Maryam Jahandideh Khangeshlaghi ·
Mohammad Reza Darafsheh

NSE characterization of the Chevalley group $G_2(4)$

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Abstract Let G be a group and $\omega(G) = \{o(g) | g \in G\}$ be the set of element orders of G . Let $k \in \omega(G)$ and $s_k = |\{g \in G | o(g) = k\}|$. Let $nse(G) = \{s_k | k \in \omega(G)\}$. In this paper, we prove that if G is a group and $G_2(4)$ is the Chevalley group such that $nse(G) = nse(G_2(4))$, then $G \cong G_2(4)$.

المخلص

ليكن G زمرة وليكن $\omega(G) = \{o(g) | g \in G\}$ مجموعة عناصر ترتيبات G . لتكن $k \in \omega(G)$ و $s_k = \{g \in G | o(g) = k\}$. لتكن $nse(G) = \{s_k | k \in \omega(G)\}$. في هذا البحث، نثبت أنه إذا كان G زمرة و $G_2(4)$ زمرة شوفالي بحيث أن $nse(G) = nse(G_2(4))$ ، فإن $G \cong G_2(4)$.

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1 Introduction

Let G be a finite group and $\omega(G)$ be the set of element orders of G . If $k \in \omega(G)$, then s_k is the number of elements of order k in G . Let $nse(G) = \{s_k | k \in \omega(G)\}$. If n is a positive integer, the set of all prime divisors of n is denoted by $\pi(n)$. The number of the Sylow p -subgroups P_p of G is denoted by n_p or $n_p(G)$. We set $\pi(G) = \pi(|G|)$. To see notations concerning finite simple group, we refer the reader to [1]. A finite group G is called a simple K_n -group, if G is a simple group and $|\pi(G)| = n$. In 1987, J. G. Thompson posed the following problem related to algebraic number fields [2]: Let $T(G) = \{(k, s_k) | k \in \omega(G), s_k \in nse(G)\}$. Suppose that $T(G) = T(H)$ for some finite group H . If G is a finite solvable group, is it true that H is also necessarily solvable? If the groups G and H have the same order type, then $|G| = |H|$ and $nse(G) = nse(H)$. Due to this problem, many results were obtained as follows:

Result 1 [3,4] Let G be a group and S a simple K_i -group, where $i = 3, 4$. Then, $G \cong S$ if and only if $|G| = |S|$ and $nse(G) = nse(S)$.

Result 2 [5,6] The two groups A_{12} and A_{13} are characterizable by order and nse .

Result 3 [7] All sporadic simple groups are characterizable by nse and order.

M. J. Khangeshlaghi (✉)
Department of Mathematics, Mahshahr Branch, Islamic Azad University, Mahshahr, Iran
E-mail: m.jahandideh@mhriau.ac.ir

M. R. Darafsheh
School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran

Result 4 [8] $L_2(2^m)$, with $2^m + 1$ prime, is $2^m - 1$ prime, is characterized by nse and order.

Result 5 [4, 9] $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the nse .

Result 6 [10] $L_3(4)$ is characterized by nse .

Result 7 [11] $L_5(2)$ is characterized by nse .

Result 8 [12] $U_3(5)$ is characterized by nse .

Up to now, some groups can be characterized by only the set $nse(G)$. The aim of this paper is to prove that the Chevalley group is characterizable by nse .

Main Theorem. Let G be a group such that $nse(G) = nse(G_2(4))$ where $G_2(4)$ is the Chevalley group of type G_2 over $GF(4)$. Then $G \cong G_2(4)$.

2 Preliminaries

In this section, we will give some lemmas which will be used in the proof of the main theorem.

Lemma 2.1 [13] Let G be a finite group and n be a positive integer dividing $|G|$. If $L_n(G) = \{g \in G \mid g^n = 1\}$, then $|n||L_n(G)|$.

Lemma 2.2 [14] Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p -subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic, the number of elements of order n is always a multiple of p^s .

Lemma 2.3 [4] Let G be a group containing more than two elements. If the maximum number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Lemma 2.4 [15] Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and h_m be the number of Hall π -subgroups of G . Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$, q_i 's should be prime numbers satisfying the following conditions for all $i \in \{1, 2, \dots, s\}$:

1. $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ for some p_j .
2. The order of some chief factor of G is divided by $q_i^{\beta_i}$. We need the structure of simple K_n -group with $n = 4, 5$.

