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ORIGINAL RESEARCH

The harmonic index of product graphs

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Abstract The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{\deg_G(u)+\deg_G(v)}$ of all edges uv of G, where $\deg_G(u)$ denotes the degree of a vertex u in G. In this paper, we investigate the harmonic index of Cartesian, lexicographic, tensor, strong, corona and edge corona product of two connected graphs.

Keywords Harmonic index · Product graphs · Inverse degree

Mathematics Subject Classification 05C07 · 05C76

Introduction

Throughout this paper, all graphs are finite, simple, undirected and connected. For a graph G, $V(G)$ and $E(G)$ denote the set of all vertices and edges, respectively. We will use P_n, C_n and K_n to denote the path, the cycle and the complete graph of order n, respectively.

The Cartesian product $G_1 \Box G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(G_2)$, or (2) $v = v'$ and $uu' \in E(G_1)$.

The lexicographic product (or composition) $G_1[G_2]$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u, v) is adjacent to (u', v') if and only if (1) $uu' \in E(G_1)$, or (2) $u = u'$ and $vv' \in E(G_2)$.

The tensor (or direct) product $G_1 \times G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u, v) is adjacent to (u', v') if and only if $uu' \in E(G_1)$ and $vv' \in E(G_2)$.

The strong (or normal) product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(G_2)$, or (2) $v = v'$ and $uu' \in E(G_1)$, or (3) $uu' \in E(G_1)$ and $vv' \in E(G_2)$. Obviously, $G_1 \boxtimes G_2 = (G_1 \Box G_2) \cup (G_1 \times G_2)$.

Let $V(G_1) = \{v_1, \ldots, v_{n_1}\}$. The corona product $G_1 \circ G_2$ of disjoint graphs G_1 and G_2 is obtained by taking n_1 copies of G_2 and joining each vertex of the ith copy of G_2 with the vertex $v_i \in V(G_1)$.

Let $E(G_1) = \{e_1, \ldots, e_{m_1}\}$. The edge corona product $G_1 \bullet G_2$ of disjoint graphs G_1 and G_2 is obtained by taking m_1 copies of G_2 and joining each vertex of the ith copy of G_2 with two end vertices of the edge $e_i \in E(G_1)$.

The following propositions easily follow from the definition and structure of product graphs.

Proposition 1.1 [8, 9] *Let G_1 and G_2 be two graphs of orders n_1 and n_2, respectively. Then*

(i) $\deg_{G_1 \Box G_2}(u, v) = \deg_{G_1}(u) + \deg_{G_2}(v)$,
(ii) $\deg_{G_1[G_2]}(u, v) = n_2 \deg_{G_1}(u) + \deg_{G_2}(v)$,
(iii) $\deg_{G_1 \times G_2}(u, v) = \deg_{G_1}(u) \deg_{G_2}(v)$,
(iv) $\deg_{G_1 \boxtimes G_2}(u, v) = \deg_{G_1}(u) + \deg_{G_2}(v) + \deg_{G_1}(u) \deg_{G_2}(v)$.

Proposition 1.2 [8, 9] *Let G_1 and G_2 be two disjoint graphs of orders n_1 and n_2, respectively. Then*

(i) $$\deg_{G_1 \circ G_2}(u) = \begin{cases} \deg_{G_1}(u) + n_2 & u \in V(G_1) \\ \deg_{G_2}(u) + 1 & u \in V(G_2), \end{cases}$$

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(ii) $$\deg_{G_1 \bullet G_2}(u) = \begin{cases} (1+n_2)\deg_{G_1}(u) & u \in V(G_1) \\ \deg_{G_2}(u)+2 & u \in V(G_2). \end{cases}$$

The inverse degree and harmonic index of a graph G are two important vertex-degree-based indices related to G, were denoted by r(G) and H(G), respectively, and defined as follows:

$$r(G) = \sum_{u \in V(G)} \frac{1}{\deg_G(u)},$$

$$H(G) = \sum_{uv \in E(G)} \frac{2}{\deg_G(u)+\deg_G(v)}.$$

In recent years, the harmonic index has been extensively studied. Shwetha et al. [9] *derived expressions for the harmonic index of the join, corona product, Cartesian product, composition and symmetric difference of graphs. Recently, Onagh investigated the harmonic index of subdivision graph S(G), t-subdivision graph $S_t(G)$, vertex-semitotal graph R(G), edge-semitotal graph Q(G), total graph T(G) and F-sum of graphs, where $F \in \{S, S_t, R, Q, T\}$* [5–7]. *More results on the harmonic index can been found in* [1–3, 10–12].

