A relatively short proof of Itô's formula for SPDEs and its applications

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Abstract We give a short proof of Itô's formula for stochastic Hilbert-space valued processes in the setting $V \subset H \subset V^*$ based on the possibility to lift the stochastic differentials, which are originally in V^* , into H . Using this result we also prove the maximum principle for second-order SPDEs in arbitrary domains.

Keywords Itô's formula · Maximum principle · Stochastic partial differential equations

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Itô's formula is one of the main tools in Stochastic Analysis and, in particular, in the theory of stochastic partial differential equations (SPDEs) of Itô type. Pardoux ([15]) was the first to consider the most general SPDEs with deterministic and stochastic terms containing the unknown function and its derivatives from an abstract point of view of stochastic Itô equations in the setting symbolically described as $V \subset H \subset V^*$, where H is a Hilbert space and V, V^* are Banach spaces (see [15] for references to previous results). One of the main steps in treating SPDEs consists of establishing Itô's formula for the square of the H -norm of solutions.

In the deterministic case (without any stochastic terms) we deal with a function v_t , that is in V for almost all $t > 0$ and its time derivative is in V^* , which is dual to V , for almost all $t > 0$ (cf. Remark 2.3). The goal is to show that there is a modification u_t of v_t which is an H -valued *continuous* function and $\|u_t\|_H^2$ is an absolutely continuous function admitting a natural formula for its time derivative. Even in this case and even if V, V^* are Hilbert spaces the formula is not completely trivial. For instance, the

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proof of Theorem 3 on p. 287 in [5] still has a tiny gap since the continuity of $\|u_t\|_H^2$ at $t = 0$ is not proved.

In the stochastic case the proof given in [15], Chap. 2, Second part, and [16] is rather involved and consists of many steps. In particular, it is based on the deterministic version of Itô's formula in Banach spaces with reference (see p. 30 in [15]) to [14] where in Remark 1.2 on p. 156 and Remark 7.9 on p. 236 one indeed finds the statement that the result is true but neither a proof or a reference to a proof is given. In [10] an approach not using the deterministic result was suggested for equations driven by continuous martingales. In contrast with the deterministic case or, for that matter, with [15] and [16], where the deterministic case was the starting point, the method of [10] is based on discretization of the time variable and some arithmetical manipulations. The method of [10] is generalized in [6] for the case of arbitrary cadlag martingales, which required a dramatic increase in what concerns arithmetics.

Most likely there is no simple proof of Itô's formula in the case of general Banach spaces and this makes it hard for a person interested in SPDEs to enter the area. On the other hand, the case that V , H , and V' are Hilbert spaces is the most common in applications and it turns out that in this case there is a simple proof presented here of the continuity of the process $\|u_t\|_H^2$ and of Itô's formula for it. The method seems to be much simpler than the ones previously known.

Basically, we lift du_t into H by using resolvent operators R_λ constructed in “Resolvent operator” section, write Itô's formula for Hilbert space-valued processes, and then pass to the limit as $\lambda \rightarrow \infty$ in “Itô's formula for the squared norm” section. In the end of “Itô's formula for the squared norm” section we give a version of Theorem 2.1 for SPDEs when V is a Sobolev-Hilbert space with integral numbers of derivatives. In “A more general Itô's formula” section we prove Itô's formula not for the squared norm in H but for more general functions as in [15, 16]. In our view the proofs given here are more straightforward than previously known. In the final “The maximum principle for second-order SPDEs” section we apply the result of “A more general Itô's formula” section to prove the maximum principle for the second-order SPDEs in arbitrary domains. To the best of the author's knowledge, in what concerns the arbitrariness of the domain and the structure of the equation this is the most general result known so far.

It seems that the maximum principle for general second order SPDEs was first proved in [9] (see also [11] for the case of random coefficients) for SPDEs in the whole space by the method of random characteristics introduced there and also in [13] (a particular case of the maximum principle appeared already in [15], p. 154). Later the method of random characteristics was used in many papers for various purposes, for instance, to prove smoothness of solutions (see, for instance, [1–3, 17] and the references therein). It was very tempting to try to use this method for proving the maximum principle for SPDEs in domains. However, the implementation of the method turns out to become extremely cumbersome and inconvenient if the coefficients of the equation are random processes. Also, it requires more regularity of solutions than actually needed.

In [8] a new method was suggested based on Itô's formula for $\|u_t^+\|_H^2$ which was derived by using mollifications in space variable. This required the underlying domain to be in C_{loc}^1 . In the present article the domain is an arbitrary open set. On the other

hand, the results of [8] are much more general in other respects. In particular, they were applied in [8] to prove the Hölder continuity of solutions up to the boundary and the results of the present article seem not to be applicable for this purpose. Still they can be applied in the same way as in [12] for investigating filtering problems in the situation of partially observable diffusion processes when the observations are only available until the unobservable component exits from a given domain. This would show that the result of [12] about filtering density before the exit time occurs is true in case of arbitrary domains.

Finally, we mention that there are many situations in which Itô's formula is known for Banach space valued processes. See, for instance, [4] and the references therein. These formulas could be more general in some respects but they do not cover our situation when the stochastic differential lives in V^* .

1 Resolvent operator

It is highly unlikely that the results of this section are new. However, the author's several attempts to find them in the literature failed and were abandoned for the reason that it takes less time to prove the results than to find them published somewhere. In addition, the proof only takes two pages.

Let V and H be two Hilbert spaces with scalar products and norms $(\cdot, \cdot)_V$, $\|\cdot\|_V$ and $(\cdot, \cdot)_H$, $\|\cdot\|_H$, respectively. Assume that $V \subset H$ and V is dense in H (in the metric of H) and $\|u\|_H \leq \|u\|_V$ for any $u \in V$.

The norm in V is obviously equivalent to

$$\left(\lambda \|u\|_H^2 + \|u\|_V^2 \right)^{1/2},$$

where $\lambda \geq 0$ is any fixed number. Then take an $f \in H$ and observe that the linear functional $(f, u)_H$ is bounded as a linear functional on V . By Riesz's representation theorem there exists a unique $v =: R_\lambda f \in V$ such that

$$(f, u)_H = \lambda (v, u)_H + (v, u)_V \quad \forall u \in V,$$

that is

$$(f, u)_H = \lambda (R_\lambda f, u)_H + (R_\lambda f, u)_V, \quad \forall u \in V, \quad (1.1)$$

$$(R_\lambda f, u)_V = ((1 - \lambda R_\lambda) f, u)_H, \quad \forall u \in V. \quad (1.2)$$

If $f, g \in H$, then (1.1) with $u = R_\lambda g$ reads

$$(f, R_\lambda g)_H = \lambda (R_\lambda f, R_\lambda g)_H + (R_\lambda f, R_\lambda g)_V,$$

where the right-hand side is symmetric in f, g . So is the left-hand side implying that R_λ is a symmetric operator in H . After that (1.1) shows that $(R_\lambda f, u)_V = (f, R_\lambda u)_V$ if $f, u \in V$, so that R_λ is also a symmetric operator in V .

