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Stability and nonstability of octadecic functional equation in multi-normed spaces

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Abstract In this paper, we introduce octadecic functional equation. Moreover, we prove the stability of the octadecic functional equation in multi-normed spaces by using the fixed point method.

Mathematics Subject Classification 39A11 · 39B52

المخلص

في هذا البحث، نتناول المعادلة الدالية الثمانية عشرية. أيضاً، نثبت الاستقرار لهذه المعادلة الدالية الثمانية عشرية في الفضاءات متعددة المعيارية، وذلك باستخدام طريقة النقطة الثابتة.

1 Introduction

In 1940, Ulam [17] proposed the following question concerning the stability of group homomorphisms:

Let G_1 be a group and (G_2, d) be a metric group. Given $\varepsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$, such that $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the next year, 1941, Hyers [8] solved the famous stability problem of Ulam in Banach spaces:

Let X be a normed space and Y be a Banach space. Suppose that for some $\varepsilon > 0$, the mapping $f : X \rightarrow Y$ satisfies $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$, such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$.

In 1978, Rassias [15] proved the following theorem:

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Let X and Y be real normed spaces with Y complete. Let $f : X \rightarrow Y$ be a mapping such that, for each fixed $x \in X$, the mapping $h(t) = f(tx)$ is continuous on $\mathbb{R}$, and let $\varepsilon \geq 0$ and $p \in [0, 1)$ be such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

holds for all $x, y \in X$. Then there exists a unique linear mapping $T : X \rightarrow Y$, such that

$$\|f(x) - T(x)\| \leq \varepsilon \frac{\|x\|^p}{1 - 2^{p-1}}$$

for all $x \in X$. Since the past few decades several stability problems of functional equations have been investigated [1, 2, 5–7, 11, 12, 14, 19]. Xu et al. [20] proved the general solution and the stability of the quintic functional equation

$$f(x+3y) - 5f(x+2y) + 10f(x+y) - 10f(x) + 5f(x-y) - f(x-2y) = 120f(y)$$

and the sextic functional equation

$$\begin{aligned} f(x+3y) - 6f(x+2y) + 15f(x+y) - 20f(x) + 15f(x-y) - 6f(x-2y) \\ + f(x-3y) = 720f(y) \end{aligned}$$

in quasi- β -normed spaces.

The general solution and the stability of the septic functional equation

$$\begin{aligned} f(x+4y) - 7f(x+3y) + 21f(x+2y) - 35f(x+y) - 35f(x) - 21f(x-y) \\ + 7f(x-2y) - f(x-3y) = 5040f(y) \end{aligned}$$

and the octic functional equation

$$\begin{aligned} f(x+4y) - 8f(x+3y) + 28f(x+2y) - 56f(x+y) - 70f(x) - 56f(x-y) \\ + 28f(x-2y) - 8f(x-3y) + f(x-4y) = 40320f(y) \end{aligned}$$

in quasi- β -normed spaces were investigated by Xu and Rassias [18]. Rassias and Eslamian [13] investigated the general solution of a nonic functional equation

$$\begin{aligned} f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) - 126f(x+y) - 126f(x) \\ + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!f(y) \end{aligned}$$

and proved the stability of nonic functional equation in quasi- β -normed spaces by using the fixed point method. A fixed point approach for the stability of decic functional equation

$$\begin{aligned} f(x+5y) - 10f(x+4y) + 45f(x+3y) - 120f(x+2y) - 210f(x+y) - 252f(x) \\ + 210f(x-y) - 120f(x-2y) + 45f(x-3y) - 10f(x-4y) + f(x-5y) = 10!f(y) \end{aligned}$$

in quasi- β -normed spaces was investigated by Ravi et al. [16].

Let $(X, \|\cdot\|)$ be a complex normed space, and $k \in \mathbb{N}$. We denote the group of permutations on k symbols by $\mathfrak{S}_k$.

