Effect of a time dependent stenosis on flow of a second grade fluid through porous medium in constricted tube using integral method

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Abstract In this article, effect of a time-dependent stenosis on the flow of a non-Newtonian fluid through porous medium in axially symmetric constricted tube is analyzed using integral method. The study is based on a second grade fluid model. An order of magnitude analysis is performed to simplify the model for mild constriction. Integral approach coupled with the fourth-order polynomial solution for the velocity profile is used. The effect of different non-dimensional parameters such as Reynolds number, non-Newtonian parameter, porous parameter and time emerging in the model on velocity profile, pressure gradient, wall shear stress, separation and reattachment data are presented and discussed graphically. Velocity of the fluid increases with an increase in time, while with the increase in porous parameter velocity of the fluid decreases. It is noted that Reynolds number provides a mechanism to control the attachment and de-attachment points for different values of porous parameter. The present study is valid only for mild stenosis.

Keywords Non-Newtonian fluids · Constricted tube · Porous medium · Wall shear stress

Introduction

Stenosis is the formation of atherosclerotic plaques in the lumen of an artery which causes serious circulatory disorders [1, 2]. In circulatory systems, reduction of blood-flow in constricted region of artery, blockage of the arteries, the presence of stenosis in major blood vessels carrying blood to the heart, brain, etc. lead to various serious arterial diseases. Hydrodynamic factors play an important role in the formation development and progression of arterial stenosis [1, 2]. Rheological and fluid dynamic properties of blood flow play a vital role in the understanding, diagnosis and treatment of such diseases [3–7]. Blood flow model through constricted arteries are discussed by many researchers such as [8–16] have reported the experimental work on the blood flow through stenotic arteries. Perkkio and Keskinen [12] analyzed the blood flow through diseased artery theoretically by considering the blood as Newtonian fluid. Merrill [17] analyzed the blood flow through arteries at low shear rates by taking it as Newtonian fluid. Blood flow is basically a function of pressure differences and resistance (Darcy's law). Darcy's model is suggested as flow transport model in porous media which is simple, consider Darcy's resistance, neglects boundary conditions, form drag and convective terms [18]. Applications of Darcy's model covers vessels blocked by cholesterol and blood clots, muscles near artery, tumors and flow in soft tissues. Gour and Gupta [19] and Dash et al. [20] analyzed the steady, laminar and incompressible flow considering blood as Casson fluid through the constricted porous artery. Dash et al. [20] considered the Brinkman model to account the porous medium and discussed the two cases of permeability, first taking the constant permeability and second the variation of permeability in radial direction. Tandon and Rana [21] have studied the flow of blood through the axi-symmetric stenosis and the results are computed using finite element method.

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Pralhad and Schultz [22] considered the blood flow as couple stress fluid and the computed results for different blood diseases are compared with the case of normal blood and for other theoretical models. Ratan [23] focuses the porous effects on the diseased artery and represent the blood as Herschel–Bulkely fluid model and find the result of partial differential equation numerically. Biswas [24] studied the steady non-Newtonian blood flow through constricted artery analytically and numerically. Vand et al. [25] has analyzed the blood flow through mildly constricted artery analytically. Mirza [26] reported the steady incompressible non-Newtonian fluid flowing through constricted artery. Siddiqui et al. [27] has analyzed the non-Newtonian fluid flow through mildly constricted artery with no slip condition at the wall and solved analytically by taking constant volume flow rate. In the aforementioned articles, the usual time-independent stenosis is taken. In all the above-mentioned articles, the pressure is assumed as constant or pulsatile which is not enough to compare with a physical phenomena so in [27, 28] the pressure is assumed to be variable. Siddiqui et al. [27] extended the work of [28] for second grade fluid through constricted tube with time-independent stenosis. Different physical conditions for non-Newtonian fluids are discussed by many authors in detail [29–34]. The aim of the present article is to extend the work of [35, 36] for second grade fluid flow through porous medium in a time-dependent mildly constricted tube and investigate the effect of stenosis on flow variables such as velocity profile, pressure, wall shear stress, separation and reattachment points. The problem under consideration is assumed to be steady for mathematical handling.

