hardy's inequality*

$$\left(\int_a^b |(Tf)(x)|^q u(x) dx \right)^{1/q} \leq C \left(\int_a^b |f(x)|^p v(x) dx \right)^{1/p},$$

holds if and only if

$$\sup_{a < x < b} \left(\int_x^b u(t) dt \right)^{1/q} \left(\int_a^x v^{1-p'}(t) dt \right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1},$$

for the case $1 < p \leq q < \infty$. Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x, t) f(t) dt,$$

with $k(x, t)$ a given kernel, u, v weight functions, and $-\infty \leq a < b \leq \infty$.

Lemma 2 (Estimates for the interpolation error) *Let $f \in H_q(\Omega)$ with $q \geq 1$, $x_j, 0 \leq j \leq M$, be the Gauss, or the Gauss–Radau, or the Gauss–Lobatto points relative to the Legendre weight $w(x) \equiv 1$ and $I_M f$ denote the polynomial of degree M that interpolates f at one of these sets of points, namely*

$$I_M f = \sum_{j=0}^M f(x_j) \mathcal{L}_j(x),$$

where $\mathcal{L}_j$ is the j th Lagrange basis function. Then

$$\|f - I_M f\|_{L_2(\Omega)} \leq CM^{-q} \|f\|_{H_{q,M}(\Omega)},$$

with $\|f\|_{H_{q,M}(\Omega)} = \left(\sum_{k=\min\{q, M+1\}}^q \left\| \frac{d^k f}{dx^k} \right\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}}$.

In all the estimates contained in this chapter, C will denote a positive constant that depends upon the type of norm involved in the estimate, but which is independent of the function u , the integer N, M , and the diameter of the domain.

Yoon proposed L_p error estimates for the so-called “shifted surface splines” for functions f is the standard Sobolev spaces.

Lemma 3 (From [22]) *Let $\mathcal{P}f(x)$ in (4) be an interpolant to f on $\mathcal{X}$ using the basis function φ in (7) and f be a function in the native space $\mathcal{F}_\varphi$. Then, for every function $f \in W_2^m$, we have*

$$\|f - \mathcal{P}f\|_{m, L_2(\Omega)} \leq \|f - \mathcal{P}f\|_\varphi \leq \|f\|_\varphi,$$

where as before stated, φ is strictly conditionally positive definite of order m .

Theorem 1 (From [22]) *Let $\mathcal{P}f(x)$ in (4) be the interpolant to f using the “shifted” surface spline φ . Assume that the parameter a in the basis function φ in (7) is chosen to be proportional to h . Then, there is a positive constant C , independent of $\mathcal{X}$, such that for every $f \in W_2^m(\Omega) \cap W_\infty^m(\Omega)$, we have an error bound of the form:*

$$\|f - \mathcal{P}f\|_{L_p(\Omega)} \leq Ch^{\gamma_p} \|f\|_{m, L_2(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty,$$

with

$$\gamma_p = \min(m, m - d/2 + d/p),$$

$$h = \sup_{\xi \in \Omega} \min_{x_j \in \mathcal{X}} |x - x_j|. \quad (20)$$

Corollary 1 *A function that is (strictly) conditionally positive definite of order m on $\mathbb{R}^d$ is also (strictly) conditionally positive definite of any higher order; therefore, Lemma 3 and Theorem 1 are satisfied for any $\mu \geq m$.*

Theorem 2 *Let $u \in \mathcal{F}_\varphi$ be the exact solution of the fractional differential equation (8) and assume that the approximate solution $\mathcal{P}u(x)$ is given by (19). If $u \in W_2^3(\Omega) \cap W_\infty^3(\Omega)$, then we have*

$$\begin{aligned} \|u\|_{L_2(\Omega)} &\leq ChM^{-1} (\|u\|_{1, L_2(\mathbb{R})} + \|u\|_{2, L_2(\mathbb{R})} + \|u\|_{3, L_2(\mathbb{R})} + \|u\|_\varphi) \\ &\quad + CM^{-1} (\|u\|_{1, L_2(\Omega)} + \|u\|_{2, L_2(\Omega)} + \|u\|_\varphi) \\ &\quad + Ch(\|u\|_\varphi + \|u\|_{1, L_2(\mathbb{R})}) + C\|u\|_\varphi, \end{aligned} \quad (21)$$

where h is defined in (20).