Definition 1.1 [3, 4, 9] A multi-norm on $\{X^k : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $\|\cdot\|_k$ is a norm on X^k for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in X$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(MN1)

$$\|x_{\sigma(1)}, \dots, x_{\sigma(k)}\|_k = \|x_1, \dots, x_k\|_k \quad (\sigma \in \mathfrak{S}_k, x_1, \dots, x_k \in X);$$

(MN2)

$$\|\alpha_1 x_1, \dots, \alpha_k x_k\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|x_1, \dots, x_k\|_k$$

$$(\alpha_1, \dots, \alpha_k \in \mathbb{C}, x_1, \dots, x_k \in X);$$

(MN3)

$$\|x_1, \dots, x_{k-1}, 0\|_k = \|x_1, \dots, x_{k-1}\|_{k-1} \quad (x_1, \dots, x_{k-1} \in X);$$

(MN4)

$$\|x_1, \dots, x_{k-1}, x_{k-1}\|_k = \|x_1, \dots, x_{k-1}\|_{k-1} \quad (x_1, \dots, x_{k-1} \in X).$$

In this case, we say that $((X^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space. If $(X, \|\cdot\|)$ is a Banach space, then $(X^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$, in this case $((X^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 1.2 Let $(X, \|\cdot\|)$ be a Banach lattice, and let us define

$$\|x_1, \dots, x_k\|_k := \| |x_1| \vee \dots \vee |x_k| \| \quad (x_1, \dots, x_k \in X).$$

Then $((X^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Let X and Y be real vector spaces and $f : X \rightarrow Y$ be a mapping. We define a mapping $Df : X^2 \rightarrow Y$ by

$$\begin{aligned} Df(x, y) := & f(x + 9y) - 18f(x + 8y) + 153f(x + 7y) - 816f(x + 6y) + 3060f(x + 5y) \\ & - 8568f(x + 4y) + 18564f(x + 3y) - 31824f(x + 2y) + 43758f(x + y) - 48620f(x) \\ & + 43758f(x - y) - 31824f(x - 2y) + 18564f(x - 3y) - 8568f(x - 4y) \\ & + 3060f(x - 5y) - 816f(x - 6y) + 153f(x - 7y) - 18f(x - 8y) + f(x - 9y) \\ & - 18!f(y), \end{aligned}$$

for all $x, y \in X$, where $18! = 6402373705728000$.

In this paper, we introduce the following octadecic functional equation:

$$Df(x, y) = 0, \tag{1.1}$$

for all $x, y \in X$. Moreover, we prove the stability of the octadecic functional equation (1.1) in multi-normed spaces by using the standard fixed point method:

Theorem 1.3 [10] *If X and Y are real vector spaces and $f : X \rightarrow Y$ is a mapping satisfying octadecic functional equation (1.1) for all $x, y \in X$, then f is an octadecic mapping, i.e., $f(x) = x^{18}$.*

2 Stability of the functional equation (1.1) in multi-normed spaces

In this section, we prove the generalized Hyers–Ulam stability of the octadecic functional equation (1.1) in multi-normed spaces. Throughout this section, we assume that X is a normed space and that Y is a Banach space. Let $((Y^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach space.

Theorem 2.1 [10] *Let $\phi : X^2 \rightarrow [0, \infty)$ be a mapping, such that there exists $L < 1$ with $\phi(2x, 2y) \leq 2^{18}L\phi(x, y)$ for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying*

$$\|Df(x, y)\| \leq \phi(x, y),$$

for all $x, y \in X$. Then there exists a unique octadecic mapping $Q : X \rightarrow Y$, such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2^{18}(1-L)} \psi(x),$$

for all $x \in X$, where

$$\psi(x) := (3201186852864000)^{-1} (24310\phi(0, x) + 43758\phi(x, x) + 31824\phi(2x, x))$$