Then observe that for $u = R_\lambda f$ Eq. (1.1) implies that

$$\lambda \|R_\lambda f\|_H^2 + \|R_\lambda f\|_V^2 = (f, R_\lambda f)_H \leq \|f\|_H \|R_\lambda f\|_H, \quad (1.3)$$

which yields the energy estimates

$$\lambda \|R_\lambda f\|_H \leq \|f\|_H, \quad \|R_\lambda f\|_V \leq \|f\|_H. \quad (1.4)$$

- Theorem 1.1** (i) *The norms of the operator λR_λ as an operator from H into H as well as an operator from V into V are less than one;*
(ii) *If $f \in H$, $\lambda \geq 0$, and $\lambda R_\lambda f = f$, then $f = 0$;*
(iii) *The set $R_\lambda H$ is dense in V in the metric of V ;*
(iv) *For any $f \in H$ we have*

$$\lim_{\lambda \rightarrow \infty} \|f - \lambda R_\lambda f\|_H = 0; \quad (1.5)$$

(v) *For $f \in V$ we have*

$$\lim_{\lambda \rightarrow \infty} \|f - \lambda R_\lambda f\|_V = 0. \quad (1.6)$$

Proof (i) We get the first part of the assertion from (1.4). Next, if $f \in V$, then for $u = f$ we get from (1.2) that

$$\begin{aligned} (R_\lambda f, f)_V &= ((1 - \lambda R_\lambda)f, (1 - \lambda R_\lambda)f)_H + ((1 - \lambda R_\lambda)f, \lambda R_\lambda f)_H \\ &= \|f - \lambda R_\lambda f\|_H^2 + (\lambda R_\lambda f, f)_H - \|\lambda R_\lambda f\|_H^2, \end{aligned}$$

where according to (1.3) we have $(\lambda R_\lambda f, f)_H - \|\lambda R_\lambda f\|_H^2 = \lambda \|R_\lambda f\|_V^2$, so that

$$\begin{aligned} (R_\lambda f, f)_V &= \|f - \lambda R_\lambda f\|_H^2 + \lambda \|R_\lambda f\|_V^2, \\ \lambda \|R_\lambda f\|_V^2 &\leq (R_\lambda f, f)_V, \quad \|\lambda R_\lambda f\|_V \leq \|f\|_V. \end{aligned} \quad (1.7)$$

- (ii) Under given conditions we have $f \in V$ and Eq. (1.1) implies that $(R_\lambda f, u)_V = 0$ for all $u \in V$. Hence $R_\lambda f = 0$ and $f = 0$ indeed.
(iii) Assume the contrary. Then there exists $u \in V$, $u \neq 0$, such that $(R_\lambda f, u)_V = 0$ for all $f \in H$. Then (1.1) shows that $(f, u)_H = (\lambda R_\lambda f, u)_H = (f, \lambda R_\lambda u)_H$ for all $f \in H$. It follows that $\lambda R_\lambda u = u$ and $u = 0$ by (ii), which is the desired contradiction.
(iv) If $f \in V$, this assertion follows from (1.7) after we let $\lambda \rightarrow \infty$, and use (i). In the general case it suffices to use the denseness of V in H and assertion (i).
(v) Owing to (iii) and (i) while proving (1.6) we may concentrate on $f = R_1 g$, where $g \in H$. Next, we observe that taking $u = (1 - \lambda R_\lambda)f$ in (1.2) leads to

$$(R_\lambda f, (1 - \lambda R_\lambda)f)_V = ((1 - \lambda R_\lambda)f, (1 - \lambda R_\lambda)f)_H,$$

which implies that

$$\begin{aligned}\|(1 - \lambda R_\lambda)f\|_V^2 &= (f, (1 - \lambda R_\lambda)f)_V - (\lambda R_\lambda f, (1 - \lambda R_\lambda)f)_V \\ &= (f, (1 - \lambda R_\lambda)f)_V - \lambda \|(1 - \lambda R_\lambda)f\|_H^2 \leq (f, (1 - \lambda R_\lambda)f)_V.\end{aligned}$$

Here, in light of (1.2), the last expression is

$$(R_1 g, (1 - \lambda R_\lambda)f)_V = ((1 - R_1)g, (1 - \lambda R_\lambda)f)_H,$$

which tends to zero as $\lambda \rightarrow \infty$ by (iv). The theorem is proved.

As a justification of the notation R_λ and its name as a resolvent operator consider the following situation.

Let G be an open set in $\mathbb{R}^d = \{x = (x^1, \dots, x^d) : x^i \in \mathbb{R}\}$ and let $m \geq 1$ be an integer. Define

$$H_2^m = \overset{0}{H}_2^m(G)$$

as the closure of $C_0^\infty = C_0^\infty(G)$ with respect to the norm

$$\|u\|_{H_2^m} := \left(\sum_{|\alpha| \leq m} \int_G |D^\alpha u|^2 dx \right)^{1/2},$$

where as usual for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$

$$D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \quad D_i = \partial / \partial x^i, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

The space $\overset{0}{H}_2^m$ is a Hilbert space with scalar product

$$(u, \phi)_{H_2^m} = \sum_{|\alpha| \leq m} \int_G (D^\alpha u) D^\alpha \phi dx.$$

We also use $L_2 = L_2(G)$, which is the space of square-integrable functions with norm and scalar product defined by

$$\|u\|_{L_2} = \left(\int_G |u|^2 dx \right)^{1/2}, \quad (u, v)_{L_2} = \int_G uv dx.$$

If above we take $H = L_2$ and $V = \overset{0}{H}_2^m$ then our hypotheses about V and H are satisfied and (1.1) becomes

$$\int_G f u dx = \lambda \int_G u R_\lambda f dx + \sum_{|\alpha| \leq m} \int_G (D^\alpha R_\lambda f) D^\alpha u dx,$$

which in the sense of generalized functions shows that $R_\lambda f$ is a solution of the equation

$$f = \lambda v - Lv,$$

where

$$L = - \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{2\alpha}.$$

Hence, R_λ is indeed a resolvent operator for L . By the way, this example shows that, generally, $R_\lambda H \neq V$.

2 Itô's formula for the squared norm

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ -fields $\mathcal{F}_t \subset \mathcal{F}$, which are complete with respect to $\mathcal{F}, P$.

In order to avoid unimportant complications we assume that $(V, (\cdot, \cdot)_V)$ is a separable Hilbert space, which is the case in many applications. Then $(H, (\cdot, \cdot)_H)$ is also separable. It is convenient that under this assumption there is no difference between weak and strong measurability.

Assume that we are given V -valued processes $v_t, v_t^*, t > 0$, which are predictable and satisfy

$$E \int_0^T \left(\|v_t\|_V^2 + \|v_t^*\|_V^2 \right) dt < \infty \quad (2.1)$$

for any $T \in (0, \infty)$. Also let $m_t, t \geq 0$, be an H -valued continuous martingale starting at the origin such that for any $s, t \geq 0$ with $s \leq t$ we have

$$\langle m \rangle_t - \langle m \rangle_s \leq t - s \quad (2.2)$$

(recall that in the theory of continuous H -valued martingales $\langle m \rangle_t$ is defined as a unique increasing continuous $\mathcal{F}_t$ -adapted process such that $|m_t|_H^2 - \langle m \rangle_t$ is a martingale).

The theory of integrating predictable Hilbert-space valued processes with respect to continuous same space-valued martingales is quite parallel to that in case the Hilbert space is just $\mathbb{R}^d$. This theory implies that under the above conditions the stochastic integral

$$h_t := \int_0^t (v_s, dm_s)_H \quad (2.3)$$

is well defined and is a continuous real-valued martingale with

$$\langle h \rangle_t \leq \int_0^t \|v_s\|_H^2 ds.$$

Suppose that v_0 is an H -valued $\mathcal{F}_0$ -measurable random vector. Finally, assume that for any $\phi \in V$ we have

$$(\phi, v_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v_s^*)_V ds + (\phi, m_t)_H \quad (2.4)$$

for almost all (ω, t) .

