Comparison of nonlinear Von Karman and Cosserat theories in very large deformation of skew plates

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Abstract

Prediction of plate behavior in large deformation is one of the important problems in plate theories. Cosserat theory is an advanced theory for simulation of plates in very large deformation, but it is complex from mathematical viewpoint. Another theory that has been used extensively for large deformation problems is nonlinear Von Karman theory which is easy for formulation and computation. In this paper, these theories were compared for rectangular and skew plates in simply supported and clamped boundary conditions to propose the acceptable range of using nonlinear Von Karman in very large deformation as a simpler theory. Higher order shear deformation plate theory was used with Von Karman nonlinearity. Whole domain method was employed for numerical solution. Each theory was validated with the literature for verification of the numerical method. Deflection and stress distribution were compared from small to very large deformations. The obtained results show that two theories were matched up to the maximum nondimensional deflection of 5 and 3 for simply supported and clamped boundary conditions, respectively. Moreover, by increasing the skew angle, the consistency of two theories would decrease even in deflections smaller than the thickness of the plate.

Keywords Cosserat theory · Von Karman nonlinearity · Large deformation · Higher order shear deformation plate theory · Skew plate

List of symbols

h	Thickness of the plate
u_{01}	Midplane displacements in x_1 direction
u_{02}	Midplane displacements in x_2 direction
u_{03}	Displacement of plate in x_3 direction
Ψ_{0x_1}	Midplane rotation of the normal around the x_1
Ψ_{0x_2}	Midplane rotation of the normal around the x_2
N_i	Membrane
Q_i	Transverse shear force
M_i	Bending moment per unit length
P_i	Higher order bending moment
R_i	Higher order shear force
δU	Virtual strain energy
δV	Virtual work
$\mathbf{s}$	Shape function matrix
$\widehat{\mathbf{x}}$	The general coordinate system
0	Superscript, before deformation
t	Superscript, after deformation

${}^0a_i$	Base vector of convective coordinate system
${}^ta_i$	Base vector of convective coordinate system
${}^0\mathbf{d}$	Director vector
${}^t\mathbf{d}$	Director vector
$J = \frac{d^tS}{d^0S}$	Jacobian transformation
F	Deformation gradient tensor
w_{ext}	Virtual work of the external forces
$\bar{W}$	Nondimensional deflection
$\bar{\sigma}_i$	Nondimensional stress
ϕ	Skew angle

Introduction

Nonlinear deformation of plates and shells has been studied as an interesting and challenging subject in recent years. Skew plates are widely used in industrial modern structures such as decks of bridges, ship hulls, and buildings as reinforced slabs, stiffened, and decks (Shahidi et al. 2007). Deflection and stress distribution is one of the fundamental problems for designing of plates with large deformation. Von Karman is one of the most applicable nonlinear theories for large deformation of plates and shells which is

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suitable for moderately large deformation (Ugural 1981). This nonlinear theory does not have any mathematical complexities. Moreover, higher order shear deformation theory, considered Von Karman term, has been employed for large deformation of plates by many researchers. Derived equations have been solved by various numerical and analytical methods. Kumar et al. (2011) and Shi (2007) worked on bending analysis of plates using higher order shear deformation plate theory (HSDT) and nonlinear Von Karman theory analytically. Tahouneh and Naei (2016a, b) investigated the effect of bidirectional continuously graded nanocomposite materials on free vibration of thick shell panels rested on elastic foundations. They utilized 2-D differential quadrature method as an efficient numerical tool to discretize governing equations and implement boundary conditions. Their new results for natural frequencies of the shell included the effects of elastic coefficients of foundation, boundary conditions, and material and geometrical parameters. Moreover, the obtained results indicated that a graded nanocomposite volume fraction in two directions had a higher capability to reduce the natural frequency than the conventional 1-D functionally graded nanocomposite materials. They also analyzed free vibration and vibrational displacements of thick laminated curved panels with finite length resting on two-parameter elastic foundations based on the three-dimensional elasticity theory. The effects of two-parameter elastic foundation modulus, and geometrical and material parameters together with the boundary conditions on the frequency parameters of the laminated functionally graded panels were investigated. They found that the outer functionally graded material layers have significant effect on the vibration behavior of cylindrical panels. Indeed, their study served as a benchmark for assessing the validity of numerical methods or two-dimensional theories used to analysis of laminated curved panels (Tahouneh and Naei 2016a, b). Tahouneh (2016) evaluated 3-D elasticity solution for free vibration analysis of continuously graded carbon nanotube-reinforced rectangular plates resting on two-parameter elastic foundations. The volume fractions of oriented, straight single-walled carbon nanotubes (SWCNTs) were assumed to be graded in the thickness direction. Moreover, an equivalent continuum model based on the Eshelby–Mori–Tanaka approach was employed to estimate the effective constitutive law of the elastic isotropic medium with oriented, straight carbon nanotubes. A semi-analytical approach composed of differential quadrature method (DQM) and series solution was adopted to solve the equations of motion. Their novelty was to exploit Eshelby–Mori–Tanaka approach to reveal the impacts of the volume fractions of oriented CNTs, different CNTs distributions, various coefficients of foundation, and different combinations of free, simply supported, and clamped boundary conditions on the vibrational characteristics of CGCNR rectangular plates. For numerical

solution of this problem, Nguyen-Xuan et al. (2013), Shariyat (2010a) and Neff (2004) can be mentioned. Analytical solution of buckling and free vibration of plates using Von Karman theory studied by Fazzolari et al. (2013), Sahmani and Ansari (2013a) and Swaminathan and Naveenkumar (2014). Phung-Van et al. (2013), Sahmani and Ansari (2013b), Sahmani et al. (2014) and Shariyat (2010b) has used numerical method for solution of this problem. Amiri Rad and Panahandeh-Shahraki (2014), Duc and Cong (2013), Duc and Tung (2011), Lee and Kim (2014), Ovesy et al. (2012) and Yang et al. (2006) have applied HSDT and nonlinear Von Karman theory for post buckling analysis of plates. Cosserat brothers suggested Cosserat theory for the first time in 1909 (Cosserat and Cosserat 1909). Naghdi shell model according to Cosserat theory is another advanced theory for very large deformation of plates and shells (Simo 1993; Simo and Fox 1989; Simo et al. 1989, 1990a, b; Simo and Kennedy 1992). Some researchers have been solved large deflection of isotropic plates and shells with this theory by numerical methods (Ghassemi et al. 2010, 2011; Shahidi et al. 2007). Despite of accurate results, this theory has mathematical complexities in formulation and computation. For the first time, comparison of Cosserat and Von Karman theories has been studied in the present work. For this aim, large deflection and stress distribution of skew plates were obtained by two methods: first, HSDT by considering Von Karman nonlinearity and second, Cosserat theory. To consider the numerical solution, the whole domain method was employed. In the proposed method, the plate was considered as one element and the shape functions were developed up to very high order. The main purpose of this paper was to make a comparison between two theories from small to very large deflection of the skew plates under transverse loading. This comparison can give acceptable range of using Von Karman theory in large deflections. By determining this range, some large deflection analysis can be done using Von Karman as a simpler theory.