Lemma 2.5 [16] Let G be a simple K_4 -group. Then G is isomorphic to one of the following groups:

1. A_7, A_8, A_9 or A_{10} .
2. M_{11}, M_{12} or J_2 .
3. One of the following:
 - (i) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1, b \geq 1, c \geq 1$, and v is a prime greater than 3.
 - (ii) $L_2(2^m)$, where $2^m - 1 = u, 2^m + 1 = 3t^b$ with $m \geq 2, u, t$ are primes, $t > 3, b \geq 1$.
 - (iii) $L_2(3^m)$, where $3^m + 1 = 4t, 3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b, 3^m - 1 = 2u$, with $m \geq 2, u, t$ are odd primes, $b \geq 1, c \geq 1$.
 - (iv) One of the following 28 simple groups:

$L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), Sz(8), Sz(32), {}^2D_4(2)$ or ${}^2F_4(2)$.

Lemma 2.6 [6] Each simple K_5 -group is isomorphic to one of the following simple groups:

1. $L_2(q)$ with $|\pi(q^2 - 1)| = 4$.
2. $L_3(q)$ with $|\pi((q^2 - 1)(q^3 - 1))| = 4$.
3. $U_3(q)$ with q satisfies $|\pi((q^2 - 1)(q^3 + 1))| = 4$.
4. $O_5(q)$ with $|\pi(q^4 - 1)| = 4$.
5. $Sz(2^{2m+1})$ with $|\pi((2^{2m+1} - 1)(2^{4m+1} + 1))| = 4$.
6. $R(q)$, where q is an odd power of 3, and $|\pi(q^2 - 1)| = 3$ and $|\pi(q^2 - q + 1)| = 1$.
7. The following 30 simple groups:

$A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, L_4(4), L_4(5), L_4(7), L_5(2), L_5(3), L_6(2), O_7(3), O_9(2), PSP_8(2), U_4(4), U_4(5), U_4(7), U_4(9), U_5(3), U_6(2), O_8^+(3), O_8^-(2), {}^3D_4(3), G_2(4), G_2(5), G_2(7)$ or $G_2(9)$.

Lemma 2.7 Let G be a simple K_n -group with $n = 4, 5$ and 13 divides $|G|$ and $|G|$ divides $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$. Then, G is one of the following groups: $G_2(4)$, $Sz(8)$, $L_2(25)$, $L_2(13)$, $L_2(27)$, ${}^2F_4(2)$, $U_3(4)$.

Proof We prove this lemma in the following two cases:

Case 1. G is a simple K_4 -group. Order consideration rules out of the cases (1) and (2) of Lemma 2.5. Hence, we use Lemma 2.5 (3). In this case, it is easily seen that G is one of the following groups:

$$Sz(8), L_2(25), L_2(13), L_2(27), {}^2F_4(2), U_3(4).$$

Case 2. G is a simple K_5 -group. By Lemma 2.6, we only have $G = G_2(4)$. □

3 Proof of the main theorem

In this section, we prove that if G is a group and $nse(G) = nse(G_2(4))$, then $G \cong G_2(4)$. However, we have some observations and lemmas before that.

Observation 3.1 If s_n is the number of elements of order n in G , then $s_n = k\varphi(n)$, such that k is the number of cyclic subgroups of order n in G .

Observation 3.2 If $n > 2$, then $\varphi(n)$ is even.

Lemma 3.3 If $m \in \omega(G)$, then $\varphi(m) | s_m$ and $m | \sum_{d|m} s_d$.

Proof It follows from Lemma 2.1. □

Lemma 3.4 Let G be a group such that $nse(G) = nse(G_2(4))$, then G is a finite group.

Proof It follows from Lemma 2.3. □

Theorem 3.5 Let G be a group with $nse(G) = nse(G_2(4)) = \{(1, 69615, 1401920, 982800, 3354624, 22276800, 11980800, 15724800, 50319360, 38707200, 67092480, 23961600)\}$, where $G_2(4)$ is the Chevalley group of type G_2 over $GF(4)$. Then $G \cong G_2(4)$.