Theorem 2.1 *Under the above assumptions there exists a continuous H -valued $\mathcal{F}_t$ -adapted process u_t and a set $\Omega' \subset \Omega$ of full probability such that*

(i) $u_t = v_t$ for almost all (ω, t) , so that

$$E \int_0^T \|u_t\|_V^2 dt < \infty$$

for any $T \in (0, \infty)$,

(ii) for all $\omega \in \Omega'$, all $\phi \in V$, and all $t \geq 0$ we have

$$(\phi, u_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v_s^*)_V ds + (\phi, m_t)_H, \quad (2.5)$$

(iii) for all $\omega \in \Omega'$ and all $t \geq 0$ we have

$$\|u_t\|_H^2 = \|v_0\|_H^2 + 2 \int_0^t (u_s, v_s^*)_V ds + \langle m \rangle_t + 2 \int_0^t (v_s, dm_s)_H. \quad (2.6)$$

Proof Inspired by (1.2) for $n = 1, 2, \dots$ define $S_n = nR_n$ and

$$u_t^n = S_n v_0 + \int_0^t n(1 - S_n)v_s^* ds + S_n m_t. \quad (2.7)$$

Here the integral makes sense as the integral of an H -valued function. Furthermore, u_t^n is obviously continuous as an H -valued function.

Also observe that (2.4) with $\phi = S_n \psi$, $\psi \in H$, and (1.2) yield that for almost all (ω, t)

$$\begin{aligned} (\psi, S_n v_t)_H &= (\psi, S_n v_0)_H + \int_0^t (\psi, n(1 - S_n)v_s^*)_H ds \\ &+ (\psi, S_n m_t)_H = (\psi, u_t^n)_H. \end{aligned} \quad (2.8)$$

This and the separability of H shows that

$$u_t^n = S_n v_t$$

for almost all (ω, t) .

Next, it follows from Doob's inequality that for any $T \in [0, \infty)$

$$E \sup_{t \leq T} \|u_t^n\|_H^2 < \infty.$$

By Itô's formula for integrals of Hilbert-space valued processes we have (a.s.)

$$\begin{aligned} \|u_t^n\|_H^2 &= \|S_n v_0\|_H^2 + 2 \int_0^t (u_s^n, n(1 - S_n)v_s^*)_H ds \\ &\quad + \langle S_n m \rangle_t + 2 \int_0^t (S_n u_s^n, dm_s)_H, \end{aligned} \quad (2.9)$$

$$\begin{aligned} \|u_t^n - u_t^k\|_H^2 &= \|(S_n - S_k)v_0\|_H^2 + \langle (S_n - S_k)m \rangle_t + 2 \int_0^t (S_n u_s^n - S_k u_s^k, dm_s)_H \\ &\quad + 2 \int_0^t (u_s^n - u_s^k, [n(1 - S_n) - k(1 - S_k)]v_s^*)_H ds \end{aligned} \quad (2.10)$$

for all $t \geq 0$.

Observe that there is Ω' with $P(\Omega') = 1$ such that for any $\omega \in \Omega'$ we have

$$u_t^n = S_n v_t, \quad n = 1, 2, \dots, \quad v_t, v_t^* \in V \quad (2.11)$$

for almost all t . It follows that in the integrands in (2.9) and (2.10) we can replace u_s^n with $S_n v_s$ if $\omega \in \Omega'$ and use (1.2). Then for $\omega \in \Omega'$ and t such that (2.11) holds we have

$$\begin{aligned} (u_s^n - u_s^k, [n(1 - S_n) - k(1 - S_k)]v_s^*)_H &= (S_n v_s - S_k v_s, n(1 - S_n)v_s^*)_H \\ &\quad - (S_n v_s - S_k v_s, k(1 - S_k)v_s^*)_H \\ &= (S_n v_s - S_k v_s, S_n v_s^*)_V \\ &\quad - (S_n v_s - S_k v_s, S_k v_s^*)_V \\ &= (S_n v_s - S_k v_s, S_n v_s^* - S_k v_s^*)_V. \end{aligned}$$

Hence, for $\omega \in \Omega'$ and all $n, k \geq 1$ and $t \geq 0$ we get that

$$\begin{aligned} \|u_t^n\|_H^2 &= \|S_n v_0\|_H^2 + 2 \int_0^t (S_n v_s, v_s^*)_V ds \\ &\quad + \langle S_n m \rangle_t + 2 \int_0^t (S_n u_s^n, dm_s)_H, \\ \|u_t^n - u_t^k\|_H^2 &= \|(S_n - S_k)v_0\|_H^2 + \langle (S_n - S_k)m \rangle_t + 2 \int_0^t (S_n^2 v_s - S_k^2 v_s, dm_s)_H \\ &\quad + 2 \int_0^t (S_n v_s - S_k v_s, S_n v_s^* - S_k v_s^*)_V ds. \end{aligned}$$

Furthermore, by Doob's inequality for any $T \in [0, \infty)$

$$E \sup_{t \leq T} \|u_t^n - u_t^k\|_H^2 \leq I_{nk}^1 + 2I_{nk}^2 + I_{nk}^3 + 4(I_{nk}^4)^{1/2}, \quad (2.12)$$

where

$$\begin{aligned} I_{nk}^1 &= E \|(S_n - S_k)v_0\|_H^2 \\ I_{nk}^2 &= E \int_0^T |((S_n - S_k)v_s, (S_n - S_k)v_s^*)_V| ds, \\ I_{nk}^3 &= E \langle (S_n - S_k)m \rangle_T = E \|(S_n - S_k)m_T\|_H^2, \\ I_{nk}^4 &= E \int_0^T \|S_n^2 v_s - S_k^2 v_s\|_H^2 ds. \end{aligned}$$

By using the dominated convergence theorem, Theorem 1.1, and the inequality

$$|((S_n - S_k)v_s, (S_n - S_k)v_s^*)_V| \leq \|(S_n - S_k)v_s\|_V^2 + \|(S_n - S_k)v_s^*\|_V^2,$$

we easily conclude that $I_{nk}^1 + 2I_{nk}^2 + I_{nk}^3 \rightarrow 0$ as $n, k \rightarrow \infty$. Furthermore, $S_n^2 - S_k^2 = (S_n + S_k)(S_n - S_k)$ so that

$$I_{nk}^4 \leq 4E \int_0^T \|(S_n - S_k)v_s\|_H^2 ds,$$

which by the dominated convergence theorem implies that $I_{nk}^4 \rightarrow 0$ as $n, k \rightarrow \infty$ as well.

We now conclude from (2.12) that its left-hand side tends to zero which implies that there is a continuous H -valued process u_t such that

$$\lim_{n \rightarrow \infty} E \sup_{t \leq T} \|u_t^n - u_t\|_H^2 = 0.$$

Next, we show that u_t is the function we need. Of course, it is $\mathcal{F}_t$ -adapted. Furthermore,

$$E \int_0^T \|u_t^n - v_t\|_H^2 dt \leq E \int_0^T \|u_t^n - v_t\|_V^2 dt = E \int_0^T \|(S_n - 1)v_t\|_V^2 dt \rightarrow 0.$$

It follows that $u_t = v_t$ for almost all (ω, t) .