The outline of this paper is as follows. In “**Kinematic equations of HSDT**”, the theory of higher order shear deformation plate by considering Von Karman nonlinearity is explained. The next section is about the kinematic relations of Cosserat theory followed by which numerical method is presented. The subsequent section gives the numerical results. Finally, conclusions are given.

Kinematic equations of HSDT

According to the HSDT, displacement field is considered as follows (Reddy 1984):

$$\begin{aligned}
u_1(x_1, x_2, x_3) &= u_{01}(x_1, x_2) + x_3 \left[\Psi_{0x_1}(x_1, x_2) - \frac{4}{3} \left(\frac{x_3}{h} \right)^2 \left(\Psi_{0x_1}(x_1, x_2) + \frac{\partial u_{03}}{\partial x_1} \right) \right] \\
u_2(x_1, x_2, x_3) &= u_{02}(x_1, x_2) + x_3 \left[\Psi_{0x_2}(x_1, x_2) - \frac{4}{3} \left(\frac{x_3}{h} \right)^2 \left(\Psi_{0x_2}(x_1, x_2) + \frac{\partial u_{03}}{\partial x_2} \right) \right] \\
u_3(x_1, x_2, x_3) &= u_{03}(x_1, x_2),
\end{aligned} \quad (1)$$

where h is the thickness of the plate, and u_{01} and u_{02} are midplane displacements in x_1 and x_2 directions, respectively.

u_{03} is displacement of plate in x_3 direction. Ψ_{0x_1} and Ψ_{0x_2} are the midplane rotation of the normal around the x_1 and x_2 axes. Figure 1 shows the geometry of applied rectangular plate.

Strain field can be achieved by displacement in Eq. (2). It is notable that Von Karman theory is added due to large deformation:

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_1^0 + x_3(\kappa_1^0 + x_3^2 \kappa_1^2), \quad \varepsilon_2 = \varepsilon_2^0 + x_3(\kappa_2^0 + x_3^2 \kappa_2^2), \quad \varepsilon_3 = 0 \\
\varepsilon_4 &= \varepsilon_4^0 + x_3^2 \kappa_4^2, \quad \varepsilon_5 = \varepsilon_5^0 + x_3^2 \kappa_5^2, \quad \varepsilon_6 = \varepsilon_6^0 + x_3(\kappa_6^0 + x_3^2 \kappa_6^2),
\end{aligned} \quad (2)$$

where

$$\begin{aligned}
\varepsilon_1^0 &= \frac{\partial u_{01}}{\partial x_1} + \frac{1}{2} \left(\frac{\partial u_{03}}{\partial x_1} \right)^2, \quad \varepsilon_2^0 = \frac{\partial u_{02}}{\partial x_2} + \frac{1}{2} \left(\frac{\partial u_{03}}{\partial x_2} \right)^2, \quad \varepsilon_4^0 = \Psi_{0x_2} + \frac{\partial u_{03}}{\partial x_2} \\
\varepsilon_5^0 &= \Psi_{0x_1} + \frac{\partial u_{03}}{\partial x_1}, \quad \varepsilon_6^0 = \frac{\partial u_{01}}{\partial x_2} + \frac{\partial u_{02}}{\partial x_1} + \frac{\partial u_{03}}{\partial x_1} \frac{\partial u_{03}}{\partial x_2}.
\end{aligned} \quad (3)$$

Midplane curvatures in Eq. (2) are as follows:

$$\begin{aligned}
\kappa_1^0 &= \frac{\Psi_{0x_1}}{\partial x_1}, \quad \kappa_2^0 = \frac{\Psi_{0x_2}}{\partial x_2}, \quad \kappa_6^0 = \frac{\Psi_{0x_1}}{\partial x_2} + \frac{\Psi_{0x_2}}{\partial x_1} \\
\kappa_1^2 &= -\frac{4}{3h^2} \left(\frac{\Psi_{0x_1}}{\partial x_1} + \frac{\partial^2 u_{03}}{\partial x_1^2} \right), \quad \kappa_2^2 = -\frac{4}{3h^2} \left(\frac{\Psi_{0x_2}}{\partial x_2} + \frac{\partial^2 u_{03}}{\partial x_2^2} \right) \\
\kappa_4^2 &= -\frac{4}{h^2} \left(\Psi_{0x_2} + \frac{\partial u_{03}}{\partial x_2} \right), \quad \kappa_5^2 = -\frac{4}{h^2} \left(\Psi_{0x_1} + \frac{\partial u_{03}}{\partial x_1} \right) \\
\kappa_6^2 &= -\frac{4}{3h^2} \left(\frac{\Psi_{0x_1}}{\partial x_2} + \frac{\Psi_{0x_2}}{\partial x_1} + 2 \frac{\partial^2 u_{03}}{\partial x_1 \partial x_2} \right).
\end{aligned} \quad (4)$$