Proof of the main theorem. It follows from Lemma 3.4, G is a finite group. According to Lemma 2.1, $\pi(G) \subseteq \{2, 3, 5, 7, 13, 17, 337, 982801, 23961601\}$. The first claim is $\pi(G) = \{2, 3, 5, 7, 13\}$. According to the Observation 3.2, $s_2 = 69615$ and $2 \in \pi(G)$. Since s_{337} is not equal to none of nse values, $337 \notin \pi(G)$. Now, we show that $982801 \notin \pi(G)$. If $982801 \in \pi(G)$, $s_{982801} = 982800$. If $2 \cdot 982801 \in \pi(G)$, $\varphi(2 \cdot 982801) | s_{2 \cdot 982801}$, and by Lemma 2.1 $s_{2 \cdot 982801} = 982800$ or 15724800 , and on the other hand, $2 \cdot 982801 | 1 + s_2 + s_{982801} + s_{2 \cdot 982801} (982800, 15724800)$, a contradiction. Now, we consider Sylow 92801-subgroup P_{982801} of G and $\Sigma = \{g \in G | o(g) = 2\}$. P_{982801} acts fixed point freely on Σ , because if $P_{982801} | g^a = \{a \in P_{982801} | g^a = g\} \neq 1$, where $g \in \Sigma$, then G has an element of order $2 \cdot 982801$. Hence, $|P_{982801}| |\Sigma| = s_2$, a contradiction. As a result, $982801 \notin \pi(G)$. Similarly, we can prove that the prime numbers 24897601 and 23961601 do not belong to $\pi(G)$. Hence we have $\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}$. We have:

1. If $2^a \in \omega(G)$, then $2^{a-1} | s_{2^a}$ and $0 \leq a \leq 15$.
2. If $3^a \in \omega(G)$, then $2 \cdot 3^{a-1} | s_{3^a}$ and $0 \leq a \leq 4$.
3. If $5^a \in \omega(G)$, then $4 \cdot 5^{a-1} | s_{5^a}$ and $0 \leq a \leq 3$.
4. If $7^a \in \omega(G)$, then $6 \cdot 7^{a-1} | s_{7^a}$ and $0 \leq a \leq 2$.
5. If $13^a \in \omega(G)$, then $12 \cdot 13^{a-1} | s_{13^a}$ and $0 \leq a \leq 2$.
6. If $17^a \in \omega(G)$, then $16 \cdot 17^{a-1} | s_{17^a}$ and $0 \leq a \leq 1$.

We show that the prime number 17 does not belong to $\pi(G)$. To prove this, we distinguish several cases.

Case 1. If $\pi(G) = \{2, 17\}$, then $s_{17} = 11980800$. If P_{17} is a Sylow 17-subgroup of G , according to Lemma 2.1, $|P_{17}| | 1 + s_{17}$ and $|P_{17}| = 17$. Thus

$$n_{17} = \frac{s_{17}}{\varphi(17)} = 748800 = [G : N_G(P_{17})] \Rightarrow 3, 5, 13 \in \pi(G),$$

which is impossible.

Case 2. If $\pi(G) = \{2, 3, 17\}$, the same as case 1, we have $5 \in \pi(G)$. But it is impossible.

Case 3. If $\pi(G) = \{2, 3, 5, 17\}$ or $\pi(G) = \{2, 3, 5, 7, 17\}$, similar to case 1, we have $13 \in \pi(G)$, which is not possible.

Case 4. If $\pi(G) = \{2, 3, 5, 7, 13, 17\}$, $|G| = 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdot 7^{m_4} \cdot 13^{m_5} \cdot 17$ and $251596800 + 1401920k_1 + 982800k_2 + 3354624k_3 + 22276800k_4 + 11980800k_5 + 15724800k_6 + 50319360k_7 + 38707200k_8 + 67092480k_9 + 23961600k_{10} = 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdot 7^{m_4} \cdot 13^{m_5} \cdot 17$ where $k_1, k_2, k_{10}, m_1, \dots, m_6$ are nonnegative integers and

$$0 < \sum_{i=1}^{10} k_i \leq 5748 \Rightarrow 251596800 < 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdot 7^{m_4} \cdot 13^{m_5} \cdot 17 < 5748 \cdot 67092480.$$

This inequality does not hold. Hence, 17 does not belong to $\pi(G)$ and $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$. Now, we prove $\pi(G) = \{2, 3, 5, 7, 13\}$. To do this, we distinguish several cases.