Governing equations

The basic equations that govern the flow of an incompressible fluid consist of the conservation of mass and momentum and in the absence of body forces are given as:

$$\nabla \cdot \tilde{\mathbf{V}} = 0, \quad (1)$$

$$\rho \frac{D\tilde{\mathbf{V}}}{Dt} = -\nabla \tilde{p} + \text{div} \tau, \quad (2)$$

where $\tilde{\mathbf{V}}$ is the velocity vector, ρ the constant density and τ is the shear stress. The constitutive equation for second grade fluid is given by,

$$\tau = \mu \tilde{\mathbf{A}}_1 + \alpha_1 \tilde{\mathbf{A}}_2 + \alpha_2 \tilde{\mathbf{A}}_1, \quad (3)$$

where μ is the dynamic viscosity, α_1 and α_2 are the material constants and $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$ are the first and second Rivlin–Ericksen tensors defined as:

$$\tilde{\mathbf{A}}_1 = \nabla \tilde{\mathbf{V}} + (\nabla \tilde{\mathbf{V}})^T, \quad (4)$$

$$\tilde{\mathbf{A}}_2 = \frac{d\tilde{\mathbf{A}}_1}{dt} + \tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}}) + (\tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}}))^T, \quad (5)$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{\mathbf{V}} \cdot \nabla. \quad (6)$$

In view of Eqs. (4), (5) and (6) in (3), we get,

$$\tau = \mu \tilde{\mathbf{A}}_1 + \alpha_1 \left[\frac{\partial \tilde{\mathbf{A}}_1}{\partial t} + (\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{A}}_1 + \tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}}) + (\nabla \tilde{\mathbf{V}})^T \tilde{\mathbf{A}}_1 \right] + \alpha_2 \tilde{\mathbf{A}}_1^2, \quad (7)$$

using Eqs. (3) and (7) in Eq. (2), we get,

$$\begin{aligned} \rho \frac{\partial \tilde{\mathbf{V}}}{\partial t} + \rho [(\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{V}}] = & -\nabla \tilde{p} + \mu \text{div} \tilde{\mathbf{A}}_1 + \alpha_1 \text{div} \\ & \left(\frac{\partial \tilde{\mathbf{A}}_1}{\partial t} + (\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{A}}_1 \right) + \alpha_1 (\tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}}) + (\nabla \tilde{\mathbf{V}})^T \tilde{\mathbf{A}}_1) + \alpha_2 \text{div} \tilde{\mathbf{A}}_1^2. \end{aligned} \quad (8)$$

The identities in Eq. (8) can be defined as:

$$(\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{V}} = \frac{\nabla \tilde{\mathbf{V}}^2}{2} - (\tilde{\mathbf{V}} \times (\nabla \tilde{\mathbf{V}})), \quad (9)$$

$$\text{div}((\tilde{\mathbf{V}} \cdot \nabla) \tilde{\mathbf{A}}_1) = (\tilde{\mathbf{V}} \cdot \nabla) \text{div} \tilde{\mathbf{A}}_1 + \text{div}(\tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}})^T), \quad (10)$$

where

$$\text{div}(\tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}})^T) = (\nabla \tilde{\mathbf{V}})^T \text{div} \tilde{\mathbf{A}}_1 + \tilde{\mathbf{A}}_1 \cdot (\nabla(\nabla \tilde{\mathbf{V}})^T), \quad (11)$$

$$\text{div}(\tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}}) + \tilde{\mathbf{A}}_1(\nabla \tilde{\mathbf{V}})^T) = \text{div} \tilde{\mathbf{A}}_1^2, \quad (12)$$

using all the above identities Eqs. (9)–(12) in Eq. (8), we get,

$$\begin{aligned} \rho (\partial \tilde{\mathbf{V}} / \partial t + \nabla(\tilde{\mathbf{V}})^2 / 2 - \tilde{\mathbf{V}} \times (\nabla \times \tilde{\mathbf{V}})) + \nabla \tilde{p} = & \alpha_1 \nabla \cdot (\partial \tilde{\mathbf{A}}_1 / \partial t) \\ & + (\alpha_1 + \alpha_2) \nabla \cdot \tilde{\mathbf{A}}_1^2 + \alpha_1 ((\tilde{\mathbf{V}} \cdot \nabla) \nabla \cdot \tilde{\mathbf{A}}_1 \\ & + (\nabla \tilde{\mathbf{V}})^T \nabla \cdot \tilde{\mathbf{A}}_1 + \tilde{\mathbf{A}}_1 \cdot \nabla(\nabla \tilde{\mathbf{V}})^T) + \mu \nabla \cdot \tilde{\mathbf{A}}_1 - \frac{\mu}{k} \tilde{\mathbf{V}}, \end{aligned} \quad (13)$$