In this paper, we study the harmonic index of Cartesian, lexicographic, tensor, strong, corona and edge corona product of two graphs G_1 and G_2 and present some bounds in terms of the harmonic index and inverse degree of G_1 and G_2.

Main results

In this section, we give some bounds for the harmonic index of graphs $G_1 \square G_2$, $G_1[G_2]$, $G_1 \times G_2$, $G_1 \boxtimes G_2$, $G_1 \circ G_2$ and $G_1 \bullet G_2$ in terms of $H(G_1)$, $H(G_2)$, $r(G_1)$ and $r(G_2)$. To do this, we need the following well-known inequality.

Jensen's inequality [4] Let f be a convex function on the interval I and $x_1, \ldots, x_n \in I$. Then

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{f(x_1)+\cdots+f(x_n)}{n},$$

with equality if and only if $x_1 = \cdots = x_n$.

Hereafter, G_1 and G_2 are two nontrivial graphs with $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, $1 \le i \le 2$.

Theorem 2.1 *Let G_1 and G_2 be two graphs. Then*

$$H(G_1 \square G_2) \le \frac{1}{4}(n_2 H(G_1) + n_1 H(G_2) + m_2 r(G_1) + m_1 r(G_2)),$$

with equality if and only if G_1 and G_2 are k-regular graphs.

Proof By definition of the harmonic index, we have

$$\begin{aligned} H(G_1 \square G_2) &= \sum_{u \in V(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{\deg_{G_1 \square G_2}(u,v) + \deg_{G_1 \square G_2}(u,v')} \\ &\quad + \sum_{v \in V(G_2)} \sum_{uu' \in E(G_1)} \frac{2}{\deg_{G_1 \square G_2}(u,v) + \deg_{G_1 \square G_2}(u',v)} \\ &= \sum_{u \in V(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{(\deg_{G_1}(u)+\deg_{G_2}(v)) + (\deg_{G_1}(u)+\deg_{G_2}(v'))} \\ &\quad + \sum_{v \in V(G_2)} \sum_{uu' \in E(G_1)} \frac{2}{(\deg_{G_1}(u)+\deg_{G_2}(v)) + (\deg_{G_1}(u')+\deg_{G_2}(v))} \\ &= \sum_{u \in V(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{2\deg_{G_1}(u) + (\deg_{G_2}(v)+\deg_{G_2}(v'))} \\ &\quad + \sum_{v \in V(G_2)} \sum_{uu' \in E(G_1)} \frac{2}{(\deg_{G_1}(u)+\deg_{G_1}(u')) + 2\deg_{G_2}(v)} \\ &:= \sum 1 + \sum 2. \end{aligned}$$

By Jensen's inequality, for every $u \in V(G_1)$ and $vv' \in E(G_2)$, we have

$$\frac{2}{2\deg_{G_1}(u) + (\deg_{G_2}(v)+\deg_{G_2}(v'))} \le \frac{1}{4}\frac{1}{\deg_{G_1}(u)} + \frac{1}{4}\frac{2}{\deg_{G_2}(v)+\deg_{G_2}(v')}, \quad (1)$$

with equality if and only if $2\deg_{G_1}(u) = \deg_{G_2}(v) + \deg_{G_2}(v')$.

Similarly, for every $v \in V(G_2)$ and $uu' \in E(G_1)$,

$$\frac{2}{(\deg_{G_1}(u)+\deg_{G_1}(u')) + 2\deg_{G_2}(v)} \le \frac{1}{4}\frac{2}{\deg_{G_1}(u)+\deg_{G_1}(u')} + \frac{1}{4}\frac{1}{\deg_{G_2}(v)}, \quad (2)$$

with equality if and only if $\deg_{G_1}(u) + \deg_{G_1}(u') = 2\deg_{G_2}(v)$.

