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Conformable fractional approximations by max-product operators using convexity

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Abstract Here, we consider the approximation of functions by a large variety of max-product operators under conformable fractional differentiability and using convexity. These are positive sublinear operators. Our study relies on our general results about positive sublinear operators. We derive Jackson-type inequalities under conformable fractional initial conditions and convexity. So our approach is quantitative by obtaining inequalities where their right hand sides involve the modulus of continuity of a high-order conformable fractional derivative of the function under approximation. Due to the convexity assumptions, our inequalities are compact and elegant with small constants.

Mathematics Subject Classification 26A33 · 41A17 · 41A25 · 41A36

المخلص

هنا نعتبر تقريب الدوال بمجموعة متنوعة من مؤثرات الضرب الأعظمي تحت المشتقات الكسرية المريحة وباستعمال التحذب. تكون هذه المؤثرات دون خطية موجبة. فدراستنا تعتمد على نتائجنا العامة حول المؤثرات دون الخطية الموجبة. نشق بعض المتراجحات من نوع جاكسون تحت شروط أولية لمشتقات كسرية مريحة وتحذب. فمقاربتنا كمية وذلك بالحصول على متراجحات ذات حدود يميني تضم معاملات استمرار برتب عالية لمشتقات كسرية مريحة للدوال الخاضعة للتقريب. وبسبب فرضية التحذب، فإن متراجحاتنا متراصة وأنيقة وبثوابت صغيرة.

1 Background

In this article, we study under convexity quantitatively the conformable fractional approximation properties of max-product operators to the unit. These are special cases of positive sublinear operators. We first present results regarding the convergence to the unit of general positive sublinear operators under convexity. The focus of our study is approximation under the presence of conformable fractional smoothness.

Under our convexity conditions, the derived conformable fractional convergence inequalities are elegant and compact with very small constants.

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Our work is inspired by [5].

We make

Remark 1.1 Let $x, y \in [a, b] \subseteq [0, \infty)$, and $g(x) = x^\alpha$, $0 < \alpha \leq 1$. Then $g'(x) = \alpha x^{\alpha-1} = \frac{\alpha}{x^{1-\alpha}}$, for $x \in (0, \infty)$. Since $a \leq x \leq b$, $\frac{1}{x} \geq \frac{1}{b} > 0$ and $\frac{\alpha}{x^{1-\alpha}} \geq \frac{\alpha}{b^{1-\alpha}} > 0$.

Assume $y > x$. By the mean value theorem, we get

$$y^\alpha - x^\alpha = \frac{\alpha}{\xi^{1-\alpha}} (y - x), \quad \text{where } \xi \in (x, y). \quad (1)$$

A similar to (1) equality when $x > y$ is true.

Then we obtain

$$\frac{\alpha}{b^{1-\alpha}} |y - x| \leq |y^\alpha - x^\alpha| = \frac{\alpha}{\xi^{1-\alpha}} |y - x|. \quad (2)$$

Thus, it holds that

$$\frac{\alpha}{b^{1-\alpha}} |y - x| \leq |y^\alpha - x^\alpha|. \quad (3)$$

Hence, we get

$$|y - x| \leq \frac{b^{1-\alpha}}{\alpha} |y^\alpha - x^\alpha|, \quad (4)$$

$\forall x, y \in [a, b] \subset [0, \infty), \alpha \in (0, 1]$.

We also make

Remark 1.2 For $0 < \alpha \leq 1, x, y, t, s \geq 0$, we have

$$2^{\alpha-1} (x^\alpha + y^\alpha) \leq (x + y)^\alpha \leq x^\alpha + y^\alpha. \quad (5)$$

Assume that $t > s$. Then

$$t = t - s + s \Rightarrow t^\alpha = (t - s + s)^\alpha \leq (t - s)^\alpha + s^\alpha,$$

and hence $t^\alpha - s^\alpha \leq (t - s)^\alpha$.

Similarly, $s > t \Rightarrow s^\alpha - t^\alpha \leq (s - t)^\alpha$.

Therefore it holds that

$$|t^\alpha - s^\alpha| \leq |t - s|^\alpha, \quad \forall t, s \in [0, \infty). \quad (6)$$

We need

Definition 1.3 [6, 7] Let $f: [0, \infty) \rightarrow \mathbb{R}$. The conformable α -fractional derivative for $\alpha \in (0, 1]$ is given by

$$D_\alpha f(t) := \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad (7)$$

$$D_\alpha f(0) = \lim_{t \rightarrow 0+} D_\alpha f(t). \quad (8)$$

If f is differentiable, then

$$D_\alpha f(t) = t^{1-\alpha} f'(t), \quad (9)$$

where f' is the usual derivative.

We define

$$D_\alpha^n f = D_\alpha^{n-1} (D_\alpha f) \quad \text{and} \quad D_\alpha^0 f = f. \quad (10)$$

If $f: [0, \infty) \rightarrow \mathbb{R}$ is α -differentiable at $t_0 > 0, \alpha \in (0, 1]$, then f is continuous at t_0 ; see [7].

We will use

Theorem 1.4 (See [4]) (Taylor formula) *Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose f is $(n + 1)$ times conformable α -fractional differentiable on $[0, \infty)$, and $s, t \in [0, \infty)$, and $D_\alpha^{n+1} f$ is assumed to be continuous on $[0, \infty)$. Then we have*

$$f(t) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) \tau^{\alpha-1} d\tau. \quad (11)$$

The case $n = 0$ follows.

Corollary 1.5 [3] *Let $\alpha \in (0, 1]$. Suppose f is α -fractional differentiable on $[0, \infty)$, and $s, t \in [0, \infty)$. Assume that $D_\alpha f$ is continuous on $[0, \infty)$. Then,*

$$f(t) = f(s) + \int_s^t D_\alpha f(\tau) \tau^{\alpha-1} d\tau. \quad (12)$$

Note 1.6 *Theorem 1.4 and Corollary 1.5 are also true for $f: [a, b] \rightarrow \mathbb{R}$, $[a, b] \subseteq [0, \infty)$, $s, t \in [a, b]$.*

We need

Definition 1.7 Let $f \in C([a, b])$. We define the first modulus of continuity of f as:

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b]: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \quad (13)$$

2 Main results

We give

Theorem 2.1 *Let $\alpha \in (0, 1]$ and $n \in \mathbb{Z}_+$. Suppose f is $(n + 1)$ times conformable α -fractional differentiable on $[a, b] \subseteq [0, \infty)$, and $t \in [a, b]$, $x_0 \in (a, b)$, and $D_\alpha^{n+1} f$ is continuous on $[a, b]$. Let $0 < h \leq \min(x_0 - a, b - x_0)$ and assume $|D_\alpha^{n+1} f|$ is convex over $[a, b]$. Furthermore, assume that $D_\alpha^k f(x_0) = 0$, $k = 1, \dots, n + 1$. Then,*

$$|f(t) - f(x_0)| \leq \left(\frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{(n+2)! \alpha^{n+2} h} \right) |t - x_0|^{(n+2)\alpha}, \quad (14)$$

$\forall t \in [a, b]$.