For an elastic isotropic material, constitutive equations are as follows:

$$\begin{aligned}
\sigma_1 &= \frac{E}{1-\nu^2} (\varepsilon_1^0 + \nu \varepsilon_2^0 + x_3[k_1 + \nu k_2]) \\
\sigma_2 &= \frac{E}{1-\nu^2} (\varepsilon_2^0 + \nu \varepsilon_1^0 + x_3[k_2 + \nu k_1]) \\
\sigma_4 &= \frac{E}{1-\nu^2} \left(\frac{1-\nu}{2} \right) (\varepsilon_4^0 + x_3^2 \kappa_4^2) \\
\sigma_5 &= \frac{E}{1-\nu^2} \left(\frac{1-\nu}{2} \right) (\varepsilon_5^0 + x_3^2 \kappa_5^2) \\
\sigma_6 &= \frac{E}{1-\nu^2} \left(\frac{1-\nu}{2} \right) (\varepsilon_6^0 + x_3 \kappa_6^0),
\end{aligned} \quad (5)$$

where

$$\begin{aligned}
k_1 &= k_1^0 + x_3^2 \kappa_1^2 \\
k_2 &= k_2^0 + x_3^2 \kappa_2^2 \\
k_6 &= k_6^0 + x_3^2 \kappa_6^2.
\end{aligned} \quad (6)$$

The stress resultants and couple stresses are given by Eq. (7):

$$(N_i, M_i, P_i) = \int_{-h/2}^{+h/2} \sigma_i(1, x_3, x_3^3) dx_3 \quad i = 1, 2, 6$$

$$(Q_2, R_2) = \int_{-h/2}^{+h/2} \sigma_4(1, x_3^2) dx_3, \quad (Q_1, R_1) = \int_{-h/2}^{+h/2} \sigma_5(1, x_3^2) dx_3, \quad (7)$$

where N_i and Q_i are the membrane and transverse shear forces and M_i is the bending moment per unit length. P_i and R_i are the higher order bending moment and shear force, respectively.

Therefore, the constitutive relations of the plate can be rewritten by substituting Eq. (5) into Eq. (7) as follows:

$$\begin{aligned}
\begin{bmatrix} N \\ M \\ P \end{bmatrix} &= \begin{bmatrix} A & B & E \\ B & D & F \\ E & F & H \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa^0 \\ \kappa^2 \end{bmatrix} \\
\begin{bmatrix} \bar{Q} \\ \bar{R} \end{bmatrix} &= \begin{bmatrix} A & D \\ D & F \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa^2 \end{bmatrix},
\end{aligned} \quad (8)$$

where the matrix coefficients are as follows:

Fig. 1 Geometry of rectangular plate subject to uniform load

For numerical solution of Eq. (10), let displacement vector as follows:

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \psi_{x_1} \\ \psi_{x_2} \end{bmatrix}. \quad (13)$$

This displacement vector can be discretized as follows:

$$\mathbf{U} = \mathbf{S}\hat{\mathbf{x}}, \quad (14)$$

where $\mathbf{S}$ is the shape function matrix and $\hat{\mathbf{x}}$ is the general coordinate system. To determine parameter $\hat{\mathbf{x}}$, the whole domain method was employed which is explained in “Numerical method”.

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \int_{-h/2}^{+h/2} (Q_{ij})(1, x_3, x_3^2, x_3^3, x_3^4, x_3^6) dx_3 \quad i, j = 1, 2, 6$$

$$(A_{ij}, D_{ij}, F_{ij}) = \int_{-h/2}^{+h/2} (Q_{ij})(1, x_3^2, x_3^4) dx_3 \quad i, j = 4, 5. \quad (9)$$

The principle of virtual work is as follows:

$$\delta U + \delta V = 0, \quad (10)$$

where δU is the virtual strain energy and this term can be written as follows:

$$\delta U = \iiint_{-h/2}^{+h/2} (\sigma_i \delta \epsilon_i) dx_1 dx_2 dx_3 \quad i = 1, 2, 4, 5, 6. \quad (11)$$

δV is the virtual work done by external forces and moments.

According to Eq. (7), virtual strain energy can be rewritten as follows:

$$\delta U = \iint (N_i \delta \epsilon_i^0 + M_i \delta \kappa_i^0 + P_i \delta \kappa_i^2 + Q_1 \delta \epsilon_5^0 + Q_2 \delta \epsilon_4^0 + R_1 \delta \kappa_5^2 + R_2 \delta \kappa_4^2) dx_1 dx_2 \quad i = 1, 2, 6. \quad (12)$$

Kinematic relations of Cosserat theory

Figure 2 shows an element of a plate before and after deformation. Superscripts “0” and “t” are implied as the status of “before deformation” and “after deformation” respectively. ${}^0a_i$ and ${}^ta_i$ are the base vectors of convective coordinate system. In addition, in this figure, ${}^0\mathbf{d}$ and ${}^t\mathbf{d}$ are the director vectors. These vectors are considered normal to the midplane status.

The base vectors in Current configuration can be written as follows (Simo et al. 1989):

$${}^ta_\alpha = {}^t\mathbf{X}_{,\alpha}, {}^ta_3 = {}^t\mathbf{a}^3 = \frac{{}^ta_1 \times {}^ta_2}{\|{}^ta_1 \times {}^ta_2\|}. \quad (15)$$

Using Eq. (15), the first and second fundamental tensors of the surface can be written as follows:

Fig. 2 An element of plate before and after deformation. **a** Reference configuration. **b** Current configuration

$${}^t a_{\alpha\beta} = {}^t \mathbf{x}_{,\alpha} \cdot {}^t \mathbf{x}_{,\beta} \quad {}^t b_{\alpha\beta} = -{}^t \mathbf{a}_{3,\alpha} \cdot {}^t \mathbf{a}_\beta = {}^t \mathbf{a}_3 \cdot {}^t \mathbf{a}_{\alpha,\beta} \quad (16)$$

By this definition membrane and bending strains are expressed as follows:

$$\begin{aligned} {}^t \varepsilon_{\alpha\beta} &= \frac{1}{2}({}^t \mathbf{x}_{,\alpha} \cdot {}^t \mathbf{x}_{,\beta} - {}^0 \mathbf{x}_{,\alpha} \cdot {}^0 \mathbf{x}_{,\beta}) \\ {}^t \kappa_{\alpha\beta} &= {}^t b_{\alpha\beta} = {}^t \mathbf{a}_3 \cdot {}^t \mathbf{x}_{,\alpha\beta} = \frac{1}{\sqrt{{}^t a}}[{}^t \mathbf{x}_{,\alpha\beta} \cdot {}^t \mathbf{x}_{,1} \times {}^t \mathbf{x}_{,2}] \\ {}^t \kappa_{\alpha\beta} &= {}^t \kappa_{\alpha\beta} - {}^0 \kappa_{\alpha\beta}. \end{aligned} \quad (17)$$

In Eq. (17), subscript “0” denotes reference configuration.