Thus,

$$\sum 1 \leq \frac{1}{4} \sum_{u\in V(G_1)} \sum_{vv'\in E(G_2)} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4} \sum_{u\in V(G_1)} \sum_{vv'\in E(G_2)} \frac{2}{\deg_{G_2}(v) + \deg_{G_2}(v')}$$
$$= \frac{1}{4} \sum_{u\in V(G_1)} \left(m_2 \times \frac{1}{\deg_{G_1}(u)} \right) + \frac{1}{4} \sum_{u\in V(G_1)} H(G_2)$$
$$= \frac{1}{4} m_2 r(G_1) + \frac{1}{4} n_1 H(G_2),$$

and

$$\sum 2 \leq \frac{1}{4} \sum_{v\in V(G_2)} \sum_{uu'\in E(G_1)} \frac{2}{\deg_{G_1}(u) + \deg_{G_1}(u')} + \frac{1}{4} \sum_{v\in V(G_2)} \sum_{uu'\in E(G_1)} \frac{1}{\deg_{G_2}(v)}$$
$$= \frac{1}{4} \sum_{v\in V(G_2)} H(G_1) + \frac{1}{4} \sum_{v\in V(G_2)} \left(m_1 \times \frac{1}{\deg_{G_2}(v)} \right)$$
$$= \frac{1}{4} n_2 H(G_1) + \frac{1}{4} m_1 r(G_2).$$

So, $H(G_1 \square G_2) \leq \frac{1}{4}\left(n_2 H(G_1) + n_1 H(G_2) + m_2 r(G_1) + m_1 r(G_2)\right)$.

Moreover, equality holds in the above inequality if and only if the inequalities (1) and (2) be equalities, i.e., G_1 and G_2 are k-regular. □

Theorem 2.2 *Let G_1 and G_2 be two graphs. Then*

$$H(G_1[G_2]) < \frac{1}{9} n_2 H(G_1) + \frac{1}{4} n_1 H(G_2) + \frac{1}{4} \frac{m_2}{n_2} r(G_1) + \frac{4}{9} n_2 m_1 r(G_2).$$

Proof Note that

$$\begin{aligned}
H(G_1[G_2]) &= \sum_{u\in V(G_1)} \sum_{vv'\in E(G_2)} \frac{2}{\deg_{G_1[G_2]}(u,v) + \deg_{G_1[G_2]}(u,v')} \\
&\quad + \sum_{v\in V(G_2)} \sum_{v'\in V(G_2)} \sum_{uu'\in E(G_1)} \frac{2}{\deg_{G_1[G_2]}(u,v) + \deg_{G_1[G_2]}(u',v')} \\
&= \sum_{u\in V(G_1)} \sum_{vv'\in E(G_2)} \frac{2}{\left(n_2\deg_{G_1}(u) + \deg_{G_2}(v)\right) + \left(n_2\deg_{G_1}(u) + \deg_{G_2}(v')\right)} \\
&\quad + \sum_{v\in V(G_2)} \sum_{v'\in V(G_2)} \sum_{uu'\in E(G_1)} \frac{2}{\left(n_2\deg_{G_1}(u) + \deg_{G_2}(v)\right) + \left(n_2\deg_{G_1}(u') + \deg_{G_2}(v')\right)} \\
&= \sum_{u\in V(G_1)} \sum_{vv'\in E(G_2)} \frac{2}{2n_2\deg_{G_1}(u) + \left(\deg_{G_2}(v) + \deg_{G_2}(v')\right)} \\
&\quad + \sum_{v\in V(G_2)} \sum_{v'\in V(G_2)} \sum_{uu'\in E(G_1)} \frac{2}{n_2\left(\deg_{G_1}(u) + \deg_{G_1}(u')\right) + \deg_{G_2}(v) + \deg_{G_2}(v')} \\
&:= \sum 1 + \sum 2.
\end{aligned}$$

One can see that for every $u \in V(G_1)$ and $vv' \in E(G_2)$,

$$\frac{2}{2n_2\deg_{G_1}(u) + \left(\deg_{G_2}(v) + \deg_{G_2}(v')\right)} \leq \frac{1}{4n_2} \frac{1}{\deg_{G_1}(u)} + \frac{1}{4} \frac{2}{\deg_{G_2}(v) + \deg_{G_2}(v')}, \tag{3}$$

with equality if and only if $2n_2\deg_{G_1}(u) = \deg_{G_2}(v) + \deg_{G_2}(v')$. □

Also, for every $v \in V(G_2)$, $v' \in V(G_2)$ and $uu' \in E(G_1)$,

$$\frac{2}{n_2\left(\deg_{G_1}(u)+\deg_{G_1}(u')\right)+\deg_{G_2}(v)+\deg_{G_2}(v')} \leq \frac{1}{9n_2}\frac{2}{\deg_{G_1}(u)+\deg_{G_1}(u')}+\frac{2}{9}\frac{1}{\deg_{G_2}(v)}+\frac{2}{9}\frac{1}{\deg_{G_2}(v')}, \quad (4)$$

with equality if and only if $n_2(\deg_{G_1}(u)+\deg_{G_1}(u'))=\deg_{G_2}(v)=\deg_{G_2}(v')$.