$$\begin{aligned}
& + 18564\phi(3x, x) + 8568\phi(4x, x) + 3060\phi(5x, x) + 816\phi(6x, x) + 153\phi(7x, x) \\
& + 18\phi(8x, x) + \phi(9x, x) + \frac{\phi(0, 2x)}{2} \\
& + \left(\frac{4537567325}{6402373705728000} + \frac{48886173}{12705510619017216000} \right) \phi(0, 0) \\
& + \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right) (\phi(x, x) + \phi(x, -x)) \\
& + \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) (\phi(2x, 2x) \\
& + \phi(2x, -2x)) + \frac{940105848}{6402373705728000} (\phi(3x, 3x) + \phi(3x, -3x)) \\
& + \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) (\phi(4x, 4x) + \phi(4x, -4x)) \\
& + \frac{115306920}{6402373705728000} (\phi(5x, 5x) + \phi(5x, -5x)) \\
& + \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) (\phi(6x, 6x) + \phi(6x, -6x)) \\
& + \frac{4538898}{6402373705728000} (\phi(7x, 7x) + \phi(7x, -7x)) \\
& + \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) (\phi(8x, 8x) + \phi(8x, -8x)) \\
& + \frac{24310(\phi(9x, 9x) + \phi(9x, -9x))}{6402373705728000} + \frac{\phi(10x, 10x) + \phi(10x, -10x)}{4184557977600} \\
& + \frac{\phi(12x, 12x) + \phi(12x, -12x)}{15692092416000} + \frac{\phi(14x, 14x) + \phi(14x, -14x)}{83691159552000} \\
& + \frac{\phi(16x, 16x) + \phi(16x, -16x)}{711374856192000} \\
& + (12804747411456000)^{-1} (\phi(18x, 18x) + \phi(18x, -18x)).
\end{aligned}$$

Theorem 2.2 Let $k \in \mathbb{N}$ and $\phi : X^{2k} \rightarrow [0, \infty)$ be a mapping, such that there exists $L < 1$ with $\phi(2x_1, \dots, 2x_k, 2y_1, \dots, 2y_k) \leq 2^{18}L\phi(x_1, \dots, x_k, y_1, \dots, y_k)$ for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$\|\text{Df}(x_1, y_1), \dots, \text{Df}(x_k, y_k)\|_k \leq \phi(x_1, \dots, x_k, y_1, \dots, y_k),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then there exists a unique octadecic mapping $Q : X \rightarrow Y$, such that

$$\|f(x_1) - Q(x_1), \dots, f(x_k) - Q(x_k)\|_k \leq \frac{1}{2^{18}(1-L)} \psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$, where

$$\begin{aligned}
\psi(x_1, \dots, x_k) &:= (3201186852864000)^{-1} (24310\phi(0, \dots, 0, x_1, \dots, x_k) \\
&+ 43758\phi(x_1, \dots, x_k, x_1, \dots, x_k) + 31824\phi(2x_1, \dots, 2x_k, x_1, \dots, x_k) \\
&+ 18564\phi(3x_1, \dots, 3x_k, x_1, \dots, x_k) + 8568\phi(4x_1, \dots, 4x_k, x_1, \dots, x_k) \\
&+ 3060\phi(5x_1, \dots, 5x_k, x_1, \dots, x_k) + 816\phi(6x_1, \dots, 6x_k, x_1, \dots, x_k) \\
&+ 153\phi(7x_1, \dots, 7x_k, x_1, \dots, x_k) + 18\phi(8x_1, \dots, 8x_k, x_1, \dots, x_k) \\
&+ \phi(9x_1, \dots, 9x_k, x_1, \dots, x_k) + \frac{\phi(0, \dots, 0, 2x_1, \dots, 2x_k)}{2} \\
&+ \left(\frac{4537567325}{6402373705728000} + \frac{48886173}{12705510619017216000} \right) \phi(0, \dots, 0, 0, \dots, 0) \\
&+ \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right)
\end{aligned}$$