Proof We have that

$$\frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(s) \tau^{\alpha-1} d\tau = \frac{D_\alpha^{n+1} f(s)}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n \tau^{\alpha-1} d\tau \quad (15)$$

(by $\frac{d\tau^\alpha}{d\tau} = \alpha \tau^{\alpha-1} \Rightarrow d\tau^\alpha = \alpha \tau^{\alpha-1} d\tau \Rightarrow \frac{1}{\alpha} d\tau^\alpha = \tau^{\alpha-1} d\tau$)

$$= \frac{D_\alpha^{n+1} f(s)}{\alpha^{n+1} n!} \int_s^t (t^\alpha - \tau^\alpha)^n d\tau^\alpha$$

(by $t \leq \tau \leq s \Rightarrow t^\alpha \leq \tau^\alpha (=z) \leq s^\alpha$)

$$\begin{aligned} &= \frac{D_\alpha^{n+1} f(s)}{\alpha^{n+1} n!} \int_{s^\alpha}^{t^\alpha} (t^\alpha - z)^n dz = \frac{D_\alpha^{n+1} f(s)}{\alpha^{n+1} n!} \frac{(t^\alpha - s^\alpha)^{n+1}}{n+1} \\ &= \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{n+1}. \end{aligned} \quad (16)$$

Therefore it holds that

$$\frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(s) \tau^{\alpha-1} d\tau = \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{n+1}. \quad (17)$$

By (11) and (12), we get:

$$f(t) = \sum_{k=0}^{n+1} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n (D_\alpha^{n+1} f(\tau) - D_\alpha^{n+1} f(s)) \tau^{\alpha-1} d\tau. \quad (18)$$

In particular, it holds that

$$f(t) = \sum_{k=0}^{n+1} \frac{1}{k!} \left(\frac{t^\alpha - x_0^\alpha}{\alpha} \right)^k D_\alpha^k f(x_0) + \frac{1}{n!} \int_{x_0}^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n (D_\alpha^{n+1} f(\tau) - D_\alpha^{n+1} f(x_0)) \tau^{\alpha-1} d\tau. \quad (19)$$

By the assumption $D_\alpha^k f(x_0) = 0, k = 1, \dots, n+1$, we can write

$$f(t) - f(x_0) = \frac{1}{n!} \int_{x_0}^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n (D_\alpha^{n+1} f(\tau) - D_\alpha^{n+1} f(x_0)) \tau^{\alpha-1} d\tau. \quad (20)$$

By assumption here $|D_\alpha^{n+1} f|$ is convex over $[a, b] \subseteq [0, \infty)$; $x_0 \in (a, b)$.

Let $h: 0 < h \leq \min(x_0 - a, b - x_0)$, by Lemma 8.1.1, p. 243 of [1] we get that

$$|D_\alpha^{n+1} f(t) - D_\alpha^{n+1} f(x_0)| \leq \frac{\omega_1(D_\alpha^{n+1} f, h)}{h} |t - x_0|, \quad (21)$$

$\forall t \in [a, b]$.

Using (4), we obtain

$$|D_\alpha^{n+1} f(t) - D_\alpha^{n+1} f(x_0)| \leq \frac{\omega_1(D_\alpha^{n+1} f, h)}{h} \frac{b^{1-\alpha}}{\alpha} |t^\alpha - x_0^\alpha|, \quad (22)$$

$\forall t \in [a, b]$.

Next, we estimate (20).

1. We observe that ($t \geq x_0$)

$$|f(t) - f(x_0)| \stackrel{(20)}{\leq} \frac{1}{n!} \int_{x_0}^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n |D_\alpha^{n+1} f(\tau) - D_\alpha^{n+1} f(x_0)| \tau^{\alpha-1} d\tau \stackrel{(22)}{\leq} \frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{n! h \alpha^{n+2}} \int_{x_0}^t (t^\alpha - \tau^\alpha)^n (\tau^\alpha - x_0^\alpha) d\tau^\alpha = \quad (23)$$

$$\begin{aligned} & \left(\frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{n! h \alpha^{n+2}} \right) \int_{x_0^\alpha}^{t^\alpha} (t^\alpha - z)^{(n+1)-1} (z - x_0^\alpha)^{2-1} dz = \\ & \left(\frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{n! h \alpha^{n+2}} \right) \frac{\Gamma(n+1) \Gamma(2)}{\Gamma(n+3)} (t^\alpha - x_0^\alpha)^{n+2} = \\ & \left(\frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{n! h \alpha^{n+2}} \right) \frac{n!}{(n+2)!} (t^\alpha - x_0^\alpha)^{n+2} = \\ & \left(\frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{(n+2)! h \alpha^{n+2}} \right) (t^\alpha - x_0^\alpha)^{n+2}. \end{aligned} \quad (24)$$

We have proved that (case of $t \geq x_0$)

$$|f(t) - f(x_0)| \leq \left(\frac{\omega_1(D_\alpha^{n+1} f, h) b^{1-\alpha}}{(n+2)! h \alpha^{n+2}} \right) (t^\alpha - x_0^\alpha)^{n+2}. \quad (25)$$