One easily obtains that for each t Eq. (2.6) holds with probability one by passing to the limit in (2.9). Since both parts of (2.6) are continuous in t , it holds on a set of full probability for all t .

Obviously (2.4) will hold for almost all (ω, t) if we replace v_t with u_t , that is, (2.5) holds for any $\phi \in V$ for almost all (ω, t) . The continuity of both parts of (2.5) with respect to t and $\phi \in V$ and the separability of V then imply that there is a set Ω' of full probability such that assertion (iii) holds.

The theorem is proved.

Remark 2.2 The reader understands, of course, that condition (2.1) can be replaced with the same condition but without expectation sign. This generalization is easily achieved by using appropriate stopping times.

Next in the setting described in the end of “Resolvent operator” section suppose that we are given an H_2^m -valued process v_t and L_2 -valued processes f_t^α , $|\alpha| \leq m$. We assume that all these processes are predictable and such that

$$E \int_0^T \left[\|v_t\|_{H_2^m}^2 + \sum_{|\alpha| \leq m} \|f_t^\alpha\|_{L_2}^2 \right] dt < \infty$$

for any $T \in (0, \infty)$. We also assume that we are given a continuous L_2 -valued martingale m_t satisfying (2.2) and v_0 is an L_2 -valued $\mathcal{F}_0$ -measurable random function. Finally, suppose that for any $\phi \in C_0^\infty$ we have

$$\int_G \phi v_t dx = \int_G \phi v_0 dx + \int_0^t \int_G \sum_{|\alpha| \leq m} f_s^\alpha D^\alpha \phi dx ds + \int_G \phi m_t dx \quad (2.13)$$

for almost all (ω, t) .

Remark 2.3 Formally (2.13) can be expressed as

$$dv_t = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_t^\alpha dt + dm_t,$$

which shows that dv_t lives in $H_2^{-m} := (H_2^m)^*$.

Theorem 2.4 Under the above assumptions there exists a continuous L_2 -valued $\mathcal{F}_t$ -adapted process u_t and a set $\Omega' \subset \Omega$ of full probability such that

(i) $u_t = v_t$ for almost all (ω, t) , so that

$$E \int_0^T \|u_t\|_{H_2^m}^2 dt < \infty$$

for any $T \in (0, \infty)$,

(ii) for all $\omega \in \Omega'$, all $\phi \in H_2^m$, and all $t \geq 0$ we have

$$(\phi, u_t)_{L_2} = (\phi, v_0)_{L_2} + \int_0^t \sum_{|\alpha| \leq m} (D^\alpha \phi, f_s^\alpha)_{L_2} ds + (\phi, m_t)_{L_2}, \quad (2.14)$$

(iii) for all $\omega \in \Omega'$ and all $t \geq 0$ we have

$$\|u_t\|_{L_2}^2 = \|v_0\|_{L_2}^2 + 2 \int_0^t \sum_{|\alpha| \leq m} (D^\alpha u_s, f_s^\alpha)_{L_2} ds + \langle m \rangle_t + 2 \int_0^t (v_s, dm_s)_{L_2}.$$

Proof To derive this result from Theorem 2.1, we first observe that in light of the denseness of C_0^∞ in $\overset{0}{H}_2^m$ Eq. (2.13) also holds for any $\phi \in \overset{0}{H}_2^m$. Then notice that for each (ω, s)

$$F_s(\phi) := \int_G \sum_{|\alpha| \leq m} f_s^\alpha D^\alpha \phi \, dx$$

is a bounded linear functional on $\overset{0}{H}_2^m$ with

$$|F_s(\phi)| \leq \|\phi\|_{\overset{0}{H}_2^m} \left(\sum_{|\alpha| \leq m} \|f_s^\alpha\|_{L_2}^2 \right)^{1/2}.$$

It follows by Riesz's representation theorem that there exists a unique $v_s^* \in \overset{0}{H}_2^m$ such that

$$F_s(\phi) = (\phi, v_s^*)_{\overset{0}{H}_2^m}, \quad \|v_s^*\|^2 \leq \sum_{|\alpha| \leq m} \|f_s^\alpha\|_{L_2}^2.$$

These relations imply that v_s^* is weakly predictable and, since $\overset{0}{H}_2^m$ is separable, it is (just) predictable. Also we have that condition (2.1) is satisfied. Hence one can rewrite (2.13) in form (2.4) and then all assertion of the present theorem follow directly from Theorem 2.1. The theorem is proved.

3 A more general Itô's formula

We suppose that all assumptions stated in “Itô's formula for the squared norm” section are satisfied.

Let $\phi(h)$ be a real-valued function on H . Assume that

- (i) For any $h, \xi \in H$ the functions $\phi(h + t\xi)$ is twice continuously differentiable as a function of t and the functions

$$\phi_{(\xi)}(h) := \frac{\partial}{\partial t} \phi(h + t\xi)|_{t=0}, \quad \phi_{(\xi)(\xi)}(h) := \frac{\partial^2}{(\partial t)^2} \phi(h + t\xi)|_{t=0}$$

are continuous as functions of $(h, \xi) \in H \times H$;

- (ii) For any $R \in (0, \infty)$ there exists a $K(R)$ such that for all $h, \xi \in H$ satisfying $\|h\|_H \leq R$ we have

$$|\phi_{(\xi)}(h)| \leq K(R)\|\xi\|_H, \quad |\phi_{(\xi)(\xi)}(h)| \leq K(R)\|\xi\|_H^2.$$

In this situation for any $h, \xi, \eta \in H$ the function $\Phi(t, s) := \phi(h + t\xi + s\eta)$ at all points (t, s) has partial derivatives with respect to t and s which are continuous

as functions of (t, s) . It follows from Calculus that $\Phi(t, s)$ is a differentiable function of (t, s) and for any numbers α, β

$$\frac{\partial}{\partial \varepsilon} \Phi(t + \alpha \varepsilon, s + \beta \varepsilon) \Big|_{\varepsilon=0} = \alpha \frac{\partial}{\partial t} \Phi(t, s) + \beta \frac{\partial}{\partial s} \Phi(t, s).$$

For $t = s = 0$ this yields

$$\phi_{(\alpha\xi+\beta\eta)}(h) = \alpha\phi_{(\xi)}(h) + \beta\phi_{(\eta)}(h).$$

Hence, the function $\phi_{(\xi)}(h)$ as a function of $\xi \in H$ is a continuous linear functional and by Riesz's representation theorem there exists an element $\phi_{(\cdot)}(h) \in H$ such that

$$\phi_{(\xi)}(h) = (\phi_{(\cdot)}(h), \xi)_H, \quad \|\phi_{(\cdot)}(h)\|_H \leq K(\|h\|_H).$$

Next, we assume that,

(iii) If $h \in V$, then $\phi_{(\cdot)}(h) \in V$ and

$$\|\phi_{(\cdot)}(h)\|_V \leq K_1(1 + \|h\|_V),$$

where K_1 is a fixed constant;

(iv) For any $v^* \in V$ the function $(\phi_{(\cdot)}(v), v^*)_V$ is a continuous function on V (in the metric of V).