For the model (13) required to be compatible with thermodynamics in the sense that all motions satisfy the Clausius–Duhem inequality and assumption that the specific Helmholtz free energy is a minimum in equilibrium, then the material parameters must meet the following conditions [37, 38],

$$\mu \geq 0, \alpha_1 \geq 0, \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \quad (14)$$

Problem formulation

An incompressible steady and laminar flow of a second grade fluid in a constricted tube of an infinite length having cosine-shaped symmetric constriction of height δ . The

radius of the unobstructed tube is R_0 and $R(\tilde{z})$ is the variable radius of the obstructed tube. The $\tilde{z}$ -axis is taken along the flow direction and $\tilde{r}$ -axis normal to it. Following [35, 36] the boundary of the tube is taken as:

$$R(\tilde{z}) = \begin{cases} R_0 - \frac{\delta}{2}(1 - e^{-t/T}) \left(1 + \cos\left(\frac{\pi\tilde{z}}{z_0}\right)\right), & -z_0 < \tilde{z} < z_0, \\ R_0 & \text{otherwise.} \end{cases} \quad (15)$$

In Eq. (15) T is the time constant and z_0 is the length of the constricted region as shown in the Fig. 1.

For steady axisymmetric flow of blood in tube, the velocity vector $\tilde{\mathbf{V}}$ is assumed to be of the form,

$$\tilde{\mathbf{V}} = [\tilde{u}(\tilde{r}, \tilde{z}), 0, \tilde{w}(\tilde{r}, \tilde{z})], \quad (16)$$

where $\tilde{u}$ and $\tilde{w}$ are the velocity components in $\tilde{r}$ -, $\tilde{z}$ -directions respectively. According to the geometry of the problem the boundary conditions are:

$$\begin{aligned} \tilde{u} = \tilde{w} = 0 \text{ at } \tilde{r} = R(\tilde{z}), \\ \frac{\partial \tilde{w}}{\partial \tilde{r}} = 0 \text{ at } \tilde{r} = 0. \end{aligned} \quad (17)$$

In view of Eq. (16) the Eqs. (1) and (13) become:

$$\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \quad (18)$$

$$\frac{\partial \tilde{h}}{\partial \tilde{r}} - \rho \tilde{w} \Omega = -\mu \frac{\partial \Omega}{\partial \tilde{z}} - \alpha_1 \tilde{w} \left(\nabla^2 \Omega - \frac{\Omega}{\tilde{r}^2} \right) - \frac{\mu}{k} \tilde{u}, \quad (19)$$

$$\frac{\partial \tilde{h}}{\partial \tilde{z}} + \rho \tilde{u} \Omega = \mu \left(\frac{\partial \Omega}{\partial \tilde{r}} + \frac{\Omega}{\tilde{r}} \right) + \alpha_1 \tilde{u} \left(\nabla^2 \Omega - \frac{\Omega}{\tilde{r}^2} \right) - \frac{\mu}{k} \tilde{w}, \quad (20)$$

where

$$\Omega = \frac{\partial \tilde{w}}{\partial \tilde{r}} - \frac{\partial \tilde{u}}{\partial \tilde{z}}, \quad (21)$$

$$\begin{aligned} \tilde{h} = \frac{\rho}{2} (\tilde{u}^2 + \tilde{w}^2) - \alpha_1 \left(\tilde{u} \nabla^2 \left(\tilde{u} - \frac{\tilde{u}}{\tilde{r}^2} \right) \right) \\ - \alpha_1 \tilde{w} \nabla^2 \tilde{w} - \frac{\alpha_1}{4} |\tilde{\mathbf{A}}_1|^2 + \tilde{p}, \end{aligned} \quad (22)$$

Fig. 1 Geometry of the problem

$$|\tilde{\mathbf{A}}_1|^2 = 4 \left(\frac{\partial \tilde{u}}{\partial \tilde{r}} \right)^2 + 4 \left(\frac{\tilde{u}}{\tilde{r}} \right)^2 + 4 \left(\frac{\partial \tilde{w}}{\partial \tilde{z}} \right)^2 + 2 \left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{r}} \right)^2, \quad (23)$$

and ∇^2 is the Laplacian parameter, $\tilde{h}$ is the generalized pressure. Introducing the dimensionless variables:

$$\begin{aligned} r = \frac{\tilde{r}}{R_0}, \quad z = \frac{\tilde{z}}{z_0}, \quad w = \frac{\tilde{w}}{U_0}, \quad u = \frac{\tilde{u} z_0}{U_0 \delta} \\ h = \frac{\tilde{h}}{\rho U_0^2}, \quad p = \frac{\tilde{p}}{\rho U_0^2}, \quad R_e = \frac{U_0 R_0 \rho}{\mu}, \quad D_a = \frac{k}{R_0^2}, \end{aligned} \quad (24)$$

where U_0 is the average velocity. An order-of-magnitude analysis is used to eliminate the negligible effect which appear in Eqs. (18)–(22). According to the order of magnitude analysis, which is also applicable for non-Newtonian fluids [4], it is noted that $\frac{\partial w}{\partial z}$ is an order of $\frac{\delta}{R_0}$, i.e. $\frac{\partial w}{\partial z} \sim o\left(\frac{\delta}{R_0}\right)$. Forrester and Young [28] assumed that for mild constriction if $\delta/R_0 \ll 1$, $\delta/R_e z_0 \ll 1$ and $R_0/z_0 \approx 1$ then axial normal stress gradient $\frac{\partial^2 w}{\partial z^2}$ is negligible as compared to the gradient of shear component. So Eqs. (18) and (22) will become,

$$\frac{\partial w}{\partial z} + \frac{\delta}{R_0} \frac{1}{r} \frac{\partial u}{\partial r} = 0. \quad (25)$$

$$\frac{\partial h}{\partial r} = 0. \quad (26)$$

$$\frac{\partial h}{\partial z} = \frac{1}{R_e} \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right] - \frac{1}{R_e D_a} w, \quad (27)$$

$$h = w^2 - \alpha^* w \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \alpha^* \frac{1}{2} \left(\frac{\partial w}{\partial r} \right)^2 + p. \quad (28)$$

where $\alpha^* = \frac{\alpha_1}{R_0^2 \rho}$. The non-dimensional form of cosine shape constriction profile is:

$$R(z) = \begin{cases} 1 - \frac{\delta^*}{2} (1 - e^{-t^*}) (1 + \cos(\pi z)), & -1 < z < 1, \\ 1, & \text{otherwise.} \end{cases} \quad (29)$$

where $\delta^* = \delta/R_0$ and $t^* = \frac{t}{T}$. Eq. (27) can be integrated from $r = 0$ to $r = R$ to get,

$$\int_0^R r \frac{\partial h}{\partial z} dr = \frac{R}{R_e} \left(\frac{\partial w}{\partial r} \right)_R - \frac{1}{R_e D_a} \int_0^R r w dr, \quad (30)$$

Exact solution of Eq. (30) cannot be obtained. To find the approximate solution, fourth-order polynomial called Karman–Pohlhausen approach [39], is assumed. Therefore,

$$\frac{w}{U} = C_1 + C_2 \left(1 - \frac{r}{R}\right) + C_3 \left(1 - \frac{r}{R}\right)^2 + C_4 \left(1 - \frac{r}{R}\right)^3 + C_5 \left(1 - \frac{r}{R}\right)^4, \quad (31)$$

where U is the centerline velocity and C_1, C_2, C_3, C_4 and C_5 are undetermined coefficients which can be evaluated from the following five conditions:

$$w = 0 \quad \text{at } r = R, \quad (32)$$

$$w = U \quad \text{at } r = 0, \quad (33)$$

$$\frac{\partial w}{\partial r} = 0 \quad \text{at } r = 0, \quad (34)$$

$$\frac{dh}{dz} = \frac{1}{R_e} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \quad \text{at } r = R, \quad (35)$$

$$\frac{\partial^2 w}{\partial r^2} = -2 \frac{U}{R^2} \quad \text{at } r = 0. \quad (36)$$

The no slip boundary conditions of zero velocity at the wall and centerline velocity U are given by Eqs. (32) and (33), condition (34) is a simple definition, condition (35) is obtained from equation (27). It is assumed that at $r = 0$, the velocity profile is parabolic at the center of the tube ($w = U \left[1 - \frac{r^2}{R^2}\right]$) so that the second derivative of w with respect to r , yields the condition (36). Thus Eq. (31) becomes,

$$w = U \left[\left(\frac{-\lambda + 10}{7} \right) \eta + \left(\frac{3\lambda + 5}{7} \right) \eta^2 + \left(\frac{-3\lambda - 12}{7} \right) \eta^3 + \left(\frac{\lambda + 4}{7} \right) \eta^4 \right], \quad (37)$$