2. We observe that $(t \leq x_0)$

$$|f(t) - f(x_0)| \stackrel{(20)}{=} \frac{1}{n!} \left| \int_{x_0}^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n (D_{\alpha}^{n+1} f(\tau) - D_{\alpha}^{n+1} f(x_0)) \tau^{\alpha-1} d\tau \right| =$$

$$\frac{1}{n! \alpha} \left| \int_t^{x_0} \left(\frac{\tau^\alpha - t^\alpha}{\alpha} \right)^n (D_{\alpha}^{n+1} f(\tau) - D_{\alpha}^{n+1} f(x_0)) d\tau^\alpha \right| \leq \quad (26)$$

$$\frac{1}{n! \alpha^{n+1}} \int_t^{x_0} (\tau^\alpha - t^\alpha)^n |D_{\alpha}^{n+1} f(\tau) - D_{\alpha}^{n+1} f(x_0)| d\tau^\alpha \stackrel{(22)}{\leq}$$

$$\frac{1}{n! \alpha^{n+1}} \left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{h} \right) \int_t^{x_0} (x_0^\alpha - \tau^\alpha) (\tau^\alpha - t^\alpha)^n d\tau^\alpha =$$

$$\left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{n! \alpha^{n+2} h} \right) \int_{t^\alpha}^{x_0^\alpha} (x_0^\alpha - z)^{2-1} (z - t^\alpha)^{(n+1)-1} dz =$$

$$\left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{n! \alpha^{n+2} h} \right) \frac{\Gamma(2) \Gamma(n+1)}{\Gamma(n+3)} (x_0^\alpha - t^\alpha)^{n+2} =$$

$$\left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{n! \alpha^{n+2} h} \right) \frac{n!}{(n+2)!} (x_0^\alpha - t^\alpha)^{n+2} =$$

$$\left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{(n+2)! \alpha^{n+2} h} \right) (x_0^\alpha - t^\alpha)^{n+2}.$$

We have proved $(t \leq x_0)$ that

$$|f(t) - f(x_0)| \leq \left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{(n+2)! \alpha^{n+2} h} \right) (x_0^\alpha - t^\alpha)^{n+2}. \quad (28)$$

In conclusion, we have established that

$$|f(t) - f(x_0)| \leq \left(\frac{\omega_1(D_{\alpha}^{n+1} f, h) b^{1-\alpha}}{(n+2)! \alpha^{n+2} h} \right) |t^\alpha - x_0^\alpha|^{n+2}, \quad \forall t \in [a, b]. \quad (29)$$

By (6), we have

$$|t^\alpha - x_0^\alpha| \leq |t - x_0|^\alpha. \quad (30)$$

Thus by (29) and (30), the claim is proved. $\square$

We rewrite the statement of Theorem 2.1 in a convenient way as follows:

Theorem 2.2 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose f is n times conformable α -fractional differentiable on $[a, b] \subseteq [0, \infty)$, and $x \in (a, b)$, and $D_{\alpha}^n f$ is continuous on $[a, b]$. Let $0 < h \leq \min(x - a, b - x)$ and assume $|D_{\alpha}^n f|$ is convex over $[a, b]$. Furthermore, assume that $D_{\alpha}^k f(x) = 0$, $k = 1, \dots, n$. Then, over $[a, b]$, we have

$$|f(\cdot) - f(x)| \leq \left(\frac{\omega_1(D_{\alpha}^n f, h) b^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) |\cdot - x|^{(n+1)\alpha}. \quad (31)$$

We need

Definition 2.3 Here, $C_+([a, b]) := \{f: [a, b] \subseteq [0, \infty) \rightarrow \mathbb{R}_+, \text{ continuous functions}\}$. Let $L_N: C_+([a, b]) \rightarrow C_+([a, b])$, operators, $\forall N \in \mathbb{N}$, such that

(i)

$$L_N(\alpha f) = \alpha L_N(f), \quad \forall \alpha \geq 0, \forall f \in C_+([a, b]), \quad (32)$$

(ii) if $f, g \in C_+([a, b]) : f \leq g$, then

$$L_N(f) \leq L_N(g), \quad \forall N \in \mathbb{N}, \quad (33)$$

(iii)

$$L_N(f + g) \leq L_N(f) + L_N(g), \quad \forall f, g \in C_+([a, b]). \quad (34)$$

We call $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators.

We need a Hölder's type inequality; see next:

Theorem 2.4 (See [2]) Let $L : C_+([a, b]) \rightarrow C_+([a, b])$, be a positive sublinear operator and $f, g \in C_+([a, b])$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L((f(\cdot))^p)(s_*)$, $L((g(\cdot))^q)(s_*) > 0$ for some $s_* \in [a, b]$. Then

$$L(f(\cdot)g(\cdot))(s_*) \leq (L((f(\cdot))^p)(s_*))^{\frac{1}{p}} (L((g(\cdot))^q)(s_*))^{\frac{1}{q}}. \quad (35)$$

We make

Remark 2.5 By [5, p. 17], we get: let $f, g \in C_+([a, b])$. Then

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f - g|)(x), \quad \forall x \in [a, b] \subseteq [0, \infty). \quad (36)$$

Furthermore, we also have that

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x) + |f(x)| |L_N(e_0)(x) - 1|, \quad (37)$$

$\forall x \in [a, b] \subseteq [0, \infty); e_0(t) = 1$.

From now on, we assume that $L_N(1) = 1$. Hence, it holds that

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x), \quad \forall x \in [a, b] \subseteq [0, \infty). \quad (38)$$

We give

Theorem 2.6 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([a, b])$ is n times conformable α -fractional differentiable on $[a, b] \subseteq [0, \infty)$, and $x \in (a, b)$, and $D_\alpha^n f$ is continuous on $[a, b]$. Let $0 < h \leq \min(x - a, b - x)$ and assume $|D_\alpha^n f|$ is convex over $[a, b]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Let $\{L_N\}_{N \in \mathbb{N}}$ from $C_+([a, b])$ into itself, positive sublinear operators such that: $L_N(1) = 1$, $\forall N \in \mathbb{N}$. Then,

$$|L_N(f)(x) - f(x)| \leq \left(\frac{\omega_1(D_\alpha^n f, h) b^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) L_N(|\cdot - x|^{(n+1)\alpha})(x), \quad \forall N \in \mathbb{N}. \quad (39)$$

Proof By (31) and (38). □

We give

Theorem 2.7 All as in Theorem 2.6. Additionally assume that $L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0$, $\forall N \in \mathbb{N}$. Then,

$$|L_N(f)(x) - f(x)| \leq \left(\frac{\omega_1(D_\alpha^n f, h) b^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) \left(L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x) \right)^{\frac{\alpha}{\alpha+1}}, \quad (40)$$

$\forall N \in \mathbb{N}$.