Case 1. $\pi(G) = \{2\}$.

In this case, $\omega(G) \subseteq \{1, 2, 2^2, 2^{15}\}$ and $251596800 + 1401920k_1 + 982800k_2 + 3354624k_3 + 22276800k_4 + 11980800k_5 + 15724800k_6 + 50319360k_7 + 38707200k_8 + 67092480k_9 + 23961600k_{10} = 2^m$ such that k_1, k_2, k_{10}, m are nonnegative integers. $0 < \sum_{i=1}^{10} k_i \leq 4$ but this equation has no solution in integers.

Case 2. $\pi(G) = \{2, 3\}$.

If $2^a \cdot 3^b \in \omega(G)$ according to Lemma 2.1, $0 \leq a \leq 15, 0 \leq b \leq 4$. We have $\exp(P_3) = 3, 9, 27$ or 81 . Suppose that $\exp(P_3) = 3$. In this case, it follows: $|P_3||1 + s_3(1401920)| \Rightarrow |P_3||1401921| \Rightarrow |P_3||27|$. If $|P_3| = 3, n_3 = \frac{s_3}{\varphi(3)} = \frac{1401920}{2} = 700960 \Rightarrow [G : N_G(P_3)] = 700960$, then $5 \in \pi(G)$, a contradiction. If $|P_3| = 9, |G| = 2^m \cdot 3^2$, such that $0 \leq m \leq 15$. In addition, calculations show: $251596800 + 1401920k_1 + 982800k_2 + 3354624k_3 + 22276800k_4 + 11980800k_5 + 15724800k_6 + 50319360k_7 + 38707200k_8 + 67092480k_9 + 23961600k_{10} = 2^m \cdot 3^2$, such that k_1, k_2, k_{10}, m are nonnegative integers. Then

$$0 < \sum_{i=1}^{10} k_i \leq 36 \Rightarrow 251596800 \Rightarrow 2^m \cdot 3^2 \leq 36 \cdot 67092480 \Rightarrow m > 15,$$

which is a contradiction.

If $|P_3| = 27, |G| = 2^m \cdot 3^2$, such that $0 \leq m \leq 15$. Similarly, as $|P_3| = 9$, we also get a contradiction. Now, we suppose that $\exp(P_3) = 9$, according to Lemma 2.1 $|P_3||1 + s_3 + s_9$. There are several cases for s_9 . Namely, 982800, 3354624, 22276800, 11980800, 15724800, 50319360, 15724800, 38707200, 67092480, and 23961600. For all cases, we have $n_3 = \frac{s_9}{\varphi(9)} = [G : N_G(P_3)]$, then $5 \in \pi(G)$, a contradiction.

If $\exp(P_3) = 27, s_{27} = 9828200, 3354624, 22276800, 11980800$ and 15724800 also $n_3 = \frac{s_{27}}{\varphi(27)} = [G : N_G(P_3)]$, then 5 or 13 or 7 $\in \pi(G)$, a contradiction.

If $\exp(P_3) = 81, s_{81} = 15724800, 50319360$ and 38707200 . Similar as in the previous case, we obtain a contradiction.

Case 3. $\pi(G) = \{2, 5\}$.

If $2^a \cdot 5^b$ belongs to $\pi(G)$, $0 \leq a \leq 15, 0 \leq b \leq 3$. In this case, $\exp(P_5) = 5, 25$ or 125 . If $\exp(P_5) = 5, |P_5||1 + s_5(3354624)|$, then $|P_5||125|$.

If $|P_5| = 5, n_5 = \frac{s_5}{\varphi(5)} = [G : N_G(P_5)] \Rightarrow 2, 7$ and $13 \in \pi(G)$, a contradiction.