where $\eta = \left(1 - \frac{r}{R}\right)$ and

$$\lambda = \frac{R^2 R_e}{U} \frac{dh}{dz}. \quad (38)$$

Here, λ is the function of z only, since R, U and h depend only on z . In Eq. (38) U and h are unknowns. If Q is the flux through the tube, then

$$Q = \int_0^R 2\pi r w dr. \quad (39)$$

Using Eq. (37) in (39), to obtain

$$Q = \frac{\pi R^2}{210} (-2\lambda U + 97U), \quad (40)$$

and centerline velocity U can also be written as,

$$U = \frac{210}{97} \frac{1}{\pi R^2} \left[Q + \frac{\pi R^4 R_e}{105} \frac{dh}{dz} \right], \quad (41)$$

Using Eq. (28) in (30), to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dz} \int_0^R r w^2 dr + \frac{\alpha^*}{2} \frac{d}{dz} \int_0^R r \left(\frac{\partial w}{\partial r} \right)^2 dr + \frac{R^2}{2} \frac{dP}{dz} \\ + \frac{1}{D_a} \int_0^R r w dr = \frac{1}{R_e} \left(\frac{\partial w}{\partial r} \right)_R, \end{aligned} \quad (42)$$

To obtain a closed solution one more approximation is taken into account that the velocity profile is parabolic, i.e.,

$$w = U \left[1 - \frac{r^2}{R^2} \right], \quad (43)$$

as discussed by Forrester and Young [28]. If non-linear terms are neglected then the flow will be a Poiseuille flow through the constriction [28]. Substitution of Eqs. (43) and (38) into Eqs. (28) and (42) yield generalized pressure and pressure gradient given by

$$\frac{dh}{dz} = 48\alpha^* \frac{Q^2}{\pi^2} \frac{1}{R^7} \frac{dR}{dz} + \frac{dp}{dz}, \quad (44)$$

$$\frac{dp}{dz} = \frac{388}{225} \frac{1}{R^5} \frac{Q}{\pi} \frac{dR}{dz} - \frac{8}{R^4 R_e} + \frac{2608}{75} \frac{Q^2}{\pi^2} \frac{\alpha^*}{R^7} \frac{dR}{dz} - \frac{97}{75} \frac{Q}{\pi} \frac{1}{D_a R^2}. \quad (45)$$

In view of Eqs. (37), (41), (44) and (45), Eq. (38) yields

$$\begin{aligned} w = \frac{1}{75} \left[\frac{4}{3} \frac{Q R_e}{\pi R^3} \frac{dR}{dz} + 64 R_e \frac{\alpha^*}{R^5} \frac{Q^2}{\pi^2} \frac{dR}{dz} - \frac{Q R_e}{\pi D_a} \right] \\ \times [-11\eta + 43\eta^2 - 45\eta^3 + 15\eta^4] + \frac{2}{R^2} \frac{Q}{\pi} [2\eta - \eta^2], \end{aligned} \quad (46)$$

where $\eta = 1 - r/R$, and the velocity w is a function of r and z through constricted tube. One can get velocity of unobstructed tube by taking R as constant or unity. The volume flow flux in unobstructed tube is $\tilde{Q} = \pi R_0^2 U_0$, which gives non-dimensional volume flux $Q = \tilde{Q}/R_0^2 U_0 = \pi$ which is same for obstructed tube. Hence velocity w and pressure gradient $\frac{dp}{dz}$ will become,

$$\begin{aligned} w = \frac{1}{75} \left[\frac{4}{3} \frac{R_e}{R^3} \frac{dR}{dz} + 64 R_e \frac{\alpha^*}{R^5} \frac{dR}{dz} - \frac{R_e}{D_a} \right] \\ [-11\eta + 43\eta^2 - 45\eta^3 + 15\eta^4] + \frac{2}{R^2} [2\eta - \eta^2], \end{aligned} \quad (47)$$

$$\frac{dp}{dz} = \frac{388}{225} \frac{1}{R^5} \frac{dR}{dz} - \frac{8}{R^4 R_e} + \frac{2608}{75} \frac{\alpha^*}{R^7} \frac{dR}{dz} - \frac{97}{75} \frac{1}{D_a R^2}. \quad (48)$$

The velocity profile for Forrester and Young [28] can readily be recovered as a special case by setting $\alpha^* = 0$ and $1/D_a = 0$ in Eq. (47).