Proof By (39) and Theorem 2.4: we have

$$L_N(|\cdot - x|^{(n+1)\alpha})(x) \leq \left(L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x) \right)^{\frac{\alpha}{\alpha+1}}, \quad (41)$$

proving the claim. □

We present

Theorem 2.8 Let $\{L_N\}_{N \in \mathbb{N}}$ from $C_+([a, b])$ into itself, positive sublinear operators, such that: $L_N(1) = 1$, $\forall N \in \mathbb{N}$. Additionally, assume that $L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0$, $\forall N \in \mathbb{N}$; $x \in (a, b)$. Here, $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([a, b])$ is n times conformable α -fractional differentiable on $[a, b] \subseteq [0, \infty)$, and $D_\alpha^n f$ is continuous on $[a, b]$. Assume here that $0 < (L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x))^{\frac{\alpha}{\alpha+1}} \leq \min(x - a, b - x)$, $\forall N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, and assume $|D_\alpha^n f|$ is convex over $[a, b]$. Furthermore assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha} \omega_1 \left(D_\alpha^n f, (L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x))^{\frac{\alpha}{\alpha+1}} \right)}{(n+1)! \alpha^{n+1}}, \quad (42)$$

$\forall N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$.
If $L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x) \rightarrow 0$, then $L_N(f)(x) \rightarrow f(x)$, as $N \rightarrow +\infty$.

Proof By (40) and choosing $h := (L_N(|\cdot - x|^{(n+1)(\alpha+1)})(x))^{\frac{\alpha}{\alpha+1}}$. $\square$

We also give

Theorem 2.9 Let $\{L_N\}_{N \in \mathbb{N}}$ from $C_+([a, b])$ into itself, positive sublinear operators: $L_N(1) = 1$, $\forall N \in \mathbb{N}$. Also $L_N(|\cdot - x|^{(n+1)\alpha})(x) > 0$, $\forall N \in \mathbb{N}$. Here $\alpha \in (0, 1]$, $n \in \mathbb{N}$ and $x \in (a, b)$; $[a, b] \subseteq [0, \infty)$. Suppose $f \in C_+([a, b])$ is n times conformable α -fractional differentiable on $[a, b]$, and $D_\alpha^n f$ is continuous on $[a, b]$. Let $0 < L_N(|\cdot - x|^{(n+1)\alpha})(x) \leq \min(x - a, b - x)$, $\forall N \geq N^*$; $N, N^* \in \mathbb{N}$, and assume $|D_\alpha^n f|$ is convex over $[a, b]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha} \omega_1 \left(D_\alpha^n f, L_N(|\cdot - x|^{(n+1)\alpha})(x) \right)}{(n+1)! \alpha^{n+1}}, \quad (43)$$

$\forall N \geq N^*$, where $N, N^* \in \mathbb{N}$.
If $L_N(|\cdot - x|^{(n+1)\alpha})(x) \rightarrow 0$, then $L_N(f)(x) \rightarrow f(x)$, as $N \rightarrow +\infty$.

Proof By (39) and choosing $h := L_N(|\cdot - x|^{(n+1)\alpha})(x)$. $\square$

3 Applications

(I) Here we apply Theorem 2.7 to well known max-product operators.

We make

Remark 3.1 In [5, p. 10], the authors introduced the basic max-product Bernstein operators

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^N p_{N,k}(x)}, \quad N \in \mathbb{N}, \quad (44)$$

where $\vee$ stands for maximum, and $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$ and $f: [0, 1] \rightarrow \mathbb{R}_+ = [0, \infty)$ is continuous.

These are nonlinear and piecewise rational operators.

We have $B_N^{(M)}(1) = 1$, and

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall x \in [0, 1], \forall N \in \mathbb{N}; \quad (45)$$

see [5, p. 31].

$B_N^{(M)}$ are positive sublinear operators and thus they possess the monotonicity property, also since $|\cdot - x| \leq 1$, then $|\cdot - x|^\beta \leq 1$, $\forall x \in [0, 1]$, $\forall \beta > 0$.

Therefore it holds that

$$B_N^{(M)}(|\cdot - x|^{1+\beta})(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall x \in [0, 1], \quad \forall N \in \mathbb{N}, \quad \forall \beta > 0. \quad (46)$$

Furthermore, clearly it holds that

$$B_N^{(M)}(|\cdot - x|^{1+\beta})(x) > 0, \quad \forall N \in \mathbb{N}, \quad \forall \beta \geq 0 \text{ and any } x \in (0, 1). \quad (47)$$

The operator $B_N^{(M)}$ maps $C_+([0, 1])$ into itself.

We give

Theorem 3.2 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_\alpha^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{(N^*+1)^{\frac{\alpha}{2(\alpha+1)}}} \leq \min(x, 1-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{6^{\frac{\alpha}{\alpha+1}}}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{(N+1)^{\frac{\alpha}{2(\alpha+1)}}} \right), \quad (48)$$

$\forall N \geq N^*, N \in \mathbb{N}$.

It holds that $\lim_{N \rightarrow +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof By (47), we get that $B_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0, \forall N \in \mathbb{N}$.

By (40), (46), we obtain that

$$\begin{aligned} \left| B_N^{(M)}(f)(x) - f(x) \right| &\leq \left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \left(B_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) \right)^{\frac{\alpha}{\alpha+1}} \leq \\ &\left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{6}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}} = \end{aligned} \quad (49)$$

(setting $h := (\frac{1}{\sqrt{N+1}})^{\frac{\alpha}{\alpha+1}}$)

$$\left(\frac{6^{\frac{\alpha}{\alpha+1}}}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{(N+1)^{\frac{\alpha}{2(\alpha+1)}}} \right),$$

proving the claim. $\square$

We continue with

Remark 3.3 The truncated Favard–Szász–Mirakjan operators are given by

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^N s_{N,k}(x)}, \quad x \in [0, 1], \quad N \in \mathbb{N}, \quad f \in C_+([0, 1]), \quad (50)$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$; see also [5, p. 11].