$$\begin{aligned}
& \times (\phi(x_1, \dots, x_k, x_1, \dots, x_k) + \phi(x_1, \dots, x_k, -x_1, \dots, -x_k)) \\
& + \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) \\
& \times (\phi(2x_1, \dots, 2x_k, 2x_1, \dots, 2x_k) + \phi(2x_1, \dots, 2x_k, -2x_1, \dots, -2x_k)) \\
& + \frac{940105848}{6402373705728000} (\phi(3x_1, \dots, 3x_k, 3x_1, \dots, 3x_k) \\
& + \phi(3x_1, \dots, 3x_k, -3x_1, \dots, -3x_k)) + \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) \\
& \times (\phi(4x_1, \dots, 4x_k, 4x_1, \dots, 4x_k) + \phi(4x_1, \dots, 4x_k, -4x_1, \dots, -4x_k)) \\
& + \frac{115306920}{6402373705728000} (\phi(5x_1, \dots, 5x_k, 5x_1, \dots, 5x_k) \\
& + \phi(5x_1, \dots, 5x_k, -5x_1, \dots, -5x_k)) + \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) \\
& \times (\phi(6x_1, \dots, 6x_k, 6x_1, \dots, 6x_k) + \phi(6x_1, \dots, 6x_k, -6x_1, \dots, -6x_k)) \\
& + \frac{4538898}{6402373705728000} (\phi(7x_1, \dots, 7x_k, 7x_1, \dots, 7x_k) \\
& + \phi(7x_1, \dots, 7x_k, -7x_1, \dots, -7x_k)) + \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) \\
& \times (\phi(8x_1, \dots, 8x_k, 8x_1, \dots, 8x_k) + \phi(8x_1, \dots, 8x_k, -8x_1, \dots, -8x_k)) \\
& + \frac{24310(\phi(9x_1, \dots, 9x_k, 9x_1, \dots, 9x_k) + \phi(9x_1, \dots, 9x_k, -9x_1, \dots, -9x_k))}{6402373705728000} \\
& + \frac{\phi(10x_1, \dots, 10x_k, 10x_1, \dots, 10x_k) + \phi(10x_1, \dots, 10x_k, -10x_1, \dots, -10x_k)}{4184557977600} \\
& + \frac{\phi(12x_1, \dots, 12x_k, 12x_1, \dots, 12x_k) + \phi(12x_1, \dots, 12x_k, -12x_1, \dots, -12x_k)}{15692092416000} \\
& + \frac{\phi(14x_1, \dots, 14x_k, 14x_1, \dots, 14x_k) + \phi(14x_1, \dots, 14x_k, -14x_1, \dots, -14x_k)}{83691159552000} \\
& + \frac{\phi(16x_1, \dots, 16x_k, 16x_1, \dots, 16x_k) + \phi(16x_1, \dots, 16x_k, -16x_1, \dots, -16x_k)}{711374856192000} \\
& + (12804747411456000)^{-1} (\phi(18x_1, \dots, 18x_k, 18x_1, \dots, 18x_k) \\
& + \phi(18x_1, \dots, 18x_k, -18x_1, \dots, -18x_k)).
\end{aligned}$$

Proof The proof is similar to the proof of Theorem 2.1. $\square$

From Theorem 2.2, we get the following corollaries:

Corollary 2.3 Let $k \in \mathbb{N}$ and α, L be positive real numbers, such that $\frac{1}{2^{18}} \leq L < 1$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \alpha,$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then there exists a unique octadecic mapping $Q : X \rightarrow Y$, such that

$$\|f(x_1) - Q(x_1), \dots, f(x_k) - Q(x_k)\|_k \leq \frac{1}{2^{18}(1-L)}\beta,$$

for all $x_1, \dots, x_k \in X$, where

$$\begin{aligned}
\beta := & (3201186852864000)^{-1} \alpha (24310 + 43758 + 31824 + 18564 + 8568 + 3060 + 816 \\
& + 153 + 18 + 1 + \frac{1}{2} + \frac{4537567325}{6402373705728000} + \frac{48886173}{12705510619017216000} \\
& + 2 \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right)
\end{aligned}$$

$$\begin{aligned}
& + 2 \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) \\
& + \frac{18802116960}{6402373705728000} + 2 \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) \\
& + \frac{230613840}{6402373705728000} + 2 \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) \\
& + \frac{9077796}{6402373705728000} + 2 \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) \\
& + \frac{48620}{6402373705728000} + \frac{2}{41845579776000} + \frac{2}{15692092416000} \\
& + 2(83691159552000)^{-1} + 2(711374856192000)^{-1} + 2(12804747411456000)^{-1}.
\end{aligned}$$