Thus,

$$\sum 1 \leq \frac{1}{4}\frac{m_2}{n_2}r(G_1)+\frac{1}{4}n_1H(G_2),$$
$$\sum 2 \leq \frac{1}{9}n_2H(G_1)+\frac{4}{9}n_2m_1r(G_2).$$

Therefore,

$$H(G_1[G_2]) \leq \frac{1}{9}n_2H(G_1)+\frac{1}{4}n_1H(G_2)+\frac{1}{4}\frac{m_2}{n_2}r(G_1)+\frac{4}{9}n_2m_1r(G_2).$$

Now, suppose that equality holds in the above inequality. Then, the inequalities (3) and (4) must be equalities. So, G_1 and G_2 are k_1-regular and k_2-regular graphs, respectively, such that $2n_2k_1=k_2+k_2$ and $n_2(k_1+k_1)=k_2$, a contradiction. □

Theorem 2.3 *Let G_1 and G_2 be two graphs. Then*

$$H(G_1\times G_2)\geq 2H(G_1)H(G_2),$$

with equality if and only if either G_1 or G_2 is a regular graph.

Proof By definition of the harmonic index, we have

$$H(G_1\times G_2)=2\sum_{uu'\in E(G_1)}\sum_{vv'\in E(G_2)}\frac{2}{\deg_{G_1}(u)\deg_{G_2}(v)+\deg_{G_1}(u')\deg_{G_2}(v')}.$$

Note that for every $uu'\in E(G_1)$ and $vv'\in E(G_2)$,

$$\frac{2}{\left(\deg_{G_1}(u)+\deg_{G_1}(u')\right)\left(\deg_{G_2}(v)+\deg_{G_2}(v')\right)} = \frac{2}{\left(\deg_{G_1}(u)\deg_{G_2}(v)+\deg_{G_1}(u')\deg_{G_2}(v')\right)+\left(\deg_{G_1}(u)\deg_{G_2}(v')+\deg_{G_1}(u')\deg_{G_2}(v)\right)} \leq \frac{1}{4}\left(\frac{2}{\deg_{G_1}(u)\deg_{G_2}(v)+\deg_{G_1}(u')\deg_{G_2}(v')}+\frac{2}{\deg_{G_1}(u)\deg_{G_2}(v')+\deg_{G_1}(u')\deg_{G_2}(v)}\right),$$

with equality if and only if

$$\deg_{G_1}(u)\deg_{G_2}(v)+\deg_{G_1}(u')\deg_{G_2}(v') = \deg_{G_1}(u)\deg_{G_2}(v')+\deg_{G_1}(u')\deg_{G_2}(v)$$

, or, equivalently,

$$(\deg_{G_1}(u)-\deg_{G_1}(u'))(\deg_{G_2}(v)-\deg_{G_2}(v'))=0$$

. On the other hand,

$$\sum_{uu'\in E(G_1)}\sum_{vv'\in E(G_2)}\frac{2}{\left(\deg_{G_1}(u)+\deg_{G_1}(u')\right)\left(\deg_{G_2}(v)+\deg_{G_2}(v')\right)} = \frac{1}{2}H(G_1)H(G_2),$$

and

$$\sum_{uu'\in E(G_1)}\sum_{vv'\in E(G_2)}\frac{2}{\deg_{G_1}(u)\deg_{G_2}(v)+\deg_{G_1}(u')\deg_{G_2}(v')} + \sum_{uu'\in E(G_1)}\sum_{vv'\in E(G_2)}\frac{2}{\deg_{G_1}(u)\deg_{G_2}(v')+\deg_{G_1}(u')\deg_{G_2}(v)} = \frac{1}{2}H(G_1\times G_2)+\frac{1}{2}H(G_1\times G_2) = H(G_1\times G_2).$$

This implies that $H(G_1\times G_2)\geq 2H(G_1)H(G_2)$. □

Moreover, equality holds in the above inequality if and only if for every $uu'\in E(G_1)$ and $vv'\in E(G_2)$,

$$(\deg_{G_1}(u)-\deg_{G_1}(u'))(\deg_{G_2}(v)-\deg_{G_2}(v'))=0$$

, i.e., either G_1 or G_2 is regular. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4