If $|P_5| = 25, |G| = 2^a \cdot 5^2$ and also $251596800 + 1401920k_1 + 982800k_2 + 3354624k_3 + 22276800k_4 + 11980800k_5 + 15724800k_6 + 50319360k_7 + 38707200k_8 + 67092480k_9 + 23961600k_{10} = 2^m \cdot 5^2$, such that $k_1, k_2, \dots, k_{10}, m$ are nonnegative integers. Then, $0 < \sum_{i=1}^{10} k_i \leq 36 \Rightarrow 251596800 \leq 2^m \cdot 5^2 \leq 36 \cdot 67092480$, then $m > 15$, which is a contradiction. If $|P_5| = 125$, similarly, we have a contradiction. If $\exp(P_5) = 25$, then $|P_5||1 + s_5 + s_{25}(982800, 22276800, 11980800, 15724800, 38707200, 67092480 \text{ or } 23961600)|$, then $|P_5||25 \cdot |P_5|$ can be 5 or 25, similar to $\exp(P_5) = 5$ we get a contradiction. Now if $\exp(P_5) = 125, s_{125} = 982800, 22276800, 11980800$ or 15724800 .

In this case, $n_5 = \frac{s_{125}}{\varphi(125)} = [G : N_G(P_5)]$, then $3 \in \pi(G)$, a contradiction. Case 4. $\pi(G) = \{2, 7\}$.

In this case, $s_7 = 11980800$ and if $7^2 \in \omega(G), s_{7^2} = 38707200, |P_7||1 + s_7 + s_{49}$. Hence $|P_7||49|$.

If $|P_7| = 7, n_7 = \frac{s_7}{\varphi(7)} = 1996800 \Rightarrow 5, 3$ and $13 \in \pi(G)$, a contradiction. If $|P_7| = 49, n_7 = \frac{s_{49}}{\varphi(49)} = 921600$, then 3 and 5 $\in \pi(G)$, a contradiction. So 7^2 does not belong to $\omega(G)$ and $|P_7||1 + s_7$. Since $|P_7| = 7$, similar as above, we have a contradiction.

Case 5. $\pi(G) = \{2, 13\}$.

We have $s_{13} = 38707200$ and $s_{13^2} = 11980800$. According to Lemma 2.1, $|P_{13}||1 + s_{13} + s_{13^2}|$ and $|P_{13}||13^2|$.

If $|P_{13}| = 13$, $n_{13} = \frac{s_{13}}{\varphi(13)} = 3225600$, then $5 \in \pi(G)$, a contradiction. If $|P_3| = 13^2$, $n_3 = \frac{s_{13^2}}{\varphi(13^2)} = 3225600$, then $5 \in \pi(G)$, a contradiction. Hence $13^2 \notin \omega(G)$ and $|G| = 2^a \cdot 13$, $|P_3| = 13$. Again we get a contradiction similar as above.

Case 6. $\pi(G) = \{2, 3, 5\}$.

In this case, $|G| = 2^a \cdot 3^b \cdot 5^c$, where $0 \leq a \leq 15$, $0 \leq b \leq 4$, $0 \leq c \leq 3$. $251596800 + 1401920k_1 + 982800k_2 + 3354624k_3 + 22276800k_4 + 11980800k_5 + 15724800k_6 + 50319360k_7 + 38707200k_8 + 67092480k_9 + 23961600k_{10} = 2^a \cdot 3^b \cdot 5^c$, such that $k_1, k_2, \dots, k_{10}, a, b, c$ are nonnegative integers. $0 < \sum_{i=1}^{10} k_i \leq 308 \Rightarrow 251596800 \leq 2^a \cdot 3^b \cdot 5^c \leq 308 \cdot 67092480$, then $a > 15$, it is a contradiction. In the remaining cases, in the same way, we obtain a contradiction. Therefore, $\pi(G) = \{2, 3, 5, 7, 13\}$. Now, we prove that $2.7 \notin \omega(G)$. If 2.7 belongs to $\omega(G)$, let P_7 and Q_7 be Sylow 7-subgroups of G , P_7 and Q_7 are conjugate in G and $C_G(P_7)$ and $C_G(Q_7)$ are conjugate in G . Therefore, $s_{14} = \varphi(14) \cdot n_7 \cdot k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_7)$. Hence, $n_7 = \frac{s_7}{\varphi(7)} = 1996800$ and $s_{14} = 6 \cdot 1996800 \cdot k \Rightarrow s_{14} = 11980800$, since 14 does not divide $1 + s_2 + s_7 + s_{14}$, we conclude that 2.7 does not belong to $\omega(G)$. It follows that the Sylow 2-group of G acts fixed point freely on the set of elements of order 7 so $|P_2||s_7$ and $|P_2||2^{12}$. Next, it is shown by the way of contradiction, $3.13 \notin \omega(G)$. If $3.13 \in \omega(G)$, let P_{13} and Q_{13} be Sylow 13-subgroups of G , then $s_{39} = \varphi(39) \cdot n_{13} \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_{13})$. Therefore, $n_{13} = \frac{s_{13}}{\varphi(13)} = 3225600$ and $s_{39} = 24 \cdot 3225600 \cdot k$. Due to the values of nse and $77414400|s_{39}|$, it is impossible and it follows that the Sylow 3-group of G acts fixed point freely on the set of elements of order 13, so $|P_3||s_{13}|$; therefore, $|P_3||3^3$. If $5.13 \in \omega(G)$, s_{65} does not belong to nse and similarly as before case $|P_5||s_{13}|$; therefore, $|P_3||3^3$. If $5.13 \in \omega(G)$, s_{65} does not belong to nse and similarly as before case $|P_5||s_{13}|$ and $|P_5||5^2$.