Corollary 2.4 Let $k \in \mathbb{N}$ and α, p, L be positive real numbers, such that $L < 1$ and $2^p \leq 2^{18}L$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$\|\text{Df}(x_1, y_1), \dots, \text{Df}(x_k, y_k)\|_k \leq \alpha(\|x_1\|^p + \dots + \|x_k\|^p + \|y_1\|^p + \dots + \|y_k\|^p),$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then there exists a unique octadecic mapping $Q : X \rightarrow Y$, such that

$$\|f(x_1) - Q(x_1), \dots, f(x_k) - Q(x_k)\|_k \leq \frac{1}{2^{18}(1-L)} \psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$, where

$$\begin{aligned}
\psi(x_1, \dots, x_k) := & (3201186852864000)^{-1} \alpha(\|x_1\|^p + \dots + \|x_k\|^p) (24310 + 87516 \\
& + 31824(2^p + 1) + 18564(3^p + 1) + 8568(4^p + 1) + 3060(5^p + 1) + 816(6^p + 1) \\
& + 153(7^p + 1) + 18(8^p + 1) + 9^p + 1 + \frac{2^p}{2} \\
& + 4 \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right) \\
& + 4(2)^p \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) \\
& + \frac{4(3)^p 9401058480}{6402373705728000} + 4(4)^p \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) \\
& + \frac{4(5)^p 115306920}{6402373705728000} + 4(6)^p \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) \\
& + \frac{4(7)^p 4538898}{6402373705728000} + 4(8)^p \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) \\
& + \frac{4(9)^p 24310}{6402373705728000} + \frac{4(10)^p}{41845579776000} + \frac{4(12)^p}{15692092416000} \\
& + \frac{4(14)^p}{83691159552000} + \frac{4(16)^p}{711374856192000} \\
& + 4(18)^p (12804747411456000)^{-1}.
\end{aligned}$$

The following example shows that the assumption $2^p \leq 2^{18}L$ cannot be omitted in Corollary 2.4. We know from Example 1.2 that if

$$\|x_1, \dots, x_k\|_k = \sup\{|x_1|, \dots, |x_k|\},$$

then $(\mathbb{R}^k, \|\cdot\|_k) : k \in \mathbb{N}$ is a multi-normed space.

Example 2.5 Let $k \in \mathbb{N}$. We define $\phi : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\phi(x) := \begin{cases} 1 & x \in [1, \infty) \\ x^{18} & x \in (-1, 1) \\ -1 & x \in (-\infty, -1]. \end{cases}$$

We consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(4^n x)}{4^{18n}}, \quad (x \in \mathbb{R}).$$

Then f satisfies the following functional inequality:

$$\|\text{Df}(x_1, y_1), \dots, \text{Df}(x_k, y_k)\|_k \leq \frac{2^{18} + 18!}{4^{18} - 1} 4^{54} (|x_1|^{18} + \dots + |x_k|^{18} + |y_1|^{18} + \dots + |y_k|^{18}), \quad (2.1)$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}$.

Proof We have

$$|f(x)| \leq \frac{4^{18}}{4^{18} - 1}$$

for all $x \in \mathbb{R}$. Therefore, we see that f is bounded. Let $x, y \in \mathbb{R}$. If $|x|^{18} + |y|^{18} = 0$ or $|x|^{18} + |y|^{18} \geq \frac{1}{4^{18}}$, then

$$|\text{Df}(x, y)| \leq \frac{(2^{18} + 18!)4^{18}}{4^{18} - 1} \leq \frac{(2^{18} + 18!)4^{18}}{4^{18} - 1} 4^{18} (|x|^{18} + |y|^{18}).$$

Now, suppose that $0 < |x|^{18} + |y|^{18} < \frac{1}{4^{18}}$. Then there exists a nonnegative integer k such that

$$\frac{1}{4^{18(k+2)}} \leq |x|^{18} + |y|^{18} < \frac{1}{4^{18(k+1)}}.$$

Hence,

$$4^k x < \frac{1}{4} \quad \text{and} \quad 4^k y < \frac{1}{4},$$

and

$$\begin{aligned} &4^n(x + 9y), 4^n(x + 8y), 4^n(x + 7y), 4^n(x + 6y), 4^n(x + 5y), 4^n(x + 4y), \\ &4^n(x + 3y), 4^n(x + 2y), 4^n(x + y), 4^n x, 4^n(x - y), 4^n(x - 2y), 4^n(x - 3y), 4^n(x - 4y), \\ &4^n(x - 5y), 4^n(x - 6y), 4^n(x - 7y), 4^n(x - 8y), 4^n(x - 9y), 4^n y \in (-1, 1) \end{aligned}$$