(i) For any $n\geq 3$ and $m\geq 3$, $H(P_n\times C_m)=\frac{4}{3}m+\frac{n-3}{2}m$,

(ii) for any $n\geq 3$ and $m\geq 2$, $H(P_n\times K_m)=\frac{4}{3}m+\frac{n-3}{2}m$,

(iii) for any $n \geq 3$ and $m \geq 3$, $H(C_n \times C_m) = \frac{nm}{2}$,
(iv) for any $n \geq 3$ and $m \geq 2$, $H(C_n \times K_m) = \frac{nm}{2}$,
(v) for any $n \geq 2$ and $m \geq 2$, $H(K_n \times K_m) = \frac{nm}{2}$.

Theorem 2.5 *Let G_1 and G_2 be two graphs. Then*

$$H(G_1 \boxtimes G_2) \leq \frac{1}{9}((n_2 + 2m_2 + r(G_2))H(G_1) + (n_1 + 2m_1 + r(G_1))H(G_2) + H(G_1 \times G_2) + m_2 r(G_1) + m_1 r(G_2)),$$

with equality if and only if G_1 and G_2 are 1-regular graphs.

Proof By definition of the harmonic index, we have

$$H(G_1 \boxtimes G_2) = \sum_{u \in V(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{\deg_{G_1 \boxtimes G_2}(u,v) + \deg_{G_1 \boxtimes G_2}(u,v')} + \sum_{v \in V(G_2)} \sum_{uu' \in E(G_1)} \frac{2}{\deg_{G_1 \boxtimes G_2}(u,v) + \deg_{G_1 \boxtimes G_2}(u',v)} + 2 \sum_{uu' \in E(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{\deg_{G_1 \boxtimes G_2}(u,v) + \deg_{G_1 \boxtimes G_2}(u',v')} := \sum 1 + \sum 2 + \sum 3.$$

Then,

$$\sum 1 = \sum_{u \in V(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{\left(\deg_{G_1}(u) + \deg_{G_2}(v) + \deg_{G_1}(u)\deg_{G_2}(v)\right) + \left(\deg_{G_1}(u) + \deg_{G_2}(v') + \deg_{G_1}(u)\deg_{G_2}(v')\right)}$$
$$= \sum_{u \in V(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{2\deg_{G_1}(u) + \deg_{G_1}(u)\left(\deg_{G_2}(v) + \deg_{G_2}(v')\right) + \left(\deg_{G_2}(v) + \deg_{G_2}(v')\right)},$$

$$\sum 2 = \sum_{v \in V(G_2)} \sum_{uu' \in E(G_1)} \frac{2}{\left(\deg_{G_1}(u) + \deg_{G_2}(v) + \deg_{G_1}(u)\deg_{G_2}(v)\right) + \left(\deg_{G_1}(u') + \deg_{G_2}(v) + \deg_{G_1}(u')\deg_{G_2}(v)\right)}$$
$$= \sum_{v \in V(G_2)} \sum_{uu' \in E(G_1)} \frac{2}{\left(\deg_{G_1}(u) + \deg_{G_1}(u')\right) + \deg_{G_2}(v)\left(\deg_{G_1}(u) + \deg_{G_1}(u')\right) + 2\deg_{G_2}(v)},$$

$$\sum 3 = 2 \sum_{uu' \in E(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{\left(\deg_{G_1}(u) + \deg_{G_2}(v) + \deg_{G_1}(u)\deg_{G_2}(v)\right) + \left(\deg_{G_1}(u') + \deg_{G_2}(v') + \deg_{G_1}(u')\deg_{G_2}(v')\right)}$$
$$= 2 \sum_{uu' \in E(G_1)} \sum_{vv' \in E(G_2)} \frac{2}{\left(\deg_{G_1}(u) + \deg_{G_1}(u')\right) + \left(\deg_{G_2}(v) + \deg_{G_2}(v')\right) + \left(\deg_{G_1}(u)\deg_{G_2}(v) + \deg_{G_1}(u')\deg_{G_2}(v')\right)}.$$