Proof According to Losada and Nieto [18], Eq. (8) has a unique solution. We can rewrite (8) as

$$u(x) = \mu \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) \tilde{u}'(t) dt - \tilde{f}(x), \quad (22)$$

where $\mu = \frac{1}{\gamma(1-\alpha)}$ and $\tilde{f}(x) = \frac{f(x)}{\gamma}$. Therefore, the numerical solution (22) can be obtained using the MQ-RBF collocation method by (18). From (18) and (13), we have

$$\begin{aligned}\hat{u}(x_k) &= \mu \sum_{\ell=0}^M \tilde{Q}(x_k, \xi_\ell) w_\ell - \mu \int_{-1}^1 \tilde{Q}(x_k, \xi) d\xi \\ &\quad + \mu \int_0^{x_k} \exp\left(\frac{-\alpha(x_k - t)}{1 - \alpha}\right) \hat{u}'(t) dt - \tilde{f}(x_k), \\ 1 &\leq k \leq N,\end{aligned}$$

which gives

$$\begin{aligned}\hat{u}(x_k) &= \mu \int_0^{x_k} \exp\left(\frac{-\alpha(x_k - t)}{1 - \alpha}\right) \hat{u}'(t) dt \\ &\quad - \tilde{f}(x_k) + y(x_k), \quad 1 \leq k \leq N,\end{aligned} \quad (23)$$

where

$$y(x) = \mu \left(\sum_{\ell=0}^M \tilde{Q}(x, \xi_\ell) w_\ell - \int_{-1}^1 \tilde{Q}(x, \xi) d\xi \right).$$

It is worth mentioning that we can substitute $\hat{u}(x_i)$ into $u(x_i)$ using interpolation condition for $1 \leq k \leq N$. The system (23) can be rewritten in matrix form:

$$(u(x_k))_{k=1}^N = \mu (s(x_k))_{k=1}^N - (\tilde{f}(x_k))_{k=1}^N + (y(x_k))_{k=1}^N, \quad (24)$$

where $s(x) = \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) \hat{u}'(t) dt$. According to (6), we add $\eta = m = 1$ zero lines to (24), we obtain

$$\begin{pmatrix} U \\ O \end{pmatrix} = \mu \begin{pmatrix} S \\ O \end{pmatrix} - \begin{pmatrix} \mathcal{F} \\ O \end{pmatrix} + \begin{pmatrix} \mathcal{Y} \\ O \end{pmatrix}, \quad (25)$$

where $U = (u_k)_{k=1}^N$, $S = (s(x_k))_{k=1}^N$, $\mathcal{F} = (\tilde{f}(x_k))_{k=1}^N$, $\mathcal{Y} = (y(x_k))_{k=1}^N$ and $O = (0)_{1 \times 1}$. Multiplying $\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}^{-1}$ $((\phi(x - x_j))_{N \times 1} \quad (\psi_1(x))_{1 \times 1})$ on both sides of (25) yield

$$\hat{u}(x) = \mu \mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) \hat{u}'(t) dt \right) - \mathcal{P}\tilde{f}(x) + \mathcal{P}y(x). \quad (26)$$

Let us suppose that $E(x)$ is the error function, we have

$$E(x) = \hat{u}(x) - u(x).$$

Therefore, (26) becomes

$$\begin{aligned}E(x) + u(x) &= \mu \mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right) \\ &\quad + \mu \mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) u'(t) dt \right) \\ &\quad - \mathcal{P}\tilde{f}(x) + \mathcal{P}y(x).\end{aligned}$$

It follows from (22)

$$\begin{aligned}E(x) &= \mu \mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right) \\ &\quad + \mathcal{P}u(x) + \mathcal{P}\tilde{f}(x) - u(x) - \mathcal{P}\tilde{f}(x) + \mathcal{P}y(x),\end{aligned}$$

which gives

$$\begin{aligned}E(x) &= \mu \mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right) + \mathcal{P}u(x) \\ &\quad - u(x) + \mathcal{P}y(x).\end{aligned} \quad (27)$$

Therefore

$$\begin{aligned}E(x) &= \mu \mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right) \\ &\quad - \mu \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \\ &\quad + \mu \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt + \mathcal{P}u(x) \\ &\quad - u(x) + \mathcal{P}y(x),\end{aligned}$$