Pressure drop across the constriction and across the whole length of the tube

The pressure distribution at any cross section z along the stenosis can be obtained by integrating Eq. (48) using boundary condition $p = p_0$ at $z = z_0$.

$$(\Delta p) = \frac{496}{75} \alpha^* \int_{R_0}^R \frac{1}{R^5} dR + \frac{388}{225} \int_{R_0}^R \frac{1}{R^5} dR - \frac{8}{R_e} \int_{z_0}^z \frac{1}{R^4} dz - \frac{97}{75} \frac{1}{D_a} \int_{z_0}^z \frac{1}{R^2} dz, \quad (49)$$

or

$$(\Delta p) = \left(\frac{124}{75} \alpha^* + \frac{97}{225} \right) \left(\frac{1}{R^4} - \frac{1}{R_0^4} \right) - \frac{16}{\pi^2 R_e} \int_0^\pi \frac{1}{[a - b \cos u]^4} du - \frac{97}{75} \frac{1}{\pi D_a} \int_0^\pi \frac{1}{[a - b \cos u]^2} du, \quad (50)$$

where

$$a = 1 - \frac{\delta^*(1 - e^{t^*})}{2}, \quad b = \frac{\delta^*(1 - e^{t^*})}{2}. \quad (51)$$

Now

$$\int_0^\pi \frac{1}{a - b \cos u} du = \pi(a^2 - b^2)^{-1/2}. \quad (52)$$

Differentiating Eq. (52) twice and thrice partially with respect to a , to get

$$\int_0^\pi \frac{1}{[a - b \cos u]^2} du = \pi a(a^2 - b^2)^{-3/2} = \pi f(\delta^*), \quad (53)$$

$$\int_0^\pi \frac{1}{[a - b \cos u]^4} du = \pi a \left(a^2 + \frac{3}{2} b^2 \right) (a^2 - b^2)^{-7/2} = \pi g(\delta^*), \quad (54)$$

where

$$f(\delta^*) = \left(1 - \frac{\delta^*(1 - e^{t^*})}{2} \right) (1 - \delta^*(1 - e^{t^*}))^{-3/2}, \\ g(\delta^*) = \left(1 - \frac{\delta^*(1 - e^{t^*})}{2} \right) \left(1 - \delta^*(1 - e^{t^*}) + \frac{5}{8} (\delta^*(1 - e^{t^*}))^2 \right) (1 - \delta^*(1 - e^{t^*}))^{-7/2}, \quad (55)$$

so that

$$(\Delta p) = \left(\frac{124}{75} \alpha^* + \frac{97}{225} \right) \left(\frac{1}{R^4} - \frac{1}{R_0^4} \right) - \frac{16z_0}{R_e R_0^4} g(\delta^*) - \frac{194z_0}{75} \frac{1}{D_a R_0^2} f(\delta^*). \quad (56)$$

When there is no constriction i.e. $\delta = 0$ and $f(\delta^*) = g(\delta^*) = 1$, the pressure drop across the normal tube is given by,

$$(\Delta p)_P = -\frac{16z_0}{R_e R_0^4} - \frac{194z_0}{75} \frac{1}{D_a R_0^2}. \quad (57)$$

In the absence of constriction, flow become Poiseuille and the subscript P denotes Poiseuille flow. If $2L$ is the length of the tube, then the expression for the pressure across the whole length of the constricted tube is,

$$\Delta p = \left(\frac{124}{75} \alpha^* + \frac{97}{225} \right) \left(\frac{1}{R^4} - \frac{1}{R_0^4} \right) + \frac{8}{R_e R_0^4} (2L - 2z_0) + \frac{97}{75} \frac{1}{D_a R_0^2} (2L - 2z_0). \quad (58)$$

In the absence of constriction, $\delta^* = 0$, the expression for the pressure of the normal tube will become,

$$[\Delta p]_P = \frac{16L}{R_e R_0^4} + \frac{194L}{75} \frac{1}{D_a R_0^2}. \quad (59)$$

It is worth mentioning that Eqs. (56) and (58) include the theoretical results of Forrester and Young [28] as a special case for $\alpha^* = 0$ and $1/D_a = 0$.