By similar argument as in the proof of Theorem 2.2, one can show that

$$\sum 1 \leq \frac{1}{9} m_2 r(G_1) + \frac{1}{9} r(G_1) H(G_2) + \frac{1}{9} n_1 H(G_2),$$

with equality if and only if $2\deg_{G_1}(u) = \deg_{G_1}(u)(\deg_{G_2}(v) + \deg_{G_2}(v')) = \deg_{G_2}(v) + \deg_{G_2}(v')$, for all $u \in V(G_1)$ and $vv' \in E(G_2)$,

$$\sum 2 \leq \frac{1}{9} n_2 H(G_1) + \frac{1}{9} r(G_2) H(G_1) + \frac{1}{9} m_1 r(G_2),$$

with equality if and only if $\deg_{G_1}(u) + \deg_{G_1}(u') = \deg_{G_2}(v)(\deg_{G_1}(u) + \deg_{G_1}(u')) = 2\deg_{G_2}(v)$, for all $v \in V(G_2)$ and $uu' \in E(G_1)$, and

$$\sum 3 \leq \frac{2}{9} m_2 H(G_1) + \frac{2}{9} m_1 H(G_2) + \frac{1}{9} H(G_1 \times G_2),$$

with equality if and only if $\deg_{G_1}(u) + \deg_{G_1}(u') = \deg_{G_2}(v) + \deg_{G_2}(v') = \deg_{G_1}(u)\deg_{G_2}(v) + \deg_{G_1}(u')\deg_{G_2}(v')$, for all $uu' \in E(G_1)$ and $vv' \in E(G_2)$. □

Therefore,

$$H(G_1 \boxtimes G_2) \leq \frac{1}{9}((n_2 + 2m_2 + r(G_2))H(G_1) + (n_1 + 2m_1 + r(G_1))H(G_2) + H(G_1 \times G_2) + m_2 r(G_1) + m_1 r(G_2)).$$

It is easy to see that equality holds in the above inequality if and only if G_1 and G_2 are 1-regular graphs. □

Theorem 2.6 *Let G_1 and G_2 be two disjoint graphs. Then*

$$H(G_1 \circ G_2) < \frac{1}{4} H(G_1) + \frac{1}{4} n_1 H(G_2) + \frac{2}{9} n_2 r(G_1) + \frac{2}{9} n_1 r(G_2) + \frac{1}{4} n_1 m_2 + \frac{1}{4}\frac{m_1}{n_2} + \frac{2}{9}\frac{n_1 n_2}{n_2 + 1}.$$

Proof Note that

$$H(G_1 \circ G_2) = \sum_{uv\in E(G_1)} \frac{2}{\deg_{G_1\circ G_2}(u) + \deg_{G_1\circ G_2}(v)} + n_1 \sum_{uv\in E(G_2)} \frac{2}{\deg_{G_1\circ G_2}(u) + \deg_{G_1\circ G_2}(v)} + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \frac{2}{\deg_{G_1\circ G_2}(u) + \deg_{G_1\circ G_2}(v)}$$
$$= \sum_{uv\in E(G_1)} \frac{2}{(\deg_{G_1}(u) + n_2) + (\deg_{G_1}(v) + n_2)} + n_1 \sum_{uv\in E(G_2)} \frac{2}{(\deg_{G_2}(u) + 1) + (\deg_{G_2}(v) + 1)} + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \frac{2}{(\deg_{G_1}(u) + n_2) + (\deg_{G_2}(v) + 1)}$$
$$= \sum_{uv\in E(G_1)} \frac{2}{(\deg_{G_1}(u) + \deg_{G_1}(v)) + 2n_2} + n_1 quad \sum_{uv\in E(G_2)} \frac{2}{(\deg_{G_2}(u) + \deg_{G_2}(v)) + 2} + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \frac{2}{\deg_{G_1}(u) + \deg_{G_2}(v) + (n_2 + 1)}$$
$$:= \sum 1 + \sum 2 + \sum 3.$$

By using a similar method, one can verify that

$$\sum 1 \le \frac{1}{4} H(G_1) + \frac{1}{4}\frac{m_1}{n_2},$$

with equality if and only if $\deg_{G_1}(u) + \deg_{G_1}(v) = 2n_2$, for all $uv \in E(G_1)$,

$$\sum 2 \le \frac{1}{4} n_1 H(G_2) + \frac{1}{4} n_1 m_2,$$

with equality if and only if $\deg_{G_2}(u) + \deg_{G_2}(v) = 2$, for all $uv \in E(G_2)$, and $\sum 3 < \frac{2}{9} n_2 r(G_1) + \frac{2}{9} n_1 r(G_2) + \frac{2}{9}\frac{n_1 n_2}{n_2+1}$. □