We rewrite (28) as follows:

$$E(x) = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4,$$

where

$$\begin{aligned}\mathcal{T}_1 &= \mu \left(\mathcal{P} \left(\int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E' dt \right) \right. \\ &\quad \left. - \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right),\end{aligned}$$

$$\mathcal{T}_2 = \mu \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt,$$

$$\mathcal{T}_3 = \mathcal{P}u(x) - u(x),$$

$$\mathcal{T}_4 = \mathcal{P}y(x),$$

consequently

$$\|E(x)\|_{L_2(\Omega)} \leq \|\mathcal{T}_1\|_{L_2(\Omega)} + \|\mathcal{T}_2\|_{L_2(\Omega)} + \|\mathcal{T}_3\|_{L_2(\Omega)} + \|\mathcal{T}_4\|_{L_2(\Omega)}. \quad (28)$$

We will obtain a suitable bound for each term of the right-hand side of (28).

It follows from L_2 error bound (Theorem 1), the Sobolev norms and seminorms and Leibniz integral rule that

$$\begin{aligned}\|\mathcal{T}_1\|_{L_2(\Omega)} &\leq Ch \left\| \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right\|_{1, L_2(\mathbb{R})} \\ &= Ch \left\| \frac{\partial}{\partial x} \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right\|_{L_2(\mathbb{R})} \\ &\leq Ch \left(\|E'(t)\|_{L_2(\mathbb{R})} + \left\| \int_0^x \frac{-\alpha}{1-\alpha} \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right\|_{L_2(\mathbb{R})} \right).\end{aligned}$$

Using generalized Hardy's inequality (Lemma 1) and Lemma 3 gives

$$\begin{aligned}\|\mathcal{T}_1\|_{L_2(\Omega)} &\leq Ch\left(\|E'\|_{L_2(\mathbb{R})} + \|E'\|_{L_2(\mathbb{R})}\right) \\ &\leq Ch|E|_{1,L_2(\mathbb{R})} \leq Ch|u|_\varphi.\end{aligned}\quad (29)$$

Similarly, based on generalized Hardy's lemma and Lemma 3, we have

$$\begin{aligned}\|\mathcal{T}_2\|_{L_2(\Omega)} &= \left\| \mu \int_0^x \exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) E'(t) dt \right\|_{L_2(\Omega)} \\ &\leq C\|E'\|_{L_2(\Omega)} = C|E|_{1,L_2(\Omega)} \leq C|u|_\varphi.\end{aligned}\quad (30)$$

From L_2 error bound for (Theorem 1), we have

$$\|\mathcal{T}_3\|_{L_2(\Omega)} \leq Ch|u|_{1,L_2(\mathbb{R})}. \quad (31)$$

Now, we have

$$\begin{aligned}\|\mathcal{T}_4\|_{L_2(\Omega)} &\leq \|\mathcal{P}y(x) - y(x)\|_{L_2(\Omega)} + \|y(x)\|_{L_2(\Omega)} \\ &\leq Ch|y(x)|_{1,L_2(\mathbb{R})} + \|y(x)\|_{L_2(\Omega)}.\end{aligned}\quad (32)$$

We may compute two upper bounds for $|y(x)|_{1,L_2(\mathbb{R})}$ and $\|y(x)\|_{L_2(\Omega)}$. According to interpolation conditions by Lagrange basis functions, we can obtain a posterior bound for the Gauss–Legendre quadrature error of Caputo–Fabrizio fractional derivative:

$$\begin{aligned}\|y(x)\|_{L_2(\Omega)} &= \left\| \mu \left(\int_{-1}^1 \tilde{Q}(x, \xi) d\xi - \sum_{\ell=0}^M \tilde{Q}(x, \xi_\ell) w_\ell \right) \right\|_{L_2(\Omega)} \\ &= \left\| \mu \left(\int_{-1}^1 \tilde{Q}(x, \xi) d\xi - \sum_{\ell=0}^M I_M \tilde{Q}(x, \xi_\ell) w_\ell \right) \right\|_{L_2(\Omega)} \\ &= \left\| \mu \int_{-1}^1 (\tilde{Q}(x, \xi) d\xi - I_M \tilde{Q}(x, \xi)) \right\|_{L_2(\Omega)}.\end{aligned}$$