Shear stress on constricted surface

The shear stress on the constricted surface is

$$\tau_w = - \left(\mu \left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial r} \right) + \alpha_1 \left[\left(\tilde{u} \frac{\partial}{\partial r} + \tilde{w} \frac{\partial}{\partial z} \right) \left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial r} \right) + 2 \frac{\partial \tilde{u}}{\partial r} \frac{\partial \tilde{u}}{\partial z} + 2 \frac{\partial \tilde{w}}{\partial r} \frac{\partial \tilde{w}}{\partial z} \right] + \alpha_2 \left[\frac{\tilde{u}}{r^2} \left(\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial r} \right) \right] \right) \Big|_R. \quad (60)$$

Using Eq. (24) in Eq. (60), wall shearing stress becomes,

$$\frac{\tau_w}{\rho U_0^2} = \frac{-1}{R_e} \left(\frac{\partial w}{\partial r} \right)_R - \alpha^* \left(\frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \right)_R. \quad (61)$$

The Eqs. (47) and (61), yields,

$$\tau_w = \left[\frac{4}{R^3} + \frac{11}{75R} \left(-64\alpha^* \frac{R_e}{R^5} \frac{dR}{dz} - \frac{4R_e}{3R^3} \frac{dR}{dz} + \frac{R_e}{D_a} \right) \right] \left[\frac{1}{R_e} + \alpha^* \frac{dR}{dz} \left(\frac{4}{R^3} + \frac{11}{75R} \left(-64\alpha^* \frac{R_e}{R^5} \frac{dR}{dz} - \frac{4R_e}{3R^3} \frac{dR}{dz} + \frac{R_e}{D_a} \right) \right) \right]. \quad (62)$$

For $\alpha^* = 0$ and $1/D_a = 0$, the theoretical results of Forrester and Young [28] for shear stress are recovered. Shear stress in unobstructed tube will be,

$$(\tau_w)_p = \frac{4}{R^3 R_e} + \frac{11}{75 R D_a}. \quad (63)$$

Separation and reattachment

Prandtl [40] has explained the phenomena of separation in such a manner that the velocity of the fluid in the boundary layer drubbed towards the wall and inside the boundary layer the kinetic energy of the fluid particles appears to be less than that at the outer edge of the boundary layer. This means that the fluid particles inside the boundary layer may not be able to get the pressure which is applied in the outer layer. Even a small rise in pressure may trigger the fluid particles near the wall to stop and turn back to form a recirculating flow region, which is the characteristic of the separated flows. The separation and reattachment points can be calculated by taking negligible effects of shear stress at the wall, i.e. $\tau_w = 0$. Therefore,

$$\left[\frac{4}{R^3} + \frac{11}{75R} \left(-64\alpha^* \frac{R_e}{R^5} \frac{dR}{dz} - \frac{4R_e}{3R^3} \frac{dR}{dz} + \frac{R_e}{D_a} \right) \right] \left[\frac{1}{R_e} + \alpha^* \frac{dR}{dz} \left(\frac{4}{R^3} + \frac{11}{75R} \left(-64\alpha^* \frac{R_e}{R^5} \frac{dR}{dz} - \frac{4R_e}{3R^3} \frac{dR}{dz} + \frac{R_e}{D_a} \right) \right) \right] = 0, \quad (64)$$

or

$$R_e = \frac{900R^3}{A}, \frac{450 \frac{dR}{dz} R^3 \alpha^* \pm 15\sqrt{B}}{11\alpha^* \frac{dR}{dz} A}, \quad (65)$$

where

$$A = 4R \frac{dR}{dz} - 3 \frac{R^5}{D_a} + 192\alpha^* \frac{dR}{dz},$$

$$B = 44R^8 \alpha^* \left(\frac{dR}{dz} \right)^2 - 33 \frac{R^{11} \alpha^*}{D_a} \frac{dR}{dz} + 3012R^6 \alpha^* \left(\frac{dR}{dz} \right)^2. \quad (66)$$