According to the above equation, consider strain vector, ${}^t \boldsymbol{\varepsilon}$, as:

$${}^t \boldsymbol{\varepsilon} = [{}^t \varepsilon_{11}, {}^t \varepsilon_{22}, 2{}^t \varepsilon_{12}, {}^t \kappa_{11}, {}^t \kappa_{22}, 2{}^t \kappa_{12}]^T. \quad (18)$$

Considering Naghdi shell model, invariant form of Cauchy stress resultants is obtained as follows:

$$\begin{aligned} \mathbf{D}_m &= \frac{Eh}{(1-\nu^2)} \begin{bmatrix} ({}^0 a^{11})^2 & \nu({}^0 a^{11})({}^0 a^{22}) + (1-\nu)({}^0 a^{12})^2 & ({}^0 a^{11})({}^0 a^{12}) \\ \text{sym} & ({}^0 a^{11})^2 & ({}^0 a^{22})({}^0 a^{12}) \\ \frac{1+\nu}{2}({}^0 a^{12})^2 + \frac{1-\nu}{2}({}^0 a^{11})({}^0 a^{22}) & & \end{bmatrix} \\ \mathbf{D}_b &= \frac{h^2}{12} \mathbf{D}_m. \end{aligned} \quad (26)$$

$$\begin{aligned} {}^t N &= {}^t N^{\alpha\beta} {}^t \mathbf{a}_\alpha \otimes {}^t \mathbf{a}_\beta \\ {}^t Q &= {}^t Q^\alpha {}^t \mathbf{a}_\alpha \\ {}^t M &= {}^t M^{\alpha\beta} {}^t \mathbf{a}_\alpha \otimes {}^t \mathbf{a}_\beta. \end{aligned} \quad (19)$$

In Eq. (19), subscript “ t ” denotes current configuration.

In the initial configuration, second Piola stress resultants can be written as follows:

$$\begin{aligned} {}^0 N &= {}^0 N^{\alpha\beta} {}^0 \mathbf{a}_\alpha \otimes {}^0 \mathbf{a}_\beta \\ {}^0 Q &= {}^0 Q^{\alpha\beta} {}^0 \mathbf{a}_\alpha \otimes {}^0 \mathbf{a}_\beta \\ {}^0 M &= {}^0 M^{\alpha\beta} {}^0 \mathbf{a}_\alpha \otimes {}^0 \mathbf{a}_\beta. \end{aligned} \quad (20)$$

In Eq. (20), subscript “0” denotes reference configuration.

The Cauchy and Piola stresses are related as follows:

$${}^t N^{\alpha\beta} = J {}^t N^{\alpha\beta}, \quad {}^t M^{\alpha\beta} = J {}^t M^{\alpha\beta}. \quad (21)$$

In Eq. (21), $J = \frac{d^t S}{d^0 S}$ is the Jacobean transformation. This term is the ratio between the element surface after and before deformation:

$$J = \det(\mathbf{F}) \quad \text{where } \mathbf{F}_i^j = {}^t \mathbf{a}_i \otimes {}^0 \mathbf{a}^j = {}^t \mathbf{x}_{,i} \otimes {}^0 \mathbf{x}_{,j}. \quad (22)$$

In Eq. (22), “ F ” is the deformation gradient tensor.

According to Eq. (20), the resultant stress vector can be defined as follows:

$${}^t \boldsymbol{\sigma} = [{}^t N^{11}, {}^t N^{22}, {}^t N^{12}, {}^t M^{11}, {}^t M^{22}, {}^t M^{12}]. \quad (23)$$

Based on the Hooke’s law for an isotropic linear elastic material, resultant stress and strain vectors are related as follows (Simo and Kennedy 1992):

$${}^t \boldsymbol{\sigma} = \mathbf{D} {}^t \boldsymbol{\varepsilon}, \quad (24)$$

where:

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_b \end{bmatrix} \quad (25)$$

Principle of virtual work can be used for equilibrium condition in current and reference configuration. These equations are written as follows:

$$\text{Current configuration : } \int_{tS} ({}^t N^{\alpha\beta} \delta {}^t \varepsilon_{\alpha\beta} + {}^t M^{\alpha\beta} \delta {}^t \kappa_{\alpha\beta}) d^t S = {}^t W_{\text{ext}} \quad (27)$$

$$\text{Reference configuration : } \int_{0S} ({}^0 N^{\alpha\beta} \delta {}^0 \varepsilon_{\alpha\beta} + {}^0 M^{\alpha\beta} \delta {}^0 \kappa_{\alpha\beta}) d^0 S = {}^0 W_{\text{ext}}. \quad (28)$$

W_{ext} is virtual work of the external forces. It can be written as follows:

$$\int_{tS} (\mathbf{F}_{\text{ext}} \cdot \delta {}^t \mathbf{U} + \mathbf{M}_{\text{ext}} \cdot \delta {}^t \mathbf{d}) d^t S = {}^t W_{\text{ext}}. \quad (29)$$

For numerical solution of Eq. (27) or (28), displacement vector should be discretized by considering

$${}^t \mathbf{X} = {}^0 \mathbf{X} + \mathbf{U}. \quad (30)$$

Displacement vector can be discretized as follows:

$$\mathbf{U} = \mathbf{S}\hat{\mathbf{x}}, \quad (31)$$

where $\mathbf{S}$ is the shape function matrix and $\hat{\mathbf{x}}$ is the general coordinate system.