It follows from generalized Hardy's inequality (Lemma 1) and estimates for the interpolation error (Lemma 2) that

$$\begin{aligned}\|y(x)\|_{L_2(\Omega)} &\leq C\|\tilde{Q}(x, \xi) - I_M \tilde{Q}(x, \xi)\|_{L_2(\Omega)} \\ &\leq CM^{-q}|\tilde{Q}(x, \xi)|_{H^{q,M}(\Omega)}.\end{aligned}$$

By setting $q = 1$, we obtain

$$\frac{\partial \tilde{Q}}{\partial \xi} = \frac{\partial \tilde{Q}}{\partial t} \frac{\partial t}{\partial \xi}.$$

Thus

$$\begin{aligned}\|y(x)\|_{L_2(\Omega)} &\leq CM^{-1} \left\| \frac{\partial}{\partial \xi} \left(\exp\left(\frac{-\alpha(x-t(x,s))}{1-\alpha}\right) \hat{u}'(t) \right) \right\|_{L_2(\Omega)} \\ &\leq CM^{-1} \left\| \frac{\alpha x^2}{4(1-\alpha)} \exp\left(\frac{-\alpha(x-t(x,s))}{1-\alpha}\right) \right\|_{L_2(\Omega)} \|\hat{u}'\|_{L_2(\Omega)} \\ &\quad + CM^{-1} \left\| \frac{x^2}{4} \exp\left(\frac{-\alpha(x-t(x,s))}{1-\alpha_i}\right) \right\|_{L_2(\Omega)} \|\hat{u}''\|_{L_2(\Omega)}.\end{aligned}$$

From $x \in (0, 1)$, we may write

$$\exp\left(\frac{-\alpha(x-t)}{1-\alpha}\right) \leq \exp\left(\frac{\alpha}{1-\alpha}\right),$$

which gives

$$\begin{aligned}\|y(x)\|_{L_2(\Omega)} &\leq CM^{-1} \left(\frac{\alpha}{1-\alpha} \|\hat{u}'\|_{L_2(\Omega)} + \|\hat{u}''\|_{L_2(\Omega)} \right) \\ &\leq CM^{-1} \left(\frac{\alpha}{1-\alpha} |\hat{u} - u|_{1,L_2(\Omega)} + \frac{\alpha}{1-\alpha} |u|_{1,L_2(\Omega)} + |\hat{u} \right. \\ &\quad \left. - u|_{2,L_2(\Omega)} + |u|_{2,L_2(\Omega)} \right).\end{aligned}$$

Using Lemma 3 and Corollary 1, we may write

$$\begin{aligned}\|y(x)\|_{L_2(\Omega)} &\leq CM^{-1} \left(\frac{\alpha}{1-\alpha} |u|_{1,L_2(\Omega)} + |u|_{2,L_2(\Omega)} + \frac{1}{1-\alpha} |u|_\varphi \right) \\ &\leq CM^{-1} (|u|_{1,L_2(\Omega)} + |u|_{2,L_2(\Omega)} + |u|_\varphi).\end{aligned}\quad (33)$$

Finally, we compute an upper bound for $|y(x)|_{1,L_2(\Omega)}$ in a similar manner with (33) as

$$\begin{aligned}|y(x)|_{1,L_2(\Omega)} &= \left\| \frac{\partial y(x)}{\partial x} \right\|_{L_2(\Omega)} \\ &= \left\| \mu \frac{\partial}{\partial x} \left(\int_{-1}^1 \tilde{Q}(x, \xi) d\xi - \sum_{\ell=0}^M \tilde{Q}(x, \xi_\ell) w_\ell \right) \right\|_{L_2(\Omega)} \\ &= \left\| \mu \left(\int_{-1}^1 \frac{\partial}{\partial x} \tilde{Q}(x, \xi) d\xi - \sum_{\ell=0}^M \frac{\partial}{\partial x} \tilde{Q}(x, \xi_\ell) w_\ell \right) \right\|_{L_2(\Omega)} \\ &\leq CM^{-q} \left| \frac{\partial}{\partial x} \tilde{Q}(x, \xi) \right|_{H^{q,M}(\Omega)}.\end{aligned}$$

Therefore

$$|y(x)|_{1,L_2(\Omega)} \leq CM^{-1} (|u|_{1,L_2(\mathbb{R})} + |u|_{2,L_2(\mathbb{R})} + |u|_{3,L_2(\mathbb{R})} + |u|_\varphi) \quad (34)$$

It follows from (32)–(34) that

$$\begin{aligned}\|\mathcal{T}_4\|_{L_2(\Omega)} &\leq ChM^{-1} (|u|_{1,L_2(\mathbb{R})} + |u|_{2,L_2(\mathbb{R})} + |u|_{3,L_2(\mathbb{R})} + |u|_\varphi) \\ &\quad + CM^{-1} (|u|_{1,L_2(\Omega)} + |u|_{2,L_2(\Omega)} + |u|_\varphi).\end{aligned}\quad (35)$$

The above estimates, together with (28), lead to (21).