Fig. 2 Effect of non-Newtonian parameter α^* on velocity profile

Results and discussions

Consider two-dimensional axial flow of a second grade fluid as a blood flowing through porous medium in a constricted tube of infinite length. This geometry, of course, is intended to simulate a time-dependent arterial stenosis, and the results are applicable to mild stenosis. The flow is assumed to be steady, laminar and incompressible. An approximate method is used to get the solution for the velocity, pressure drop across the constriction length, across the whole length of the tube and shear stress on the constricted surface. The effect of different flow parameters on velocity profile, pressure gradient, shear stress, separation and reattachment points are discussed graphically. In Fig. 2 the variation of non-Newtonian parameter α^* on the non-dimensional velocity profile is described at $z = 0.475$ taking $R_e = 5$, $\delta^* = 0.0836$, $t^* = 3$, $D_a = 2$. It is noted that velocity increases with an increase in non-Newtonian parameter. It can be seen from Fig. 3 that with an increase in Reynolds number, velocity of the fluid increases near the throat of the stenosis, however, it decreases in the diverging region, physically it means that viscous forces are dominant over inertia forces. The results for Forrester and Young [28] also recovered which are evident from Figs. 3 and 4. It is depicted from Fig. 5 that velocity of the fluid increases with an increase in time. Effect of porous parameter D_a for non-Newtonian fluids can be seen from Fig. 6, which shows that velocity decreases with an increase in porous parameter for $0 \leq r \leq 0.575$ and converse effect is noted for $0.575 \leq r \leq 0.95$. The effect of Reynolds number on dimensionless pressure gradient between $z = \pm 1$ is shown in Fig. 7.

It is well-mentioned that the pressure gradient increases upto the throat of the constriction and then decreases in the

Fig. 3 Effect of R_e on velocity profile

Fig. 4 Effect of R_e on velocity profile of Forrester and Young [28] and present study

Fig. 5 Effect of time t^* on velocity

diverging region for both non-Newtonian. In the meanwhile it is evident from Fig. 7 that the pressure gradient decreases with increasing Reynolds number. Figure 8 verify the present work with the Forrester and Young [28].

Fig. 6 Effect of porous parameter D_a on velocity

Fig. 7 Effect of R_e on pressure gradient

Fig. 8 Effect of R_e on pressure gradient of Forrester and Young [28] and present work

Behavior of porous parameter D_a on the pressure gradient can be observed in Fig. 9, which shows that pressure decreases with an increase in porous parameter D_a . Effects of non-Newtonian parameter α^* on pressure gradient is

Fig. 9 Effect of porous parameter D_a on pressure gradient

Fig. 12 Effect of t^* on pressure gradient

Fig. 10 Effect of α^* on pressure gradient

Fig. 13 Effect of R_e on wall shear stress

Fig. 11 Effect of δ^* on pressure gradient

Fig. 14 Effect of R_e on wall shear stress of Forrester and Young [28] and present study

given in Fig. 10 that the pressure increases as non-Newtonian parameters increases. Same behavior for constriction height δ^* on the pressure gradient is observed in Fig. 11.

Figure 12 shows the effect of variation of time on pressure gradient. It is observed from the Fig. 13 that for any Reynolds number, the shearing stress reaches a

Fig. 15 Effect of t^* on wall shear stress Fig. 18 Separation point for D_a in converging region Fig. 16 Effect of non-Newtonian parameter α^* on wall shear stress Fig. 19 Reattachment point D_a in diverging region Fig. 17 Effect of porous parameter D_a on wall shear stress

maximum value on the throat and then rapidly decreases in the diverging region. Figure 14 verify the present work with the Forrester and Young [28], and It is also

noted that shear stress decreases with an increase in Reynolds number. It means that Reynolds number provides a mechanism to control the wall shear stress. Figure 15 shows that wall shear stress increases with an increase in time t^* .

From Fig. 16 it is well-mentioned that as non-Newtonian parameter α^* increases wall shear stress decreases in the converging region and converse behavior is observed in the diverging region. In Fig. 17, effect of porous parameter on wall shear stress is depicted. It is observed that wall shear stress decreases with an increase in porous parameter. Figures 18 and 19 give the influence of porous parameter on the separation and reattachment points respectively. It is observed, as naturally expected, that separation point moves upstream with an increase in non-Newtonian parameter α^* while reattachment point moves downstream.

Conclusion

In the present study, an incompressible laminar and steady flow of a second grade fluid through porous medium in constricted tube is modeled and analyzed theoretically. The fluid is assumed to be blood flowing through the constricted artery with time-dependent stenosis and the results are applicable to mild stenosis. The expressions for velocity field, pressure gradient, wall shear stress and separation phenomena for the geometry are presented. An integral momentum method is applied for the solution of the problem. The summary of findings of the present work is as follows:

- Velocity increases with an increase in non-Newtonian parameter.
- Inertia forces are dominant over viscous forces near the throat of the constriction, however, opposite effect is observed in the diverging region.
- Reynolds number and non-Newtonian parameter are economical parameters to control the wall shear stress.
- Reynolds number also provides a mechanism to control the separation and reattachment points.
- The separation and reattachment points strongly depend upon constriction height.

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