So:

$$\delta^t e_{\alpha\beta} = (\mathbf{x}_{,\alpha}^T \mathbf{s}_{,\beta} + \mathbf{x}_{,\beta}^T \mathbf{s}_{,\alpha}) \delta \hat{\mathbf{x}} = \mathbf{E}_{\alpha\beta} \delta \hat{\mathbf{x}} \quad (32)$$

$$\delta^t \kappa_{\alpha\beta} = \left(\frac{1}{\sqrt{t}a} \Gamma_{\alpha\beta} - \frac{\gamma_{\alpha\beta}}{2\sqrt{t}a^3} \mathbf{A} \right) \delta \hat{\mathbf{x}} = \mathbf{K}_{\alpha\beta} \delta \hat{\mathbf{x}} \quad (33)$$

$$\delta^t \mathbf{d} = \delta^t \mathbf{a}_3 = \left(\frac{1}{\sqrt{t}a} \boldsymbol{\beta}_{12}^T - \frac{1}{2\sqrt{t}a^3} \mathbf{A}^T (\mathbf{x}_{,1} \times \mathbf{x}_{,2})^T \right) \delta \hat{\mathbf{x}} = \mathbf{Y} \delta \hat{\mathbf{x}}, \quad (34)$$

where

$$\Gamma_{\alpha\beta} = ({}^t\mathbf{X}_{,1} \times {}^t\mathbf{X}_{,2})^T \mathbf{S}_{,\alpha\beta} + ({}^t\mathbf{X}_{,2} \times {}^t\mathbf{X}_{,\alpha\beta})^T \mathbf{S}_{,1} + ({}^t\mathbf{X}_{,\alpha\beta} \times {}^t\mathbf{X}_{,1})^T \mathbf{S}_{,2} \quad (35)$$

$$\gamma_{\alpha\beta} = {}^t\mathbf{X}_{,\alpha\beta} \cdot {}^t\mathbf{X}_{,1} \times {}^t\mathbf{X}_{,2}. \quad (36)$$

$$\mathbf{A} = 2{}^t a_{11} {}^t\mathbf{X}_{,2}^T \cdot \mathbf{S}_{,2} + 2{}^t a_{22} {}^t\mathbf{X}_{,1}^T \cdot \mathbf{S}_{,1} - 2{}^t a_{12} {}^t\mathbf{X}_{,2}^T \cdot \mathbf{S}_{,1} - 2{}^t a_{12} {}^t\mathbf{X}_{,1}^T \cdot \mathbf{S}_{,2} \quad (37)$$

$$\boldsymbol{\beta}_{12} = \begin{bmatrix} \mathbf{0} & {}^t x_{3,2} S_{2,1} - {}^t x_{3,1} S_{2,2} & {}^t x_{2,1} S_{3,2} - {}^t x_{2,2} S_{3,1} \\ {}^t x_{3,1} S_{1,2} - {}^t x_{3,2} S_{1,1} & \mathbf{0} & {}^t x_{1,2} S_{3,1} - {}^t x_{1,1} S_{3,2} \\ {}^t x_{2,2} S_{1,1} - {}^t x_{2,1} S_{1,2} & {}^t x_{1,1} S_{2,2} - {}^t x_{1,2} S_{2,1} & \mathbf{0} \end{bmatrix}. \quad (38)$$

Fig. 3 Mapping an arbitrary skew plate into the standard square computational domain

Fig. 4 Roller boundary condition of rectangular plate under Sinusoidal load

By substitution of Eqs. (32)–(36) in Eq. (28), the following relation can be achieved:

$$\int_{0S} {}^t_0 N^{\alpha\beta} \mathbf{E}_{\alpha\beta} + {}^t_0 M^{\alpha\beta} \mathbf{K}_{\alpha\beta} d^0 S = \int_{tS} {}^t \mathbf{F}_{\text{ext}}^T \mathbf{s} + {}^t \mathbf{M}_{\text{ext}}^T \mathbf{Y} d^t S, \quad (39)$$

where $\mathbf{E}_{\alpha\beta}$, $\mathbf{K}_{\alpha\beta}$, and $\mathbf{Y}$ would be determined through the Eqs. (32)–(34), respectively.

Numerical method

Whole domain method was applied for numerical solution of virtual work equations for both theories. In this method, whole domain of the plate will be considered as one element. First, the domain of the plate is mapped to a standard square (Fig. 3).

Hierarchical finite-element shape function was employed for interpolation of displacement field. Equations (40)–(42) show shape functions of free, simply supported, and clamped boundary conditions in one direction:

1. Free

$$S_i = [1 \quad \xi_i \quad \xi_i(1 - \xi_i^2) \quad \xi_i^2(1 - \xi_i^2) \quad \dots] \quad i = 1 \text{ or } 2. \quad (40)$$

2. Simply support

$$S_i = [1 - \xi_i^2 \quad \xi_i(1 - \xi_i^2) \quad \xi_i^2(1 - \xi_i^2) \quad \xi_i^3(1 - \xi_i^2) \quad \dots] \quad i = 1 \text{ or } 2. \quad (41)$$

3. Clamped

$$S_i = [(1 - \xi_i^2)^2 \xi_i(1 - \xi_i^2)^2 \xi_i^2(1 - \xi_i^2)^2 \xi_i^3(1 - \xi_i^2)^2 \dots] \quad i = 1 \text{ or } 2. \quad (42)$$

For two-dimensional cases, shape functions matrix is as follows:

$$S = S_1 S_2^T. \quad (43)$$

Therefore, displacement in each direction can be discretized as follows:

$$U_i = S_i \hat{X}. \quad (44)$$

Numerical results

To validate the present numerical method, the results of the two theories are compared separately with the literatures. Then, the difference between two theories is compared to rectangular and skew plates from small to very large deformation.

Validation the numerical result of HSDT by consideration Von Karman nonlinearity

In this section, HSDT which is solved by whole domain method is compared with exact solution presented by

Table 1 Comparison of the nondimensional deflection at the center of a rectangular plate under distributed load