Numerical experiments

In this section, we will use the MQ-RBF collocation method to solve Caputo–Fabrizio FDEs. These examples are considered, because exact solutions are available for them. All computations were performed on a running Mathematica software. We consider the MQ-RBF as (19) with shape parameter $a = 1$ for a good balance between accuracy and stability. In these experiments, we have used zeros of the shifted Legendre polynomials on $(0, 1)$ as the

centers of MQ-RBFs and the Gauss–Legendre quadrature method with $M = 2N$. We tested the algorithm using var-

where

$$f(x) = \frac{(\alpha - \alpha^2 - 1) \exp\left(\frac{x}{\alpha-1}\right) - (1 + 2\alpha(-1 + \alpha))(-1 + 2\alpha x) - \alpha((1 - 3\alpha + 4\alpha^2) \cos x + \alpha \sin x)}{\alpha + 2\alpha^2(\alpha - 1)}.$$

ious differential equations. All tests were performed with three different values for α , namely $\alpha = 0.1, 0.5$, and 0.9 . In the following, we show a representative selection of our results.

Example 1 For our first example, we consider the fractional differential equation:

$${}^{\text{CF}}D^\alpha u(x) - f(x) = 2u(x), \quad u(0) = 1,$$

Fig. 1 Comparison of approximation errors for Example 1 with several values of N and for different values of α

Table 1 Numerical results using different α and comparison of errors and condition numbers in Example 1

α	N				
	6	8	10	12	14
0.1					
Max. err.	4.366E-4	3.896E-5	3.564E-6	3.310E-7	3.104E-8
Cond. num.	3.039E+6	3.727E+8	3.942E+10	3.858E+12	3.617E+14
0.5					
Max. err.	8.905E-3	9.778E-4	1.062E-5	1.141E-6	1.213E-7
Cond. num.	2.783E+7	5.219E+9	7.907E+11	1.066E+14	1.336E+16
0.9					
Max. err.	5.655E-3	3.163E-4	1.929E-5	1.262E-6	8.734E-8
Cond. num.	4.905E+5	5.014E+7	4.979E+9	4.704E+11	4.314E+13

The exact solution is $u(x) = x + \cos x$.

Numerical results of MQ-RBF collocation method with several values of N and 40 digits precision are displayed in Fig. 1 and Table 1.

Example 2 For our second example, we consider the fractional differential equation:

Fig. 2 Comparison of approximation errors for Example 2 with several values of N and for different values of α

Table 2 Numerical results using different α and comparison of errors and condition numbers in Example 2

α	N				
	6	8	10	12	14
0.1					
Max. err.	1.155E−3	1.049E−4	9.842E−6	9.403E−7	9.078E−8
Cond. num.	1.378E+7	1.657E+9	1.729E+11	1.676E+13	1.560E+15
0.5					
Max. err.	1.278E−3	1.134E−4	1.049E−5	9.921E−7	9.507E−8
Cond. num.	1.562E+6	1.870E+8	1.946E+10	1.884E+12	1.753E+14
0.9					
Max. err.	5.431E−3	3.007E−4	1.847E−5	1.579E−6	1.424E−7
Cond. num.	2.278E+5	2.439E+7	2.401E+9	2.251E+11	2.052E+13

$${}^{\text{CF}}D^\alpha u(x) + f(x) = u(x), \quad u(0) = 0,$$

where

$$f(x) = \alpha \exp(x) \left(-1 + \exp\left(\frac{x}{-1 + \alpha}\right) \right).$$

The exact solution is $u(x) = x \exp(x)$.

Numerical results of MQ-RBF collocation method with several values of N and 40 digit precision are displayed in Fig. 2 and Table 2.