So,

$$H(G_1 \circ G_2) < \frac{1}{4} H(G_1) + \frac{1}{4} n_1 H(G_2) + \frac{2}{9} n_2 r(G_1) + \frac{2}{9} n_1 r(G_2) + \frac{1}{4} n_1 m_2 + \frac{1}{4}\frac{m_1}{n_2} + \frac{2}{9}\frac{n_1 n_2}{n_2 + 1}.$$

This completes the proof. □

Theorem 2.7 *Let G_1 and G_2 be two disjoint graphs. Then*

$$H(G_1 \bullet G_2) < \frac{1}{n_2 + 1} H(G_1) + \frac{1}{4} m_1 H(G_2) + \frac{8}{9} m_1 r(G_2) + \frac{4}{9} n_2 m_1 + \frac{1}{8} m_1 m_2 + \frac{4}{9}\frac{n_1 n_2}{n_2 + 1}.$$

Proof Note that

$$H(G_1 \bullet G_2) = \sum_{uv\in E(G_1)} \frac{2}{\deg_{G_1\bullet G_2}(u) + \deg_{G_1\bullet G_2}(v)} + m_1 \sum_{uv\in E(G_2)} \frac{2}{\deg_{G_1\bullet G_2}(u) + \deg_{G_1\bullet G_2}(v)} + 2\sum_{uv\in E(G_1)} \sum_{x\in V(G_2)} \left(\frac{2}{\deg_{G_1\bullet G_1}(u) + \deg_{G_1\bullet G_2}(x)} + \frac{2}{\deg_{G_1\bullet G_1}(v) + \deg_{G_1\bullet G_2}(x)} \right)$$
$$= \sum_{uv\in E(G_1)} \frac{2}{(1+n_2)\deg_{G_1}(u) + (1+n_2)\deg_{G_1}(v)} + m_1 \sum_{uv\in E(G_2)} \frac{2}{(\deg_{G_2}(u)+2) + (\deg_{G_2}(v)+2)} + 2\sum_{uv\in E(G_1)} \sum_{x\in V(G_2)} \left(\frac{2}{(1+n_2)\deg_{G_1}(u) + (\deg_{G_2}(x)+2)} + \frac{2}{(1+n_2)\deg_{G_1}(v) + (\deg_{G_2}(x)+2)} \right)$$
$$= \frac{1}{1+n_2} \sum_{uv\in E(G_1)} \frac{2}{\deg_{G_1}(u) + \deg_{G_1}(v)} + m_1 \sum_{uv\in E(G_2)} \frac{2}{(\deg_{G_2}(u) + \deg_{G_2}(v)) + 4} + 2\sum_{uv\in E(G_1)} \sum_{x\in V(G_2)} \left(\frac{2}{(1+n_2)\deg_{G_1}(u) + \deg_{G_2}(x) + 2} + \frac{2}{(1+n_2)\deg_{G_1}(v) + \deg_{G_2}(x) + 2} \right)$$
$$:= \frac{1}{n_2+1} H(G_1) + \sum 1 + \sum 2.$$

Similarly, one can prove that $\sum 1 \le \frac{1}{4} m_1 H(G_2) + \frac{1}{8} m_1 m_2$, with equality if and only if $\deg_{G_1}(u) + \deg_{G_1}(v) = 4$, for all $uv \in E(G_1)$. □

Also, $\sum 2 \le \frac{4}{9}\frac{n_1 n_2}{n_2+1} + \frac{8}{9} m_1 r(G_2) + \frac{4}{9} n_2 m_1$, with equality if and only if $(1+n_2)\deg_{G_1}(u) = (1+n_2)\deg_{G_1}(v) = \deg_{G_2}(x) = 2$, for all $uv \in E(G_1)$ and $x \in V(G_2)$.

Therefore,

$$H(G_1 \bullet G_2) \le \frac{1}{n_2 + 1} H(G_1) + \frac{1}{4} m_1 H(G_2) + \frac{8}{9} m_1 r(G_2) + \frac{4}{9} n_2 m_1 + \frac{1}{8} m_1 m_2 + \frac{4}{9}\frac{n_1 n_2}{n_2 + 1}.$$

It is easy to show that equality cannot occur in the above inequality. □

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