$\frac{a}{h}$	$\bar{W}$			$\frac{a}{h}$	$\bar{W}$			$\frac{a}{h}$	$\bar{W}$		
	Mechab et al. (2010)	Present model	Error (%)		Mechab et al. (2010)	Present model	Error (%)		Mechab et al. (2010)	Present model	Error (%)
2	0.6993	0.7371	5.4078	8	0.3047	0.3256	6.8494	14.5	0.2890	0.2954	2.2366
2.5	0.5733	0.5943	3.6577	8.5	0.3084	0.3205	3.9481	15	0.2890	0.2944	1.8742
3	0.5213	0.5073	2.6791	9	0.2894	0.3155	9.0158	15.5	0.2890	0.2935	1.5713
3.5	0.4588	0.4623	0.7596	9.5	0.2878	0.3122	8.4606	16	0.2852	0.2927	2.6336
4	0.4068	0.4243	4.2961	10	0.2863	0.3089	7.8965	16.5	0.2702	0.2920	8.0825
4.5	0.3792	0.3967	4.6061	10.5	0.2847	0.3073	7.9414	17	0.2710	0.2911	7.4115
5	0.3552	0.3831	7.8684	11	0.2849	0.3075	7.9365	17.5	0.2711	0.2904	7.1533
5.5	0.3416	0.3660	7.1588	11.5	0.2794	0.3042	8.8869	18	0.2711	0.2898	6.8954
6	0.3385	0.3525	4.1308	12	0.2904	0.3026	4.1916	18.5	0.2711	0.2892	6.7014
6.5	0.3249	0.3494	7.5294	13	0.2889	0.2994	3.6275	19	0.2711	0.2887	6.5078
7	0.3161	0.3374	6.7402	13.5	0.2889	0.2977	3.0224	19.5	0.2711	0.2866	5.7344
7.5	0.3063	0.3306	7.9501	14	0.2890	0.2965	2.5994	20	0.2713	0.2795	3.0263

Table 2 Comparison of the nondimensional stress at the center of a rectangular plate for two aspect ratios

a/b	$\bar{x}_3 = \frac{x_3}{h}$	$\bar{\sigma}_y$			a/b	$\bar{x}_3 = \frac{x_3}{h}$	$\bar{\sigma}_y$		
		Mechab et al. (2010)	Present model	Error (%)			Mechab et al. (2010)	Present model	Error (%)
2	− 0.50	− 0.6522	− 0.6430	1.4061	3	− 0.50	− 0.3478	− 0.3390	2.5386
	− 0.40	− 0.5870	− 0.5790	1.3561		− 0.40	− 0.3261	− 0.3180	2.4809
	− 0.35	− 0.5435	− 0.5400	0.6403		− 0.35	− 0.2826	− 0.2740	3.0466
	− 0.25	− 0.4783	− 0.4690	1.9362		− 0.25	− 0.2609	− 0.2520	3.4002
	− 0.15	− 0.4130	− 0.4080	1.2202		− 0.15	− 0.2391	− 0.2260	5.4907
	0.00	− 0.2609	− 0.2570	1.4835		0.00	− 0.1304	− 0.1280	1.8706
	0.10	− 0.1739	− 0.1690	2.8233		0.10	− 0.0435	− 0.0410	5.7038
	0.25	0.2391	0.2270	5.0726		0.25	0.1739	0.1660	4.5483
	0.30	0.4348	0.4290	1.3294		0.30	0.3043	0.2920	4.0547
	0.35	0.6957	0.6880	1.0997		0.35	0.4783	0.4660	2.5635
	0.40	1.0000	1.0800	8.0000		0.40	0.6087	0.5940	2.4134
	0.45	1.5870	1.5740	0.8167		0.45	0.7826	0.7760	0.8433
	0.50	1.9565	1.9490	0.3844		0.50	1.0870	1.0710	1.4683

Mechab et al. (2010). To this aim, consider a rectangular plate with roller boundary conditions as follows (Fig. 4):

$$\begin{aligned} u_3(x, b) &= u_3(x, 0) = u_3(a, y) = u_3(0, y) \\ u_2(0, y) &= u_2(a, y) = 0 \\ u_1(x, b) &= u_1(x, 0) = 0. \end{aligned} \quad (45)$$

The Young's modulus of the plate is $E = 2 \times 10^{11}$ (Pa) and the poisson ratio $\nu = 0.3$. Transverse loading q is considered as follows:

$$q = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \quad (46)$$

The dimensionless formulation of deflection and stresses are as follows:

$$\bar{W} = \frac{10h^3 E}{a^4 q_0} u_3\left(\frac{a}{2}, \frac{b}{2}\right), \quad \bar{\sigma} = \frac{h}{a q_0} \sigma_i\left(\frac{a}{2}, \frac{b}{2}, \frac{h}{3}\right). \quad (47)$$

Table 1 reveals the nondimensional deflection at the center of a rectangular plate for various thicknesses. As

Fig. 5 Simply supported boundary condition of rectangular plate under uniform load

Table 3 Comparison of the nondimensional deflection at the center of a rectangular plate under distributed load

$\frac{q_0 a^4}{Dh} \times 10^{-6}$	$\bar{W}$			$\frac{q_0 a^2}{Dh} \times 10^{-6}$	$\bar{W}$		
	Shahidi et al. (2007)	Present model	Error (%)		Shahidi et al. (2007)	Present model	Error (%)
0	3.7975	3.7973	0.0045	10	28.0576	28.4550	1.4164
0.5	11.6429	11.8497	1.7762	12	28.8104	29.8691	3.6747
1	13.6655	14.0583	2.8744	14	29.8150	31.4102	5.3503
2	15.6881	16.1039	2.6504	16	30.8189	31.9304	3.6066
3	17.9613	18.1174	0.8691	20	32.3225	33.3278	3.1102
4	19.9832	20.6076	3.1246	25	35.3376	36.3831	2.9586
5	21.7513	22.1562	1.8615	30	37.3441	38.5839	3.3199
6	23.7720	24.2320	1.9350	35	39.8550	41.7791	4.8278
7	25.2862	25.5262	0.9491	40	42.1126	43.7732	3.9432
8	26.7999	26.8560	0.2093	50	47.3847	48.5269	2.4105

Fig. 6 Simply supported boundary condition of rectangular plate under concentrated load

Table 4 Comparison of the center deflection for a rectangular plate under concentrated load

$\frac{q_0 a^4}{Dh} \times 10^{-6}$	$\bar{W}$			$\frac{q_0 a^4}{Dh} \times 10^{-6}$	$\bar{W}$		
	Shahidi et al. (2007)	Present model	Error (%)		Shahidi et al. (2007)	Present model	Error (%)
1	17.0950	18.1126	5.9526	24	62.6816	63.5268	1.3484
1.5	22.7933	23.8332	4.5623	29	67.3743	67.607	0.3454
3.5	28.4916	29.3005	2.8391	35	71.3966	72.0926	0.9748
6	34.1899	35.6996	4.4156	43	76.4246	77.0712	0.8461
7.5	40.2235	41.1421	2.2837	50	80.1117	81.0354	1.1530
10	45.2514	46.4480	2.6443	60	85.4749	85.4461	0.0337
12	49.6089	49.9284	0.6440	70	89.162	89.8055	0.7217
15	53.6313	54.2571	1.1669	80	93.1844	93.9422	0.8132
19	57.9888	58.4119	0.7296	90	97.8771	98.1924	0.3221

Fig. 7 Comparison of the center deflection for rectangular plate under uniform load: (a) main figure; (b) better view, roller (R), fully clamped (C)

shown in this table, the presented numerical results are very close to exact solution.