Let $w_t^1, w_t^2, \dots$ be a finite or infinite sequence of independent Wiener processes on $(\Omega, \mathcal{F}, P)$, which are Wiener processes with respect to $\{\mathcal{F}_t\}$. We assume that we are given a sequence of predictable H -valued processes σ_t^i such that for any $T \in (0, \infty)$

$$\sum_k E \int_0^T \|\sigma_t^k\|_H^2 dt < \infty.$$

Under this assumption it is well known that the series

$$\sum_k \int_0^t \sigma_s^k dw_s^k$$

converges in H uniformly on finite time intervals in probability and we assume that the series converges to m_t . From the continuity of the scalar product in H it follows also that for any $h \in H$ we have (a.s.) for all t

$$(h, m_t)_H = \sum_k \int_0^t (h, \sigma_s^k)_H dw_s^k,$$

where the series converges uniformly on finite time intervals in probability. Then Eq. (2.14) is equivalent to saying that the function u_t satisfies

$$(\phi, u_t)_H = (\phi, v_0)_H + \int_0^t (\phi, v_s^*)_V ds + \sum_k \int_0^t (\phi, \sigma_s^k)_H dw_s^k \quad (3.1)$$

for each $\phi \in V$ (a.s.) for all t .

The following result can be found in [15], p. 65, in a more general situation. Our innovation is a different and shorter proof.

Theorem 3.1 *Under the above assumptions (a.s.) for all t*

$$\begin{aligned} \phi(u_t) &= \phi(u_0) + \sum_k \int_0^t \phi_{(\sigma_s^k)}(u_s) dw_s^k \\ &\quad + \int_0^t \left[(\phi_{(\cdot)}(u_s), v_s^*)_V + (1/2) \sum_k \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s) \right] ds, \end{aligned} \quad (3.2)$$

where u_t is taken from Theorem 2.1 and the series of stochastic integrals converges uniformly on finite time intervals in probability.

Proof The last assertion of the theorem follows from the fact that the series of quadratic variations of the stochastic integrals in (3.2) converges:

$$\begin{aligned} \sum_k \int_0^t |\phi_{(\sigma_s^k)}(u_s)|^2 ds &\leq \sum_k \int_0^t \|\phi_{(\cdot)}(u_s)\|_H^2 \|\sigma_s^k\|_H^2 ds \\ &\leq K^2 \left(\sup_{s \leq t} \|u_s\|_H \right) \sum_k \int_0^t \|\sigma_s^k\|_H^2 ds < \infty. \end{aligned}$$

It is also worth noting that other terms in (3.2) make sense as well. Indeed,

$$\begin{aligned} \int_0^t \sum_k |\phi_{(\sigma_s^k)(\sigma_s^k)}(u_s)| ds &\leq K^2 \left(\sup_{s \leq t} \|u_s\|_H \right) \sum_k \int_0^t \|\sigma_s^k\|_H^2 dt < \infty, \\ \int_0^t |(\phi_{(\cdot)}(u_s), v_s^*)_V| ds &\leq K_1 \int_0^t (1 + \|u_s\|_V) \|v_s^*\|_V ds < \infty. \end{aligned}$$

This argument shows that the right-hand side of (3.2) is a continuous process (a.s.). So is its left-hand side and, to prove that (3.2) holds (a.s.) for all t , it suffices to prove that (3.2) holds for each t (a.s.).

The rest of the proof we split into a few steps.

Step 1. Consider the case that $V = H$ and the number of the Wiener processes is finite, say, equal to p . Take an orthonormal basis $\{e_i\}$ in H , denote by Π_n the orthogonal projection operator on $\text{Span}\{e_1, \dots, e_n\}$, and set

$$u_t^n := \Pi_n u_t = \Pi_n u_0 + \int_0^t \Pi_n v_s^* ds + \sum_{k \leq p} \int_0^t \Pi_n \sigma_s^k dw_s^k, \quad \phi_n(h) = \phi(\Pi_n h).$$

The function ϕ_n , as a continuous function on a finite-dimensional Euclidean space, has two continuous directional derivatives in any direction. Therefore, it is twice continuously differentiable and by the classical Itô's formula

$$\begin{aligned} \phi(u_t^n) = \phi_n(u_t^n) = \phi(\Pi_n u_0) &+ \sum_{k \leq p} \int_0^t \phi_{(\Pi_n \sigma_s^k)}(\Pi_n u_s) dw_s^k \\ &+ \int_0^t \left[\phi_{(\Pi_n v_s^*)}(\Pi_n u_s) + (1/2) \sum_{k \leq p} \phi_{(\Pi_n \sigma_s^k)(\Pi_n \sigma_s^k)}(\Pi_n u_s) \right] ds. \end{aligned} \quad (3.3)$$

Here

$$|\phi_{(\Pi_n \sigma_s^k)(\Pi_n \sigma_s^k)}(\Pi_n u_s)| \leq \|\sigma_s^k\|_H^2 K \left(\max_{s \leq t} \|u_s\|_H \right)$$

and on an event of full probability on which u_s is an H -valued continuous function

$$\phi_{(\Pi_n \sigma_s^k)(\Pi_n \sigma_s^k)}(\Pi_n u_s) \rightarrow \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s)$$

for all s and k . It follows by the dominated convergence theorem that

$$\int_0^t \phi_{(\Pi_n \sigma_s^k)(\Pi_n \sigma_s^k)}(\Pi_n u_s) ds \rightarrow \int_0^t \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s) ds$$

for any t and k (a.s.). Similarly, for any t (a.s.)

$$\int_0^t \phi_{(\Pi_n v_s^*)}(\Pi_n u_s) ds \rightarrow \int_0^t \phi_{(v_s^*)}(u_s) ds.$$

Finally, by the same reasons as above

$$\int_0^t |\phi_{(\Pi_n \sigma_s^k)}(\Pi_n u_s) - \phi_{(\sigma_s^k)}(u_s)|^2 ds \rightarrow 0$$

for any t and k (a.s.).

This allows us to pass to the limit in (3.3) and conclude that

$$\begin{aligned} \phi(u_t) = \phi(u_0) &+ \sum_{k \leq p} \int_0^t \phi_{(\sigma_s^k)}(u_s) dw_s^k \\ &+ \int_0^t \left[\phi_{(v_s^*)}(u_s) + (1/2) \sum_{k \leq p} \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s) \right] ds \end{aligned} \quad (3.4)$$

for any t (a.s.).

Step 2. Again let $V = H$ but suppose that the number of the Wiener processes is infinite. Then introduce

$$u_t^n = u_0 + \int_0^t v_s^* ds + \sum_{k \leq n} \int_0^t \sigma_s^k dw_s$$

and observe that, as we pointed out before the theorem, for any (finite) $t, \varepsilon > 0$,

$$P(\sup_{s \leq t} \|u_s - u_s^n\|_H > \varepsilon) \rightarrow 0 \quad (3.5)$$

as $n \rightarrow \infty$. By the result of Step 1

$$\begin{aligned} \phi(u_t^n) &= \phi(u_0) + \sum_{k \leq n} \int_0^t \phi_{(\sigma_s^k)}(u_s^n) dw_s^k \\ &\quad + \int_0^t \left[\phi_{(v_s^*)}(u_s^n) + (1/2) \sum_{k \leq n} \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s^n) \right] ds \end{aligned} \quad (3.6)$$

for any t (a.s.).