By [5, p. 178–179], we get that

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall x \in [0, 1], \quad \forall N \in \mathbb{N}. \quad (51)$$

Clearly, it holds that

$$T_N^{(M)}(|\cdot - x|^{1+\beta})(x) \leq \frac{3}{\sqrt{N}}, \quad \forall x \in [0, 1], \quad \forall N \in \mathbb{N}, \quad \forall \beta > 0. \quad (52)$$

The operators $T_N^{(M)}$ are positive sublinear operators mapping $C_+([0, 1])$ into itself, with $T_N^{(M)}(1) = 1$. Furthermore, it holds that

$$T_N^{(M)}(|\cdot - x|^\lambda)(x) = \frac{\bigvee_{k=0}^N \frac{(Nx)^k}{k!} \left| \frac{k}{N} - x \right|^\lambda}{\bigvee_{k=0}^N \frac{(Nx)^k}{k!}} > 0, \quad \forall x \in (0, 1], \quad \forall \lambda \geq 1, \quad \forall N \in \mathbb{N}. \quad (53)$$

We give

Theorem 3.4 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_\alpha^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{(N^*)^{\frac{\alpha}{2(\alpha+1)}}} \leq \min(x, 1-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{3^{\frac{\alpha}{\alpha+1}}}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{\alpha}{2(\alpha+1)}}} \right), \quad (54)$$

$\forall N \geq N^*, N \in \mathbb{N}$.

It holds that $\lim_{N \rightarrow +\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof By (53), we have $T_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0, \forall N \in \mathbb{N}$.

By (40), (52), we get that

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{3}{\sqrt{N}} \right)^{\frac{\alpha}{\alpha+1}} =$$

(setting $h := \left(\frac{1}{\sqrt{N}} \right)^{\frac{\alpha}{\alpha+1}}$)

$$\left(\frac{3^{\frac{\alpha}{\alpha+1}}}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{\alpha}{2(\alpha+1)}}} \right), \quad (55)$$

proving the claim. $\square$

We make

Remark 3.5 Next, we study the truncated max-product Baskakov operators (see [5, p. 11])

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^N b_{N,k}(x)}, \quad x \in [0, 1], \quad f \in C_+([0, 1]), \quad N \in \mathbb{N}, \quad (56)$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}. \quad (57)$$

From [5, pp. 217–218], we get ($x \in [0, 1]$)

$$\left(U_N^{(M)}(|\cdot - x|) \right)(x) \leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}, \quad N \geq 2, \quad N \in \mathbb{N}. \quad (58)$$

Let $\lambda \geq 1$, clearly then it holds that

$$\left(U_N^{(M)}(|\cdot - x|^\lambda) \right)(x) \leq \frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}, \quad \forall N \geq 2, \quad N \in \mathbb{N}. \quad (59)$$

Also, it holds that $U_N^{(M)}(1) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0, 1])$ into itself. Furthermore, it holds that

$$U_N^{(M)}(|\cdot - x|^\lambda)(x) > 0, \quad \forall x \in (0, 1], \quad \forall \lambda \geq 1, \quad \forall N \in \mathbb{N}. \quad (60)$$

We give

Theorem 3.6 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_\alpha^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N} - \{1\}$ such that $\frac{1}{(N^*+1)^{\frac{\alpha}{2(\alpha+1)}}} \leq \min(x, 1-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| U_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{(2\sqrt{3}(\sqrt{2}+2))^{\frac{\alpha}{\alpha+1}}}{(n+1)!\alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{(N+1)^{\frac{\alpha}{2(\alpha+1)}}} \right), \quad (61)$$

$\forall N \in \mathbb{N} : N \geq N^* \geq 2$.

It holds that $\lim_{N \rightarrow +\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof By (60), we have that $U_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0$, $\forall N \in \mathbb{N}$.

By (40) and (59), we get

$$\left| U_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)!\alpha^{n+1}h} \right) \left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}} = \quad (62)$$

(setting $h := (\frac{1}{\sqrt{N+1}})^{\frac{\alpha}{\alpha+1}}$)

$$\left(\frac{(2\sqrt{3}(\sqrt{2}+2))^{\frac{\alpha}{\alpha+1}}}{(n+1)!\alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{(N+1)^{\frac{\alpha}{2(\alpha+1)}}} \right),$$

proving the claim. $\square$

We continue with

Remark 3.7 Here, we study the max-product Meyer–Köning and Zeller operators (see [5, p. 11]) defined by

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^N s_{N,k}(x)}, \quad \forall N \in \mathbb{N}, f \in C_+([0, 1]), \quad (63)$$

$$s_{N,k}(x) = \binom{N+k}{k} x^k, x \in [0, 1].$$

By [5, p. 253], we get that

$$Z_N^{(M)}(|\cdot - x|)(x) \leq \frac{8(1+\sqrt{5})}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \quad \forall x \in [0, 1], \forall N \geq 4, N \in \mathbb{N}. \quad (64)$$

As before, we get that (for $\lambda \geq 1$)

$$Z_N^{(M)}(|\cdot - x|^\lambda)(x) \leq \frac{8(1+\sqrt{5})}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \quad (65)$$

$\forall x \in [0, 1], N \geq 4, N \in \mathbb{N}$.

Also, it holds that $Z_N^{(M)}(1) = 1$, and $Z_N^{(M)}$ are positive sublinear operators from $C_+([0, 1])$ into itself. Also, it holds that

$$Z_N^{(M)}(|\cdot - x|^\lambda)(x) > 0, \quad \forall x \in (0, 1), \forall \lambda \geq 1, \forall N \in \mathbb{N}. \quad (66)$$

We give

Theorem 3.8 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_\alpha^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N}$, $N^* \geq 4$, such that $\frac{1}{(N^*)^{\frac{1}{2(\alpha+1)}}} \leq \min(x, 1-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| Z_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{8(1+\sqrt{5})}{3} \sqrt{x}(1-x) \right)^{\frac{\alpha}{\alpha+1}} \left(\frac{\omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{1}{2(\alpha+1)}}} \right)}{(n+1)! \alpha^{n+1}} \right), \quad (67)$$

$\forall N \geq N^* \geq 4, N \in \mathbb{N}$.

It holds that $\lim_{N \rightarrow +\infty} Z_N^{(M)}(f)(x) = f(x)$.

Proof By (66), we get that $Z_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0, \forall N \in \mathbb{N}$.