for all $n = 0, 1, \dots, k - 1$. Thus we get

$$\begin{aligned} \frac{|\text{Df}(x, y)|}{|x|^{18} + |y|^{18}} &\leq \sum_{n=k}^{\infty} \frac{2^{18} + 18!}{4^{18n} (|x|^{18} + |y|^{18})} \leq \sum_{n=0}^{\infty} \frac{2^{18} + 18!}{4^{18n} 4^{18(k+2)} (|x|^{18} + |y|^{18})} 4^{36} \\ &\leq \sum_{n=0}^{\infty} \frac{2^{18} + 18!}{4^{18n}} 4^{36} = \frac{2^{18} + 18!}{4^{18} - 1} 4^{54}, \end{aligned}$$

or

$$|\text{Df}(x, y)| \leq \frac{2^{18} + 18!}{4^{18} - 1} 4^{54} (|x|^{18} + |y|^{18}).$$

Hence f satisfies (2.1) for all $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{R}$. Now, we claim that the octadecic functional equation (1.1) is not stable for $p = 18$ in Corollary 2.4. Suppose on the contrary that there exists an octadecic mapping $C : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\|f(x_1) - C(x_1), \dots, f(x_k) - C(x_k)\|_k \leq \beta(|x_1|^{18} + \dots + |x_k|^{18})$$

for some $\beta \in \mathbb{R}$ and all $x_1, \dots, x_k \in \mathbb{R}$. So $|f(x) - C(x)| \leq \delta|x|^{18}$ for some constant $\delta > 0$ and all $x \in \mathbb{R}$. Then there exists $\gamma \in \mathbb{R}$ for which $C(x) = \gamma x^{18}$ for all $x \in \mathbb{Q}$. Therefore,

$$\frac{|f(x)|}{x^{18}} \leq \delta + |\gamma|, \quad (x \in \mathbb{Q}).$$

Let $M \in \mathbb{N}$ be such that $M > \delta + |\gamma|$. If x is a rational number in $(0, \frac{1}{4^{M-1}})$, then we have $4^n x \in (0, 1)$ for each $n = 0, 1, 2, \dots, M-1$. Consequently, for such an x we have

$$\frac{f(x)}{x^{18}} = \sum_{n=0}^{\infty} \frac{\phi(4^n x)}{4^{18n} x^{18}} \geq \sum_{n=0}^{M-1} \frac{4^{18n} x^{18}}{4^{18n} x^{18}} = M > \delta + |\gamma|,$$

which leads to a contradiction. $\square$

Corollary 2.6 Let $k \in \mathbb{N}$ and α, p, L be positive real numbers, such that $L < 1$ and $2^{2kp} \leq 2^{18}L$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$\|\text{Df}(x_1, y_1), \dots, \text{Df}(x_k, y_k)\|_k \leq \alpha \|x_1\|^p \dots \|x_k\|^p \|y_1\|^p \dots \|y_k\|^p,$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then there exists a unique octadecic mapping $Q : X \rightarrow Y$, such that

$$\|f(x_1) - Q(x_1), \dots, f(x_k) - Q(x_k)\|_k \leq \frac{1}{2^{18}(1-L)} \psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$, where

$$\begin{aligned} \psi(x_1, \dots, x_k) := & (3201186852864000)^{-1} \alpha \|x_1\|^{2p} \dots \|x_k\|^{2p} (43758 + 31824(2)^{kp} \\ & + 18564(3)^{kp} + 8568(4)^{kp} + 3060(5)^{kp} + 816(6)^{kp} + 153(7)^{kp} + 18(8)^{kp} + 9^{kp} \\ & + 2 \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right) \\ & + 2(4)^{kp} \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) \\ & + \frac{2(9)^{kp} 9401058480}{6402373705728000} + 2(16)^{kp} \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) \\ & + \frac{2(25)^{kp} 115306920}{6402373705728000} + 2(36)^{kp} \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) \\ & + \frac{2(49)^{kp} 4538898}{6402373705728000} + 2(64)^{kp} \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) \\ & + \frac{2(81)^{kp} 24310}{6402373705728000} + \frac{2(100)^{kp}}{4184557977600} + \frac{2(144)^{kp}}{15692092416000} \\ & + \frac{2(196)^{kp}}{83691159552000} + \frac{2(256)^{kp}}{711374856192000} \\ & + 2(324)^{kp} (12804747411456000)^{-1}). \end{aligned}$$