Next, owing to (3.5) there is a subsequence $n(j) \rightarrow \infty$ as $j \rightarrow \infty$ such that for any $t \in (0, \infty)$ (a.s.)

$$\sup_{s \leq t} \|u_s - u_s^{n(j)}\|_H \rightarrow 0.$$

Then, of course,

$$\int_0^t \phi_{(v_s^*)}(u_s^{n(j)}) ds \rightarrow \int_0^t \phi_{(v_s^*)}(u_s) ds \quad (\text{a.s.}).$$

Furthermore, (a.s.)

$$\phi_{(\sigma_s^k)}(u_s^{n(j)}) \rightarrow \phi_{(\sigma_s^k)}(u_s)$$

because of the continuity of $\phi_{(\xi)}(h)$ on $H \times H$. In addition,

$$\sum_k |\phi_{\sigma_s^k}(u_s^{n(j)})|^2 \leq K^2 \left(\sup_{s \leq t, r \geq 1} \|u_s^{n(r)}\|_H \right) \sum_k \|\sigma_s^k\|_H^2 ds$$

and the right-hand side has a finite integral over $[0, t]$ (a.s.). It follows by the dominated convergence theorem that the quadratic variation at time t of the difference

$$\sum_{k \leq n} \int_0^t \phi_{(\sigma_s^k)}(u_s^{n(j)}) dw_s^k - \sum_k \int_0^t \phi_{(\sigma_s^k)}(u_s) dw_s^k$$

tends to zero (a.s.) as $k \rightarrow \infty$ and the difference itself goes to zero in probability.

For similar reasons

$$\sum_k \int_0^t |\phi_{(\sigma_s^k)(\sigma_s^k)}(u_s^{n(j)}) - \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s)| ds \rightarrow 0$$

(a.s.) and we conclude from (3.6) that (3.2) holds (a.s.).

Step 3. Now we consider the general case. As in the proof of Theorem 2.1 we introduce u_t^n by (2.7) and observe that the computation (2.8) shows that (a.s.) $u_t^n = S_n u_t$ for all t . According to Step 2 for any t (a.s.)

$$\begin{aligned} \phi(S_n u_t) &= \phi(S_n u_0) + \sum_k \int_0^t \phi_{(S_n \sigma_s^k)}(S_n u_s) dw_s^k \\ &\quad + \int_0^t \left[(\phi_{(\cdot)}(S_n u_s), n(1 - S_n)v_s^*)_H + (1/2) \sum_k \phi_{(S_n \sigma_s^k)(S_n \sigma_s^k)}(S_n u_s) \right] ds. \end{aligned} \quad (3.7)$$

Here (a.s.) for all s

$$\phi_{(S_n \sigma_s^k)}(S_n u_s) \rightarrow \phi_{(\sigma_s^k)}(u_s)$$

as $n \rightarrow \infty$ because of the continuity of $\phi_{(\xi)}(h)$ on $H \times H$. Furthermore,

$$|\phi_{(S_n \sigma_s^k)}(S_n u_s)| \leq K(\|u_s\|_H) \|S_n \sigma_s^k\|_H \leq K(\|u_s\|_H) \|\sigma_s^k\|_H.$$

As before this implies that the series of stochastic integrals in (3.7) converges to that in (3.2) in probability as $n \rightarrow \infty$.

Next, owing to (1.2) and the fact that $S_n u_s \in V$ and $\phi_{(\cdot)}(S_n u_s) \in V$

$$(\phi_{(\cdot)}(S_n u_s), n(1 - S_n)v_s^*)_H = (\phi_{(\cdot)}(S_n u_s), S_n v_s^*)_V.$$

With probability one $u_s \in V$ for almost all s for which also $\phi_{(\cdot)}(S_n u_s) \rightarrow \phi_{(\cdot)}(u_s)$ weakly in V , owing to assumption (iv), whereas $S_n v_s^* \rightarrow v_s^*$ strongly in V . Hence with probability one for almost all s

$$(\phi_{(\cdot)}(S_n u_s), n(1 - S_n)v_s^*)_H \rightarrow (\phi_{(\cdot)}(u_s), v_s^*)_V.$$

as $n \rightarrow \infty$. Furthermore,

$$|(\phi_{(\cdot)}(S_n u_s), n(1 - S_n)v_s^*)_H| = |(\phi_{(\cdot)}(S_n u_s), S_n v_s^*)_V| \leq K_1(1 + \|u_s\|_V) \|v_s^*\|_V$$

by assumption (iii). It follows by the dominated convergence theorem that (a.s.)

$$\int_0^t (\phi_{(\cdot)}(S_n u_s), n(1 - S_n)v_s^*)_H ds \rightarrow \int_0^t (\phi_{(\cdot)}(u_s), v_s^*)_V ds.$$

Finally, (a.s.) for all s

$$\phi_{(S_n \sigma_s^k)(S_n \sigma_s^k)}(S_n u_s) \rightarrow \phi_{(\sigma_s^k)(\sigma_s^k)}(u_s)$$

because of assumption (i) and

$$|\phi_{(S_n \sigma_s^k)(S_n \sigma_s^k)}(S_n u_s)| \leq K (\sup_{s \leq t} \|u_s\|_H) \|\sigma_s^k\|_H^2$$

in light of assumption (ii). This allows us to pass to the limit in the remaining expression in (3.7) and brings the proof of the theorem to an end.

4 The maximum principle for second-order SPDEs

In “A more general Itô’s formula” section take a domain $G \subset \mathbb{R}^d$, $V = \overset{0}{H}_2^1 = \overset{0}{H}_2^1(G)$, and $H = L_2 = L_2(G)$.

Take an infinitely differentiable function $r(x)$, $x \in \mathbb{R}$, such that $|r(x)| \leq N|x|^2$, $|r'(x)| \leq N|x|$, and $|r''| \leq N$, where N is a constant. For $h \in L_2$ define

$$\phi(h) = \int_G r(h(x)) dx.$$

As is easy to see, assumptions (i) and (ii) of “A more general Itô’s formula” section are satisfied and for $h, \xi \in L_2$

$$\begin{aligned} \phi_{(\xi)}(h) &= \int_G r'(h(x)) \xi(x) dx, \quad \phi_{(\cdot)}(h) = r'(h(x)), \\ \phi_{(\xi)(\xi)}(h) &= \int_G r''(h(x)) \xi^2(x) dx. \end{aligned}$$

Furthermore, if $h \in \overset{0}{H}_2^1$, then there exists a sequence of $h_n \in C_0^\infty$ such that $h_n \rightarrow h$ in the norm of $\overset{0}{H}_2^1$. Almost obviously $\phi_{(\cdot)}(h_n) = r'(h_n(x)) \in C_0^\infty$ and $r'(h_n(x)) \rightarrow r'(h(x))$ in the $\overset{0}{H}_2^1$ -norm. Hence $\phi_{(\cdot)}(h) \in \overset{0}{H}_2^1$ if $h \in \overset{0}{H}_2^1$. One can also easily verify that

$$\|\phi_{(\cdot)}(h)\|_{\overset{0}{H}_2^1} \leq N \|h\|_{\overset{0}{H}_2^1},$$

where N is the constant from above, so that assumption (iii) of “A more general Itô’s formula” section is satisfied as well. Finally, it is not hard to check that for $v^* \in \overset{0}{H}_2^1$

$$(r'(h), v^*)_{\overset{0}{H}_2^1} = \int_G r'(h(x)) v^*(x) dx + \sum_{|\alpha|=1} \int_G r''(h(x)) (D_i h(x)) D_i v^*(x) dx$$

is continuous as a function of h on the space $\overset{0}{H}_2^1$ and this is what is required in assumption (iv) of “A more general Itô’s formula” section.