Table 2 shows nondimensional stresses in x_2 direction at $x_1 = \frac{a}{2}$, $x_2 = \frac{b}{2}$. Two aspect ratios of $\frac{a}{b} = 2, 3$ are tested. As the results shown in Table 2, the stresses are in close agreement with literature.

Validation the numerical results of Cosserat theory

In this section, results of Cosserat numerical solution are compared for two examples. At the first example, consider a simply supported (SSSS) rectangular plate under uniform distributed load (Fig. 5).

In Table 3, $\bar{W}$ is as follows:

$$\bar{W} = \frac{1}{h} u_3 \left(\frac{a}{2}, \frac{b}{2} \right). \quad (48)$$

Cosserat theory gives acceptable results in very large deformation (Cosserat and Cosserat 1909). Table 3 shows

Fig. 8 Comparison of the membrane force at the center of rectangular plate under uniform load, roller (R), fully clamped (C)

the nondimensional deflection ($\bar{W}$) of the plate under uniform pressure. The pressure was increased to $\bar{W} = 49$. According to the revealed results in Table 3, there is a good agreement with the literature.

As the second example, consider a rectangular plate under concentrated load (Fig. 6). Table 4 shows nondimensional deflection at the center of the plate. According to Table 4, by presented numerical solution, deflection of the plate is very close to literature up to very large deformation.

Comparison of Von Karman and Cosserat theories from small to very large deformation

Nonlinear Von Karman theory results are acceptable for moderately large deformation (Ugural 1981), and noticeably, this theory is not very complex from mathematical point of view. Cosserat theory is a nonlinear advanced theory for simulation of plate behavior in very large deformation, but it is mathematically complicated. In this section, Von Karman and Cosserat theories are compared from small to very large deformation for rectangular and skew plates. This

comparison can give acceptable range of Von Karman non-linear theory for large deformation problems.

Rectangular plate under uniform pressure

Consider a rectangular plate with $\frac{a}{h} = 0.01$. The problem is solved for two boundary conditions: (1) all edges are roller and (2) fully clamped.

Figure 7 shows nondimensional deflection ($\bar{W}$) at the center of the plate using two theories. In addition, a better view of Fig. 7 is drawn. It can be seen from Fig. 7b that two theories are very close to each other until $\bar{W} = 3$ for fully clamped boundary condition and $\bar{W} = 5$ for roller boundary condition. As the compared results of membrane force shown in Fig. 8, for clamped case, membrane forces until $\frac{Na^2}{D \times 10^4} = 1.3$ are close to each other and for simply supported case up to $\frac{Na^2}{D \times 10^4} = 1.7$, the results of the two theories are matched. It can be seen from Fig. 8 that membrane forces of the two theories are matched up to a range of large deformation.

Fig. 9 Skew plate under uniform load with simply supported boundary condition

Fig. 10 Comparison of the central deflection of skew plate under uniform load with simply supported boundary condition: (a) main figure; (b) better view

Fig. 11 Comparison of the membrane force of skew plate under uniform load with simply supported boundary condition

Skew plate under uniform pressure

Consider a skew plate according to Fig. 9. In this section, two theories are compared for five skew plates by $\phi = 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$. The plate boundary condition is considered simply supported.

Figure 10 shows nondimensional deflection at the center of the plate versus nondimensional uniform pressure. Numerical solution was applied from small to very large deformation. The results showed that two theories are matched as follows:

For $\phi = 15^\circ$ up to $\bar{w} = 3$

For $\phi = 30^\circ$ up to $\bar{w} = 2.5$

For $\phi = 45^\circ$ up to $\bar{w} = 2$

For $\phi = 60^\circ$ up to $\bar{w} = 1$

For $\phi = 75^\circ$ up to $\bar{w} = 0.5$.

It is obvious that by increasing the skew angle, consistency of two theories decreases. In addition, Fig. 11 indicates nondimensional membrane force in x_1 direction versus nondimensional deflection at the center of the plate. This figure was drawn up to large deformation. According to Fig. 11, membrane force of two theories is matched as follows:

For $\phi = 15^\circ$ up to $\frac{\bar{N}a^2}{D \times 10^3} = 0.8$ For $\phi = 60^\circ$ up to $\frac{\bar{N}a^2}{D \times 10^3} = 0.5$

For $\phi = 30^\circ$ up to $\frac{\bar{N}a^2}{D \times 10^3} = 0.7$ For $\phi = 75^\circ$ up to $\frac{\bar{N}a^2}{D \times 10^3} = 0.3$

For $\phi = 45^\circ$ up to $\frac{\bar{N}a^2}{D \times 10^3} = 0.6$

Conclusions

In this study, two theories of HSDT (considering Von Karman) and Cosserat were compared up to large deformation ranges of the skew plates in simply supported and clamped boundary conditions for all edges. Whole domain method was applied for numerical solution. It is notable that according to the obtained results, Von Karman theory was simpler than Cosserat theory in formulation and computation. This comparison suggested acceptable range of Von Karman theory in very large deformation. Moreover, for the rectangular plates, the results obtained by the two theories matched when the maximum nondimensional deflection was approximately $\bar{w} = 5$ for simply supported boundary conditions and $\bar{w} = 3$ in clamped boundary conditions. Hence, Von Karman theory was valid for deflections greater than thickness up to a specific range for presented boundary condition. In addition, for the skew plates, consistency of two theories depended on skew angle. By increasing the skew angle, consistency of two theories decreases. The results illustrated that for plates with great skew angle, Von Karman theory deviated from Cosserat theory even in deflections smaller than thickness.