By (40) and (65), we obtain

$$\left| Z_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{8(1+\sqrt{5})}{3} \sqrt{x}(1-x) \right)^{\frac{\alpha}{\alpha+1}} \left(\frac{1}{\sqrt{N}} \right)^{\frac{\alpha}{\alpha+1}} = \quad (68)$$

(setting $h := (\frac{1}{\sqrt{N}})^{\frac{\alpha}{\alpha+1}}$)

$$\left(\frac{8(1+\sqrt{5})}{3} \sqrt{x}(1-x) \right)^{\frac{\alpha}{\alpha+1}} \left(\frac{\omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{1}{2(\alpha+1)}}} \right)}{(n+1)! \alpha^{n+1}} \right),$$

proving the claim. $\square$

We make

Remark 3.9 Here, we deal with the max-product truncated sampling operators (see [5, p. 13]) defined by

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^N \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \quad (69)$$

and

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^N \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \quad (70)$$

$\forall x \in [0, \pi], f: [0, \pi] \rightarrow \mathbb{R}_+$ a continuous function.

Following [5, p. 343], and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin(Nx-k\pi)}{Nx-k\pi}$, we get that $s_{N,k}(\frac{k\pi}{N}) = 1$, and $s_{N,k}(\frac{j\pi}{N}) = 0$, if $k \neq j$, and furthermore $W_N^{(M)}(f)(\frac{j\pi}{N}) = f(\frac{j\pi}{N})$, for all $j \in \{0, \dots, N\}$.

Clearly, $W_N^{(M)}(f)$ is a well-defined function for all $x \in [0, \pi]$, and it is continuous on $[0, \pi]$, also $W_N^{(M)}(1) = 1$.

By [5, p. 344], $W_N^{(M)}$ are positive sublinear operators.

Call $I_N^+(x) = \{k \in \{0, 1, \dots, N\}; s_{N,k}(x) > 0\}$, and set $x_{N,k} := \frac{k\pi}{N}, k \in \{0, 1, \dots, N\}$.

We see that

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} s_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} s_{N,k}(x)}. \quad (71)$$

By [5, p. 346], we have

$$W_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N}, \quad \forall N \in \mathbb{N}, \quad \forall x \in [0, \pi]. \quad (72)$$

Notice also $|x_{N,k} - x| \leq \pi, \forall x \in [0, \pi]$.

Therefore, $(\lambda \geq 1)$ it holds that

$$W_N^{(M)}(|\cdot - x|^\lambda)(x) \leq \frac{\pi^{\lambda-1}\pi}{2N} = \frac{\pi^\lambda}{2N}, \quad \forall x \in [0, \pi], \quad \forall N \in \mathbb{N}. \quad (73)$$

If $x \in \left(\frac{j\pi}{N}, \frac{(j+1)\pi}{N}\right)$, with $j \in \{0, 1, \dots, N\}$, we obtain $nx - j\pi \in (0, \pi)$ and thus $s_{N,j}(x) = \frac{\sin(Nx - j\pi)}{Nx - j\pi} > 0$; see [5, pp. 343–344].

Consequently, it holds that $(\lambda \geq 1)$

$$W_N^{(M)}(|\cdot - x|^\lambda)(x) = \frac{\bigvee_{k \in I_N^+(x)} s_{N,k}(x) |x_{N,k} - x|^\lambda}{\bigvee_{k \in I_N^+(x)} s_{N,k}(x)} > 0, \quad \forall x \in [0, \pi], \quad (74)$$

such that $x \neq x_{N,k}$, for any $k \in \{0, 1, \dots, N\}$.

We give

Theorem 3.10 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, \pi])$ is n times conformable α -fractional differentiable on $[0, \pi]$, and $x \in (0, \pi)$, such that $x \neq \frac{k\pi}{N}, k \in \{0, 1, \dots, N\}, \forall N \in \mathbb{N}$, and $D_\alpha^n f$ is continuous on $[0, \pi]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{(N^*)^{\frac{\alpha}{\alpha+1}}} \leq \min(x, \pi - x)$ and assume $|D_\alpha^n f|$ is convex over $[0, \pi]$. Furthermore, assume that $D_\alpha^k f(x) = 0, k = 1, \dots, n$. Then,

$$\left| W_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\pi^{n\alpha+1}}{2^{\frac{\alpha}{\alpha+1}} (n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{\alpha}{\alpha+1}}} \right), \quad (75)$$

$\forall N \in \mathbb{N} : N \geq N^*$.

It holds that $\lim_{N \rightarrow +\infty} W_N^{(M)}(f)(x) = f(x)$.

Proof By (74) we have $W_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0, \forall N \in \mathbb{N}$.

By (40) and (73), we obtain

$$\begin{aligned} \left| W_N^{(M)}(f)(x) - f(x) \right| &\leq \left(\frac{\omega_1(D_\alpha^n f, h) \pi^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{\pi^{(n+1)(\alpha+1)}}{2N} \right)^{\frac{\alpha}{\alpha+1}} = \\ &\left(\frac{\pi^{n\alpha+1} \omega_1(D_\alpha^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{1}{2N} \right)^{\frac{\alpha}{\alpha+1}} = \end{aligned} \quad (76)$$

(setting $h := (\frac{1}{N})^{\frac{\alpha}{\alpha+1}}$)

$$\left(\frac{\pi^{n\alpha+1}}{2^{\frac{\alpha}{\alpha+1}} (n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{\alpha}{\alpha+1}}} \right),$$

proving the claim. $\square$

We make

Remark 3.11 Here, we continue with the max-product truncated sampling operators (see [5, p. 13]) defined by

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^N \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \quad (77)$$

$\forall x \in [0, \pi]$, $f: [0, \pi] \rightarrow \mathbb{R}_+$ a continuous function.

Following [5, p. 350], and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, and $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, if $k \neq j$; furthermore, $K_N^{(M)}(f)\left(\frac{j\pi}{N}\right) = f\left(\frac{j\pi}{N}\right)$, for all $j \in \{0, \dots, N\}$.