By Theorem 3.1 we now conclude that

$$\begin{aligned} \int_G r(u_t) dx &= \int_G r(u_0) dx + \sum_k \int_0^t \int_G r'(u_s) \sigma_s^k dx dw_s^k \\ &\quad + \int_0^t \int_G r'(u_s) v_s^* dx ds + \sum_{|\alpha|=1} \int_0^t \int_G r''(u_s) (D_i u_s) D_i v_s^* dx ds \\ &\quad + (1/2) \int_0^t \int_G r''(u_s) \sum_k |\sigma_s^k|^2 dx ds. \end{aligned} \quad (4.1)$$

Next, we generalize this formula for r from a wider class. Denote by $\mathcal{R}$ the set of real-valued functions $r(x)$ on $\mathbb{R}$ such that

- (i) r is continuously differentiable, $r(0) = r'(0) = 0$,
- (ii) r' is absolutely continuous, its derivative r'' is bounded and left continuous, that is usual r'' which exists almost everywhere is bounded and there is a left-continuous function with which r'' coincides almost everywhere.

For $r \in \mathcal{R}$ by r'' we will always mean the left-continuous modification of the usual second-order derivative of r .

It turns out (see Remark 2.1 in [8]) that for any $r \in \mathcal{R}$ there exists a sequence $r_n \in \mathcal{R}$ of infinitely differentiable functions such that $|r_n(x)| \leq N|x|^2$, $|r'_n(x)| \leq N|x|$, and $|r''_n| \leq N$ with $N < \infty$ independent of $x \in \mathbb{R}$ and n , and $r_n, r'_n, r''_n \rightarrow r, r', r''$ on $\mathbb{R}$. By using this fact one easily shows that $r(u) \in \overset{0}{H}_2^1$ if $u \in \overset{0}{H}_2^1$ and (4.1) also holds for $r \in \mathcal{R}$.

In particular, we can apply (4.1) with $r(x) = (x^+)^2$ and by using the well-known fact that

$$D_i(u^+) = I_{u>0} D_i u \quad (4.2)$$

we then obtain that

$$\begin{aligned} \int_G (u_t^+)^2 dx &= \int_G (u_0^+)^2 dx + 2 \sum_k \int_0^t \int_G u_s^+ \sigma_s^k dx dw_s^k \\ &\quad + 2 \int_0^t \int_G u_s^+ v_s^* dx ds + 2 \sum_{i=1}^d \int_0^t \int_G (D_i u_s^+) D_i v_s^* dx ds \\ &\quad + \int_0^t I_{u_s>0} \sum_k |\sigma_s^k|^2 dx ds = \int_G (u_0^+)^2 dx + 2 \sum_k \int_0^t (u_s^+, \sigma_s^k)_{L_2} dw_s^k \\ &\quad + 2 \int_0^t (u_s^+, v_s^*)_{H_2^1} ds + \int_0^t I_{u_s>0} \sum_k |\sigma_s^k|^2 dx ds. \end{aligned} \quad (4.3)$$

Next, assume that instead of (3.1) we have that for any $\phi \in H_2^1$ (a.s.) for all t

$$\begin{aligned} (\phi, u_t) = & (\phi, u_0) + \int_0^t (\phi, \sigma_s^{ik} D_i u_s + v_s^k u_s)_{L_2} dw_s^k \\ & + \int_0^t \left[(D_i \phi, -a_s^{ij} D_j u_s - a_s^i u_s)_{L_2} + (\phi, b_s^i D_i u_s + c_s u_s + f_s)_{L_2} \right] ds, \end{aligned} \quad (4.4)$$

where the summation with respect to repeated indices is understood. We assume that $a_t^{ij}(x)$, $b_t^i(x)$, $a_t^i(x)$, $c_t(x)$, $f_t(x)$, $\sigma_t^{ik}(x)$, and $v_t^k(x)$ are real-valued functions defined for $i, j = 1, \dots, d$, $k = 1, 2, \dots$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$ and also depending on $\omega \in \Omega$.

Equation (4.4) is written frequently in a more transparent albeit informal way as

$$\begin{aligned} du_s = & (\sigma_s^{ik} D_i u_s + v_s^k u_s) dw_s^k \\ & + \left[D_j (a_s^{ij} D_j u_s + a_s^i u_s) + b_s^i D_i u_s + c_s u_s + f_s \right] ds. \end{aligned}$$

Assumption 4.1 For all values of the arguments

- (i) $\sigma^i := (\sigma^{i1}, \sigma^{i2}, \dots)$, $v := (v^1, v^2, \dots) \in \ell_2$;
- (ii) for all $\lambda \in \mathbb{R}^d$

$$(2a^{ij} - \alpha^{ij})\lambda^i \lambda^j \geq 0,$$

where $\alpha^{ij} = (\sigma^i, \sigma^j)_{\ell_2}$.

Assumption 4.1 (ii) is just the usual parabolicity assumption.

We need one more function $K_t \geq 0$ defined on $\Omega \times [0, \infty)$.

Assumption 4.2 (i) The functions $a_t^{ij}(x)$, $b_t^i(x)$, $a_t^i(x)$, $c_t(x)$, $\sigma_t^{ik}(x)$, $v_t^k(x)$, and $K(t)$ are measurable with respect to (ω, t, x) and $\mathcal{F}_t$ -adapted for each x ;

- (ii) the functions $a_t^{ij}(x)$, $b_t^i(x)$, $a_t^i(x)$, $c_t(x)$, and $|v_t(x)|_{\ell_2}$ are bounded;
- (iii) for each ω, t the functions

$$\eta_t^i := a_t^i - b_t^i - (\sigma_t^i, v_t)_{\ell_2}$$

are once continuously differentiable on D , have bounded derivatives, and satisfy

$$D_i \eta^i + 2c + |v|_{\ell_2}^2 \leq K \quad (4.5)$$

for all values of arguments;

- (iv) the process f_t , is L_2 -valued $\mathcal{F}_t$ -adapted and jointly measurable; and for all $T \in [0, \infty)$

$$E \int_0^T \left(\|f_s\|_{L_2}^2 + K_s \right) ds < \infty;$$

- (v) The function $u_t, t \geq 0$, is a predictable and continuous L_2 -valued function such that $u \in L_2((0, T) \times \Omega, \overset{0}{H}_2^1)$ for any $T \in (0, \infty)$.

Under these assumptions the stochastic integrals in (4.4) have exactly the same form as in (3.1) if in the latter we replace σ_s^k with

$$\sum_{i=1}^d \sigma_s^{ik} D_i u_s + v_s^k u_s, \quad (4.6)$$

for which

$$\sum_k \left| \sum_{i=1}^d \sigma_s^{ik} D_i u_s + v_s^k u_s \right|^2 = \alpha_s^{ij} (D_i u_s) D_j u_s + 2(\sigma_s^i, v_s)_{\ell_2} u_s D_i u_s + |v_s|_{\ell_2}^2 u_s^2. \quad (4.7)$$

The processes (4.6) are predictable L_2 -valued processes satisfying

$$\begin{aligned} E \sum_k \int_0^T \left\| \sum_i \sigma_s^{ik} D_i u_s + v_s^k u_s \right\|_{L_2}^2 ds &\leq NE \sum_i \int_0^T \int_G \alpha_s^{ii} (D_i u_s)^2 dx ds \\ &\quad + NE \int_0^T \int_G |v_s|_{\ell_2}^2 u_s^2 dx ds < \infty \end{aligned}$$

for any $T \in (0, \infty)$, where N are absolute constants.