Corollary 2.7 Let $k \in \mathbb{N}$ and α, p, L be positive real numbers, such that $L < 1$ and $4^{kp} \leq 2^{18}L$. Let $f : X \rightarrow Y$ be a mapping satisfying

$$\begin{aligned} \|\text{Df}(x_1, y_1), \dots, \text{Df}(x_k, y_k)\|_k & \leq \alpha (\|x_1\|^p \dots \|x_k\|^p \|y_1\|^p \dots \|y_k\|^p \\ & + \|x_1\|^{2kp} + \dots + \|x_k\|^{2kp} + \|y_1\|^{2kp} + \dots + \|y_k\|^{2kp}), \end{aligned}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in X$. Then there exists a unique octadecic mapping $Q : X \rightarrow Y$, such that

$$\|f(x_1) - Q(x_1), \dots, f(x_k) - Q(x_k)\|_k \leq \frac{1}{2^{18}(1-L)} \psi(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in X$, where

$$\begin{aligned} \psi(x_1, \dots, x_k) := & (3201186852864000)^{-1} \alpha \|x_1\|^{2p} \dots \|x_k\|^{2p} (43758 + 31824(2)^{kp} \\ & + 18564(3)^{kp} + 8568(4)^{kp} + 3060(5)^{kp} + 816(6)^{kp} + 153(7)^{kp} + 18(8)^{kp} + 9^{kp} \\ & + 2 \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right) \\ & + 2(4)^{kp} \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) \\ & + \frac{2(9)^{kp} 9401058480}{6402373705728000} + 2(16)^{kp} \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) \\ & + \frac{2(25)^{kp} 115306920}{6402373705728000} + 2(36)^{kp} \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) \\ & + \frac{2(49)^{kp} 4538898}{6402373705728000} + 2(64)^{kp} \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) \\ & + \frac{2(81)^{kp} 24310}{6402373705728000} + \frac{2(100)^{kp}}{4184557977600} + \frac{2(144)^{kp}}{15692092416000} \\ & + \frac{2(196)^{kp}}{83691159552000} + \frac{2(256)^{kp}}{711374856192000} \\ & + 2(324)^{kp} (12804747411456000)^{-1} \\ & + (3201186852864000)^{-1} \alpha (\|x_1\|^{2kp} + \dots + \|x_k\|^{2kp}) (24310 + 87516 \\ & + 31824(4^{kp} + 1) + 18564(9^{kp} + 1) + 8568((16)^{kp} + 1) + 3060((25)^{kp} + 1) \\ & + 816((36)^{kp} + 1) + 153((49)^{kp} + 1) + 18((64)^{kp} + 1) + (81)^{kp} + 1 + \frac{2^{2kp}}{2} \\ & + 4 \left(\frac{3234988548}{6402373705728000} + \frac{1}{2324754432000} + \frac{1}{355687428096000} \right) \\ & + 4(4)^{kp} \left(\frac{1922913180}{6402373705728000} + \frac{1}{41845579776000} + \frac{1}{292626432000} \right) \\ & + \frac{4(9)^{kp} 9401058480}{6402373705728000} + 4(16)^{kp} \left(\frac{371153520}{6402373705728000} + \frac{1}{402361344000} \right) \\ & + \frac{4(25)^{kp} 115306920}{6402373705728000} + 4(36)^{kp} \left(\frac{27123330}{6402373705728000} + \frac{1}{689762304000} \right) \\ & + \frac{4(49)^{kp} 4538898}{6402373705728000} + 4(64)^{kp} \left(\frac{481338}{6402373705728000} + \frac{1}{1494484992000} \right) \\ & + \frac{4(81)^{kp} 24310}{6402373705728000} + \frac{4(100)^{kp}}{4184557977600} + \frac{4(144)^{kp}}{15692092416000} \\ & + \frac{4(196)^{kp}}{83691159552000} + \frac{4(256)^{kp}}{711374856192000} \\ & + 4(324)^{kp} (12804747411456000)^{-1}). \end{aligned}$$

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