Since $s_{N,j}\left(\frac{j\pi}{N}\right) = 1$, it follows that $\bigvee_{k=0}^N s_{N,k}\left(\frac{j\pi}{N}\right) \geq 1 > 0$, for all $j \in \{0, 1, \dots, N\}$. Hence, $K_N^{(M)}(f)$ is a well-defined function for all $x \in [0, \pi]$, and it is continuous on $[0, \pi]$, also $K_N^{(M)}(1) = 1$. By [5, p. 350], $K_N^{(M)}$ are positive sublinear operators.

Denote $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0, 1, \dots, N\}$.

By [5, p. 352], we have

$$K_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N}, \quad \forall N \in \mathbb{N}, \quad \forall x \in [0, \pi]. \quad (78)$$

Notice also $|x_{N,k} - x| \leq \pi$, $\forall x \in [0, \pi]$.

Therefore, $(\lambda \geq 1)$ it holds that

$$K_N^{(M)}(|\cdot - x|^\lambda)(x) \leq \frac{\pi^{\lambda-1}\pi}{2N} = \frac{\pi^\lambda}{2N}, \quad \forall x \in [0, \pi], \quad \forall N \in \mathbb{N}. \quad (79)$$

If $x \in \left(\frac{j\pi}{N}, \frac{(j+1)\pi}{N}\right)$, with $j \in \{0, 1, \dots, N\}$, we obtain $Nx - j\pi \in (0, \pi)$ and thus $s_{N,j}(x) = \frac{\sin^2(Nx-j\pi)}{(Nx-j\pi)^2} > 0$; see [5, pp. 350].

Consequently, it holds that $(\lambda \geq 1)$

$$K_N^{(M)}(|\cdot - x|^\lambda)(x) = \frac{\bigvee_{k=0}^N s_{N,k}(x) |x_{N,k} - x|^\lambda}{\bigvee_{k=0}^N s_{N,k}(x)} > 0, \quad \forall x \in [0, \pi], \quad (80)$$

such that $x \neq x_{N,k}$, for any $k \in \{0, 1, \dots, N\}$.

We give

Theorem 3.12 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, \pi])$ is n times conformable α -fractional differentiable on $[0, \pi]$, and $x \in (0, \pi)$, such that $x \neq \frac{k\pi}{N}$, $k \in \{0, 1, \dots, N\}$, $\forall N \in \mathbb{N}$, and $D_\alpha^n f$ is continuous on $[0, \pi]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{(N^*)^{\frac{\alpha}{\alpha+1}}} \leq \min(x, \pi - x)$ and assume $|D_\alpha^n f|$ is convex over $[0, \pi]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| K_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\pi^{n\alpha+1}}{2^{\frac{\alpha}{\alpha+1}} (n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N^{\frac{\alpha}{\alpha+1}}} \right), \quad (81)$$

$\forall N \in \mathbb{N} : N \geq N^*$.

It holds that $\lim_{N \rightarrow +\infty} K_N^{(M)}(f)(x) = f(x)$.

Proof By (80), we have $K_N^{(M)}(|\cdot - x|^{(n+1)(\alpha+1)})(x) > 0$, $\forall N \in \mathbb{N}$.

By (40) and (79), we obtain

$$\begin{aligned} \left| K_N^{(M)}(f)(x) - f(x) \right| &\leq \left(\frac{\omega_1(D_\alpha^n f, h) \pi^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{\pi^{(n+1)(\alpha+1)}}{2N} \right)^{\frac{\alpha}{\alpha+1}} = \\ &\left(\frac{\pi^{n\alpha+1} \omega_1(D_\alpha^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \left(\frac{1}{2N} \right)^{\frac{\alpha}{\alpha+1}} = \end{aligned} \quad (82)$$

(setting $h := (\frac{1}{N})^{\frac{\alpha}{\alpha+1}}$)

$$\left(\frac{\pi^{n\alpha+1}}{2^{\frac{\alpha}{\alpha+1}} (n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_{\alpha}^n f, \frac{1}{N^{\frac{\alpha}{\alpha+1}}} \right),$$

proving the claim. $\square$

(II) Here, we apply Theorem 2.6 to the well-known max-product operators in the case of $(n+1)\alpha \geq 1$, that is, when $\frac{1}{n+1} \leq \alpha \leq 1$, where $n \in \mathbb{N}$.

We give

Theorem 3.13 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_{\alpha}^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{\sqrt{N^*+1}} \leq \min(x, 1-x)$ and assume $|D_{\alpha}^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_{\alpha}^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{6}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_{\alpha}^n f, \frac{1}{\sqrt{N+1}} \right), \quad (83)$$

$\forall N \geq N^*, N \in \mathbb{N}$.

It holds that $\lim_{N \rightarrow +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof By (39), (46), we obtain that

$$\begin{aligned} \left| B_N^{(M)}(f)(x) - f(x) \right| &\leq \left(\frac{\omega_1(D_{\alpha}^n f, h)}{(n+1)! \alpha^{n+1} h} \right) B_N^{(M)}(|\cdot - x|^{(n+1)\alpha})(x) \leq \\ &\left(\frac{\omega_1(D_{\alpha}^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \frac{6}{\sqrt{N+1}} = \end{aligned} \quad (84)$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$\left(\frac{6}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_{\alpha}^n f, \frac{1}{\sqrt{N+1}} \right),$$

proving the claim. $\square$

Theorem 3.14 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_{\alpha}^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{\sqrt{N^*}} \leq \min(x, 1-x)$ and assume $|D_{\alpha}^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_{\alpha}^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{3}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_{\alpha}^n f, \frac{1}{\sqrt{N}} \right), \quad (85)$$

$\forall N \geq N^*, N \in \mathbb{N}$.

It holds that $\lim_{N \rightarrow +\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof By (39), (52), we get that

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\omega_1(D_{\alpha}^n f, h)}{(n+1)! \alpha^{n+1} h} \right) \frac{3}{\sqrt{N}} =$$

(setting $h := \frac{1}{\sqrt{N}}$)

$$\left(\frac{3}{(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_{\alpha}^n f, \frac{1}{\sqrt{N}} \right), \quad (86)$$

proving the claim. $\square$

Theorem 3.15 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_\alpha^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N} - \{1\}$ such that $\frac{1}{\sqrt{N^*+1}} \leq \min(x, 1-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, 1]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| U_N^{(M)}(f)(x) - f(x) \right| \leq \frac{(2\sqrt{3}(\sqrt{2}+2))}{(n+1)!\alpha^{n+1}} \omega_1 \left(D_\alpha^n f, \frac{1}{\sqrt{N+1}} \right), \quad (87)$$

$\forall N \in \mathbb{N} : N \geq N^* \geq 2$.