At the first sight, the usual integral in (4.4) does not look like the one in (3.1). However, observe that on $A := \{(\omega, s) : u_s \in \overset{0}{H}_2^1\}$ the function

$$(D_i \phi, -a_s^{ij} D_j u_s - a_s^i u_s)_{L_2} + (\phi, b_s^i D_i u_s + c_s u_s + f_s)_{L_2}$$

as a function on $\overset{0}{H}_2^1$ is continuous. By Riesz's representation theorem there exists a unique v^* such that

$$I_A(D_i \phi, -a_s^{ij} D_j u_s - a_s^i u_s)_{L_2} + (\phi, b_s^i D_i u_s + c_s u_s + f_s)_{L_2} = (\phi, v_s^*)_{\overset{0}{H}_2^1}. \quad (4.8)$$

Since u_s is an L_2 -continuous process and $\overset{0}{H}_2^1$ is a Borel subset of L_2 , the set A is predictable. Also notice that $D_j u_s$ could be defined as the limits of finite differences. Hence, $I_A D_j u_s$ are also predictable and formula (4.8) (along with the separability of $\overset{0}{H}_2^1$) shows that v_s^* is an $\overset{0}{H}_2^1$ -valued predictable process. Furthermore, the absolute value of the left-hand side of (4.8) is obviously less than

$$NI_A \|\phi\|_{H_2^1} (\|u_s\|_{H_2^1} + \|f_s\|_{L_2}),$$

where N depends only on d and the sup norms of the coefficients. It follows that

$$E \int_0^T \|v_s^*\|_{H_2^1}^2 ds < \infty$$

for any T .

Summing up all the above comments we conclude that under Assumptions 4.1 and 4.2, if (4.4) is satisfied for any $\phi \in \overset{0}{H}_2^1$ (a.s.) for all t at once, then u_t satisfies (3.1) with σ and v^* specified above. We are not going to discuss the possibility of existence of such u_t , that is the existence of solutions of (4.4) in the class of functions u_t as in Assumption 4.2 (v). By the way, generally, such solutions may not even exist. For instance, if all the coefficients and f_s in (4.4) vanish identically, we have $u_t = u_0$ and, if $u_0 \notin \overset{0}{H}_2^1$, we do not have $u_t \in \overset{0}{H}_2^1$ for almost all (ω, t) .

Still it is probably worth mentioning that the reader interested in existence theorems in various settings and various spaces can consult [7] and the references therein. An initial piece of information is a result contained already in [15], p. 120, and [16] that if Assumptions 4.1 (i), 4.2 (i), (ii) are satisfied and for all values of the arguments and $\lambda \in \mathbb{R}^d$ we have

$$(2a^{ij} - \alpha^{ij})\lambda^i \lambda^j \geq \varepsilon |\lambda|^2,$$

where $\varepsilon > 0$ is a constant (uniform parabolicity condition), then there exists a unique predictable and continuous L_2 -valued function $u_t, t \in [0, T]$, such that $u \in L_2((0, T) \times \Omega, \overset{0}{H}_2^1)$ and for any $\phi \in \overset{0}{H}_2^1$ (a.s.) for all $t \in [0, T]$ at once (4.4) holds true. Actually, in [15] and [16] more general equations are considered when a free term in front of dw^k is allowed.

A result similar to the following theorem in a different setting can be found already in [15], p. 154 (see also the references therein).

Theorem 4.3 (*maximum principle*). *Let Assumptions 4.1 and 4.2 be satisfied and let (4.4) be satisfied for any $\phi \in \overset{0}{H}_2^1$ (a.s.) for all t at once. Suppose that $u_0 \leq 0$ and $f_t \leq 0$ for almost all (ω, t) . Then (a.s.) for all t we have $u_t \leq 0$.*

Proof According to what has been explained before the theorem and formulas (4.3), (4.7), and (4.8) we have that (a.s.) for all t

$$\begin{aligned} \|u_t^+\|_{L_2} &= M_t + 2 \int_0^t \left[(D_i u_s^+, -a_s^{ij} D_j u_s - a_s^i u_s)_{L_2} + (u_s^+, b_s^i D_i u_s + c_s u_s + f_s)_{L_2} \right] ds \\ &\quad + \int_0^t \int_G I_{u_t > 0} \left[\alpha_s^{ij} (D_i u_s) D_j u_s + 2(\sigma_s^i, v_s)_{\ell_2} u_s D_i u_s + |v_s|_{\ell_2}^2 u_s^2 \right] dx ds \end{aligned} \quad (4.9)$$

where M_t is a martingale.

According to (4.2)

$$\begin{aligned}(a_s^{ij}(D_i u_s^+), D_j u_s)_{L_2} &= (a_s^{ij}(D_i u_s^+), D_j u_s^+)_{L_2}, \\ (I_{u_t > 0} \alpha_s^{ij}(D_i u_s), D_j u_s)_{L_2} &= (\alpha_s^{ij}(D_i u_s^+), D_j u_s^+)_{L_2},\end{aligned}$$

so that

$$-2(a_s^{ij}(D_i u_s^+), D_j u_s)_{L_2} + (I_{u_t > 0} \alpha_s^{ij}(D_i u_s), D_j u_s)_{L_2} \leq 0$$

in light of Assumption 4.1 (ii). Furthermore, at points where $u_s^+ \in \overset{0}{H}_2^1$, we have

$$\begin{aligned}I_s &:= -(D_i u_s^+, a_s^i u_s)_{L_2} + (u_s^+, b_s^i D_i u_s)_{L_2} + ((\sigma_s^i, v_s)_{\ell_2} u_s^+, D_i u_s)_{L_2} \\ &= -(\eta_s^i u_s^+, D_i u_s^+)_{L_2} = (D_i \eta_s^i u_s^+, u_s^+)_{L_2},\end{aligned}$$

where the last equality is obtained by integrating by parts, which is justified by approximating $u_s^+ \in \overset{0}{H}_2^1$ by C_0^∞ -functions and passing to the limit. At this point the reader can understand that, actually, we only need η_t^i to be Lipschitz continuous rather than continuously differentiable. In any case by also observing that $u_s^+ f_s \leq 0$ and using Assumption 4.2 (iii) we conclude from (4.9) that (a.s.)

$$\begin{aligned}d\|u_t^+\|_{L_2} &\leq dM_t + K_t \|u_t^+\|_{L_2}, \\ d\left[\|u_t^+\|_{L_2} \exp\left(-\int_0^t K_s ds\right)\right] &\leq \exp\left(-\int_0^t K_s ds\right) dM_t, \\ \|u_t^+\|_{L_2} \exp\left(-\int_0^t K_s ds\right) &\leq \int_0^t \exp\left(-\int_0^s K_r dr\right) dM_r.\end{aligned}$$

In the last relation the left-hand side is nonnegative and the right-hand side is a martingale starting from zero. It follows that, with probability one, the martingale is zero and so is $\|u_t^+\|_{L_2}$ which proves the theorem.

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