Therefore, $|G| = 2^m \cdot 3^n \cdot 5^k \cdot 7 \cdot 13$ and also $\sum_{s_k \in nse(G)} s_k = 251596800 = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \leq |G| = 2^m \cdot 3^n \cdot 5^k \cdot 7 \cdot 13 \leq 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$; hence, $|G| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$. Therefore, we proved $|G| = |G_2(4)|$. Now, if we supposed that $G \not\cong G_2(4)$, we first prove that G is not solvable. If G is solvable, then, by Lemma 2.4, for every prime p , $p^{p^j} \equiv 1 \pmod{q}$ for some q . Thus, we have $7 \equiv 1 \pmod{13}$, a contradiction.

Therefore, G is nonsolvable. Hence, G has a normal series $1 \triangleleft K \triangleleft L \triangleleft G$, such that $\frac{L}{K}$ is isomorphic to a simple K_i -group, where $i = 3, 4, 5$.

If $\frac{L}{K}$ is isomorphic to one of the simple K_3 -group, by [16], $\frac{L}{K}$ is isomorphic to $A_5, A_6, L_2(7), L_2(8), U_3(3)$ or $U_4(2)$. Suppose that $\frac{L}{K} \cong A_5$, hence, $n_5(\frac{L}{K}) = n_5(A_5) = 6$. Therefore, $n_5(G) = 6t$, such that 5 does not divide t ; then, $t = \frac{s_5}{24} \cdot 6$. There are several values for s_5 , but for all cases $5|t$ and it is a contradiction. Similarly, for the groups $A_6, L_2(7), L_2(8), U_3(3)$ and $U_4(2)$, we also can rule out these cases.

If $\frac{L}{K}$ is isomorphic to a simple K_n -group with $n = 4, 5$, then by Lemma 2.7, L/K is isomorphic to $G_2(4), Sz(8), L_2(25), L_2(13), L_2(27), {}^2F_4(2), U_3(4)$.

If $\frac{L}{K} \cong L_2(13)$; then from [3], $14 = n_{13}(\frac{L}{K}) = n_{13}(L_2(13))$, and so $n_{13}(G) = 14t$, such that 13 does not divide t for some integer t . Then, $s_{13} = 12 \cdot 14t = 12 \cdot 3225600$, and then, $t = 230400$. Therefore $2^{10} \cdot 5^2 \cdot 3^2 || K || 2^{10} \cdot 3^2 \cdot 5^2$; hence, $|K| = 2^{10} \cdot 3^2 \cdot 5^2$ and $N_K(P_{13}) = 1$. Therefore, $K \cap N_K(P_{13}) = K \cap C_K(P_{13})$, which means that $K \times P_{13}$ is a Frobenius group and so $|P_{13}||\text{Aut}(K)|$, a contradiction. Similarly, for the other cases, we can get a contradiction. The last case $\frac{L}{K} \cong G_2(4)$. In this case, $G_2(4) \cong \frac{L}{K} \leq \frac{G}{K}$ and $|G| = |G_2(4)|$, then $K = 1$ and $G \cong G_2(4)$. It is a contradiction to the primary hypothesis. In general, we recognized all steps and obtained that $G \cong G_2(4)$. This complete the proof.

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