It holds that $\lim_{N \rightarrow +\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof By (39), (59), we get

$$\left| U_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)!\alpha^{n+1}h} \right) \left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}} \right) = \quad (88)$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$\frac{(2\sqrt{3}(\sqrt{2}+2))}{(n+1)!\alpha^{n+1}} \omega_1 \left(D_\alpha^n f, \frac{1}{\sqrt{N+1}} \right),$$

proving the claim. $\square$

Theorem 3.16 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, 1])$ is n times conformable α -fractional differentiable on $[0, 1]$, $x \in (0, 1)$, and $D_\alpha^n f$ is continuous on $[0, 1]$. Let $N^* \in \mathbb{N}$, $N^* \geq 4$, such that $\frac{1}{\sqrt{N^*}} \leq \min(x, 1-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, 1]$. Furthermore assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| Z_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{8(1+\sqrt{5})}{3} \sqrt{x}(1-x) \right) \left(\frac{\omega_1 \left(D_\alpha^n f, \frac{1}{\sqrt{N}} \right)}{(n+1)!\alpha^{n+1}} \right), \quad (89)$$

$\forall N \geq N^* \geq 4$, $N \in \mathbb{N}$.

It holds that $\lim_{N \rightarrow +\infty} Z_N^{(M)}(f)(x) = f(x)$.

Proof By (39) and (65), we obtain

$$\left| Z_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\omega_1(D_\alpha^n f, h)}{(n+1)!\alpha^{n+1}h} \right) \left(\frac{8(1+\sqrt{5})}{3} \sqrt{x}(1-x) \right) \frac{1}{\sqrt{N}} = \quad (90)$$

(setting $h := \frac{1}{\sqrt{N}}$)

$$\frac{8(1+\sqrt{5})}{3} \sqrt{x}(1-x) \left(\frac{\omega_1 \left(D_\alpha^n f, \frac{1}{\sqrt{N}} \right)}{(n+1)!\alpha^{n+1}} \right),$$

proving the claim. $\square$

Theorem 3.17 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, \pi])$ is n times conformable α -fractional differentiable on $[0, \pi]$, and $x \in (0, \pi)$, such that $x \neq \frac{k\pi}{N}$, $k \in \{0, 1, \dots, N\}$, $\forall N \in \mathbb{N}$, and $D_\alpha^n f$ is continuous on $[0, \pi]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{N^*} \leq \min(x, \pi-x)$ and assume $|D_\alpha^n f|$ is convex over $[0, \pi]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| W_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\pi^{n\alpha+1}}{2(n+1)!\alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N} \right), \quad (91)$$

$\forall N \in \mathbb{N} : N \geq N^*$.

It holds that $\lim_{N \rightarrow +\infty} W_N^{(M)}(f)(x) = f(x)$.

Proof By (39) and (73), we obtain

$$\begin{aligned} \left| W_N^{(M)}(f)(x) - f(x) \right| &\leq \left(\frac{\omega_1(D_\alpha^n f, h) \pi^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) \frac{\pi^{(n+1)\alpha}}{2N} = \\ &\left(\frac{\pi^{n\alpha+1} \omega_1(D_\alpha^n f, h)}{2(n+1)! \alpha^{n+1} h} \right) \frac{1}{N} = \end{aligned} \quad (92)$$

(setting $h := \frac{1}{N}$)

$$\left(\frac{\pi^{n\alpha+1}}{2(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N} \right),$$

proving the claim. $\square$

Theorem 3.18 Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose $f \in C_+([0, \pi])$ is n times conformable α -fractional differentiable on $[0, \pi]$, and $x \in (0, \pi)$, such that $x \neq \frac{k\pi}{N}$, $k \in \{0, 1, \dots, N\}$, $\forall N \in \mathbb{N}$, and $D_\alpha^n f$ is continuous on $[0, \pi]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{N^*} \leq \min(x, \pi - x)$ and assume $|D_\alpha^n f|$ is convex over $[0, \pi]$. Furthermore, assume that $D_\alpha^k f(x) = 0$, $k = 1, \dots, n$. Then,

$$\left| K_N^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{\pi^{n\alpha+1}}{2(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N} \right), \quad (93)$$

$\forall N \in \mathbb{N} : N \geq N^*$.

It holds that $\lim_{N \rightarrow +\infty} K_N^{(M)}(f)(x) = f(x)$.

Proof By (39) and (79), we obtain

$$\begin{aligned} \left| K_N^{(M)}(f)(x) - f(x) \right| &\leq \left(\frac{\omega_1(D_\alpha^n f, h) \pi^{1-\alpha}}{(n+1)! \alpha^{n+1} h} \right) \frac{\pi^{(n+1)\alpha}}{2N} = \\ &\left(\frac{\pi^{n\alpha+1} \omega_1(D_\alpha^n f, h)}{2(n+1)! \alpha^{n+1} h} \right) \frac{1}{N} = \end{aligned} \quad (94)$$

(setting $h := \frac{1}{N}$)

$$\left(\frac{\pi^{n\alpha+1}}{2(n+1)! \alpha^{n+1}} \right) \omega_1 \left(D_\alpha^n f, \frac{1}{N} \right),$$

proving the claim. $\square$

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References

1. Anastassiou, G.: Moments in Probability and Approximation Theory. Pitman Research Notes in Mathematics Series. Longman Group UK, New York (1993)
2. Anastassiou, G.: Approximation by Sublinear Operators (2017, submitted)
3. Anastassiou, G.: Conformable Fractional Approximation by Max-Product Operators (2017, submitted)
4. Anderson, D.: Taylor's Formula and Integral Inequalities for Conformable Fractional Derivatives. Contributions in Mathematics and Engineering. In Honor of Constantin Carathéodory, pp. 25–43. Springer, Berlin (2016)
5. Bede, B.; Coroianu, L.; Gal, S.: Approximation by Max-Product Type Operators. Springer, Heidelberg (2016)
6. Abu Hammad, M.; Khalil, R.: Abel's formula and Wronskian for conformable fractional differential equations. Int. J. Differ. Equ. Appl. **13**(3), 177–183 (2014)
7. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)

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