As we know the accuracy of the approximation solution depends on the value of the number of centers N , the distances between them [2, 21]. Increasing the number of data points has a severe effect on the stability of the linear system. For a fixed α , the condition number of the matrix in the linear system grows exponentially as the number of data points is increased [21]. Numerical experiments indicate that we can obtain good numerical results with small N in a short time and the value of α influences the error rate. According to concept of “effective condition number”, we can achieve good results with huge condition number using the arbitrary precision ability of the Mathematica software [17].

Conclusion

In the present paper, MQ-RBF method for a Caputo–Fabrizio fractional-order ordinary differential equation was described. The error estimate was proved and numerical examples were presented. These results indicate that the desired accuracy is obtained and MQ-RBF collocation method is an effective method for solving the fractional-order ordinary differential equations. Radial basis functions in their usual form lead to the solution of an ill-conditioned system of equations. To avoid such limitation of radial basis function collocation schemes, we can explained by the concept of “effective condition number”.

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References

- Adams, R.A., Fournier, J.F.: Sobolev Spaces. Elsevier Ltd, Amsterdam (2003)
- Bayona, V., Moscoso, M., Kindelan, M.: Optimal constant shape parameter for multiquadric based RBF-FD method. J. Comput. Phys. **230**, 73847399 (2011)
- Buhmann, M.D.: Radial Basis Functions: Theory and Implementations. Cambridge University Press, Cambridge (2003)
- Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A.: Spectral Methods, Fundamentals in Single Domains. Springer, Berlin (2006)
- Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Progr. Fract. Differ. Appl. **1**, 73–85 (2015)
- Caputo, M., Fabrizio, M.: Damage and fatigue described by a fractional derivative model. J. Comput. Phys. **293**, 400–408 (2014)
- Fakhr Kazemi, B., Ghoreishi, F.: Error estimate in fractional differential equations using multiquadric radial basis functions. J. Comput. Appl. Math. **245**, 133–147 (2013)
- Fasshauer, G.E.: Solving differential equations with radial basis functions: multilevel methods and smoothing. Adv. Comput. Math. **11**, 139–159 (1999)
- Fasshauer, G.E.: Meshfree Approximation Methods with MATLAB, Interdisciplinary Mathematical Sciences. World Scientific Publishing Company, Singapore (2007)
- Gogatishvili, A., Lang, J.: The generalized hardy operator with kernel and variable integral limits in Banach function spaces. J. Inequal. Appl. **4**(1), 116 (1999)
- Hardy, R.L.: Theory and applications of the multiquadric-biharmonic method. 20 years of discovery 1968–1988. Comput. Math. Appl. **19**(8–9), 163–208 (1990)
- Hon, Y.C., Cheung, K.F., Mao, X.Z., Kansa, E.J.: Multiquadric solution for shallow water equations. ASCE J. Hydraul. Eng. **125**(5), 524–533 (1999)
- Jafari, H., Seifi, S.: Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation. Commun. Nonlinear Sci. Numer. Simul. **14**(5), 2006–2012 (2009)

14. Kansa, E.J.: Multiquadrics-a scattered data approximation scheme with applications to computational fluid-dynamics. I. Surface approximations and partial derivative estimates. *Comput. Math. Appl.* **19**(8–9), 127–145 (1990)
15. Kansa, E.J.: Multiquadrics-a scattered data approximation scheme with applications to computational fluid-dynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations. *Comput. Math. Appl.* **19**(8–9), 147–161 (1990)
16. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V, Amsterdam (2006)
17. Li, Z.-C., Lu, T.-T., Hu, H.-Y., Cheng, A.H.-D.: *Trefftz and Collocation Method*. WIT Press, Southampton (2008)
18. Losada, J., Nieto, J.J.: Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **1**, 87–92 (2015)
19. Momani, S.: Analytical approximate solution for fractional heat-like and wavelike equations with variable coefficients using the decomposition method. *Appl. Math. Comput.* **165**, 459–472 (2005)
20. Podlubny, I.: *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*. Academic Press, Inc., San Diego (1999)
21. Sarra, S.A.: Radial basis function approximation methods with extended precision floating point arithmetic. *Eng. Anal. Bound. Elem.* **35**, 6876 (2011)
22. Yoon, J.: L_p -error estimates for “shifted” surface spline interpolation on Sobolev space. *Math. Comput.* **72**(243), 1349–1367 (2003)
23. Zerroukat, M., Power, H., Chen, C.S.: A numerical method for heat transfer problems using collocation and radial basis functions. *Int. J. Numer. Methods Eng.* **42**, 1263–1278 (1998)