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References

- Amiri Rad A, Panahandeh-Shahraki D (2014) Buckling of cracked functionally graded plates under tension. *Thin Walled Struct* 84:26–33
- Cosserat E, Cosserat F (1909) *Théorie Des Corps Déformables*. Hermann, Paris
- Duc ND, Cong PH (2013) Nonlinear postbuckling of symmetric S-FGM plates resting on elastic foundations using higher order shear deformation plate theory in thermal environments. *Compos Struct* 100:566–574
- Duc ND, Tung HV (2011) Mechanical and thermal postbuckling of higher order shear deformable functionally graded plates on elastic foundations. *Compos Struct* 93(11):2874–2881
- Fazzolari FA, Banerjee JR, Boscolo M (2013) Buckling of composite plate assemblies using higher order shear deformation theory—an exact method of solution. *Thin Walled Struct* 71:18–34
- Ghassemi A, Shahidi A, Farzin M (2010) A new element for analyzing large deformation of thin Naghdi shell model. Part 1: Elastic. *Appl Math Model* 34(12):4267–4277
- Ghassemi A, Shahidi A, Farzin M (2011) A new element for analyzing large deformation of thin Naghdi shell model. Part II: Plastic. *Appl Math Model* 35(6):2650–2668
- Kumar JS, Reddy BS, Reddy CE (2011) Nonlinear bending analysis of functionally graded plates using higher order theory. *Int J Eng Sci Technol* 3(4):3010–3022

- Lee C-Y, Kim J-H (2014) Degradation of thermal postbuckling behaviors of functionally graded material in aero-hydrothermal environments. *Compos Struct* 118:228–233
- Mechab I, Atmane H, Tounsi A, Belhadj H, Bedia E (2010) A two variable refined plate theory for the bending analysis of functionally graded plates. *Acta Mech Sin* 26(6):941–949
- Neff P (2004) A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus. *Contin Mech Thermodyn* 16(6):577–628
- Nguyen-Xuan H, Thai CH, Nguyen-Thoi T (2013) Isogeometric finite element analysis of composite sandwich plates using a higher order shear deformation theory. *Compos B Eng* 55:558–574
- Ovesy HR, Taghizadeh M, Kharazi M (2012) Post-buckling analysis of composite plates containing embedded delaminations with arbitrary shape by using higher order shear deformation theory. *Compos Struct* 94(3):1243–1249
- Phung-Van P, Nguyen-Thoi T, Tran LV, Nguyen-Xuan H (2013) A cell-based smoothed discrete shear gap method (CS-DSG3) based on the C0-type higher-order shear deformation theory for static and free vibration analyses of functionally graded plates. *Comput Mater Sci* 79:857–872
- Reddy JN (1984) A simple higher order theory for laminated composite plates. *J Appl Mech* 51(4):745–752
- Sahmani S, Ansari R (2013a) On the free vibration response of functionally graded higher-order shear deformable microplates based on the strain gradient elasticity theory. *Compos Struct* 95:430–442
- Sahmani S, Ansari R (2013b) Size-dependent buckling analysis of functionally graded third-order shear deformable microbeams including thermal environment effect. *Appl Math Model* 37(23):9499–9515
- Sahmani S, Bahrami M, Ansari R (2014) Nonlinear free vibration analysis of functionally graded third-order shear deformable microbeams based on the modified strain gradient elasticity theory. *Compos Struct* 110:219–230
- Shahidi AR, Mahzoon M, Saadatpour MM, Azhari M (2007) Nonlinear static analysis of arbitrary quadrilateral plates in very large deflections. *Commun Nonlinear Sci Numer Simul* 12(5):832–848
- Shariyat M (2010a) A generalized high-order global–local plate theory for nonlinear bending and buckling analyses of imperfect sandwich plates subjected to thermo-mechanical loads. *Compos Struct* 92(1):130–143
- Shariyat M (2010b) Non-linear dynamic thermo-mechanical buckling analysis of the imperfect sandwich plates based on a generalized three-dimensional high-order global–local plate theory. *Compos Struct* 92(1):72–85
- Shi G (2007) A new simple third-order shear deformation theory of plates. *Int J Solids Struct* 44(13):4399–4417
- Simo JC (1993) On a stress resultant geometrically exact shell model. Part VII: shell intersections with 56-DOF finite element formulations. *Comput Methods Appl Mech Eng* 108(3–4):319–339
- Simo JC, Fox DD (1989) On a stress resultant geometrically exact shell model. Part I: formulation and optimal parametrization. *Comput Methods Appl Mech Eng* 72(3):267–304
- Simo JC, Kennedy JG (1992) On a stress resultant geometrically exact shell model. Part V. Nonlinear plasticity: formulation and integration algorithms. *Comput Methods Appl Mech Eng* 96(2):133–171
- Simo JC, Fox DD, Rifai MS (1989) On a stress resultant geometrically exact shell model. Part II: the linear theory; computational aspects. *Comput Methods Appl Mech Eng* 73(1):53–92
- Simo JC, Fox DD, Rifai MS (1990a) On a stress resultant geometrically exact shell model. Part III: computational aspects of the nonlinear theory. *Comput Methods Appl Mech Eng* 79(1):21–70
- Simo JC, Rifai MS, Fox DD (1990b) On a stress resultant geometrically exact shell model. Part IV: variable thickness shells with through-the-thickness stretching. *Comput Methods Appl Mech Eng* 81(1):91–126
- Swaminathan K, Naveenkumar DT (2014) Higher order refined computational models for the stability analysis of FGM plates—analytical solutions. *Eur J Mech A Solids* 47:349–361
- Tahoun V (2016) Using an equivalent continuum model for 3D dynamic analysis of nanocomposite plates. *Steel Compos Struct* 20(3):623–649
- Tahoun V, Naei MH (2016a) The effect of multi-directional nanocomposite materials on the vibrational response of thick shell panels with finite length and rested on two-parameter elastic foundations. *Int J Adv Struct Eng* 8(1):11–28
- Tahoun V, Naei MH (2016b) Free vibration and vibrational displacements analysis of thick elastically supported laminated curved panels with power-law distribution functionally graded layers and finite length via 2D GDQ method. *J Sandw Struct Mater* 18(3):263–293
- Ugural AC (1981) *Stresses in plates and shells*. McGraw-Hill, New York
- Yang J, Liew KM, Kitipornchai S (2006) Imperfection sensitivity of the post-buckling behavior of higher-order shear deformable functionally graded plates. *Int J Solids Struct* 43(17):5247–5266

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