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Turán type inequalities for the q -exponential functions

Received: 16 April 2016 / Accepted: 1 August 2017 / Published online: 21 August 2017
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Abstract In this paper, our aim is to deduce some sharp Turán type inequalities for the remainder q -exponential functions. Our results are shown to be generalizations of results which were obtained by Alzer (Arch Math 55, 462–464, 1990).

Mathematics Subject Classification 33B10 · 39B62

المخلص

هدفنا في هذه الورقة هو استنتاج بعض المتراجحات الحادة من نوع توران (Turan) لباقي الدوال q -أسية. سوف نبين أن نتائجنا هي تعميم لتلك النتائج التي حصل عليها ألزر (Alzer) من قبل.

1 Introduction

Paul Turán [11] established, in 1950, an interesting result in the special function theory, which states that

$$[P_n(x)]^2 - P_{n+1}(x)P_{n-1}(x) \geq 0, \quad (1)$$

where $-1 < x < 1$, $n \in \mathbb{N}$ and $P_n(\cdot)$ stands for the classical Legendre polynomial. The inequality (1) has been extended in several directions for various orthogonal polynomials and special functions: for example, Laguerre and Hermite polynomials [10], Kummer hypergeometric functions [6, 9], different types of Bessel functions [2, 3], Dunkl kernel and q -Dunkl kernel [8], q -Kummer hypergeometric functions [7], and some others.

In this paper, we consider two q -analogues of the exponential function and investigates the tails of Taylor expansions of the form $I_n(z; q) = \sum_{k=n}^{\infty} a_k(q)z^k$. The main results are Turán type inequalities, i.e., inequalities of the form

$$c_n(q)I_n(z; q)^2 \leq I_{n-1}(z; q)I_{n+1}(z; q) \leq I_n(z; q)^2.$$

The limit relation, as q tends to 1 from below, is a relation proved by Alzer [1].

This paper is organized as follows: in Sect. 2, we present some preliminary results and notations that will be useful in the sequel. In Sect. 3, we investigate some Turán type inequalities for the q -exponential functions.

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2 Notations and preliminaries

Throughout this paper, we fix $q \in (0, 1)$. We refer to [4] and [5] for the definitions, notations, and properties of the q -shifted factorials and the q -analogues of exponential functions. However, for the reader's convenience, we give a few of these relations below.

2.1 Basic symbols

Let $a \in \mathbb{R}$, the q -shifted factorials are defined by the following:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),$$

and we write

$$(a_1, a_2, \dots, a_p; q) = (a_1; q)_n (a_2; q)_n \dots (a_p; q)_n, \quad n = 0, 1, 2, \dots$$

Note that for $q \rightarrow 1$, the expression $\frac{(q^a; q)_n}{(1-q)^n}$ tends to $(a)_n = a(a+1) \dots (a+n-1)$.

2.2 q -analogues of exponential functions

For $q \in (0, 1)$, the q -analogues of the exponential function are given by the following:

$$e(q; z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1. \quad (2)$$

and

$$E(q; z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{(q; q)_n} = (-z; q)_\infty = \prod_{k=0}^{\infty} (1 + zq^k), \quad z \in \mathbb{C}. \quad (3)$$

We denote by $I_n(q; z)$ and $\mathcal{I}_n(q; z)$ the differences

$$I_n(q; z) = e(q; z) - \sum_{k=0}^n \frac{z^k}{(q; q)_k}, \quad 0 < z < 1 \quad (4)$$

and

$$\mathcal{I}_n(q; z) = e(q; z) - \sum_{k=0}^n q^{\frac{k(k-1)}{2}} \frac{z^k}{(q; q)_k}, \quad z > 0, \quad (5)$$

where n is a nonnegative integer.

3 Turán types inequalities for q -analogues of the exponential functions

Our first main result is the next theorem.

Theorem 3.1 For every $n \in \mathbb{N}$, $q \in (0, 1)$, and $0 < z < 1$, the following Turán type inequalities:

$$\frac{1 - q^{n+1}}{1 - q^{n+2}} (I_n(q; z))^2 < I_{n-1}(q; z) I_{n+1}(q; z) < (I_n(q; z))^2, \quad (6)$$

hold, where $\frac{1 - q^{n+1}}{1 - q^{n+2}}$ is the best possible constant.

Proof Let $n \in \mathbb{N}$ and $q \in (0, 1)$, we have

$$I_{n-1}(q; z) = I_n(q; z) + \frac{z^n}{(q; q)_n} \quad (7)$$

and

$$I_{n+1}(q; z) = I_n(q; z) - \frac{z^{n+1}}{(q; q)_{n+1}}. \quad (8)$$

Thus

$$\begin{aligned} I_{n-1}(q; z)I_{n+1}(q; z) - I_n^2(q; z) &= I_n(q; z) \left(\frac{z^n}{(q; q)_n} - \frac{z^{n+1}}{(q; q)_{n+1}} \right) - \frac{z^{2n+1}}{(q; q)_n(q; q)_{n+1}} \\ &= \frac{z^n}{(q; q)_n} \left(\frac{z^{n+1}}{(q; q)_{n+1}} + \sum_{k=n+2}^{\infty} \frac{z^k}{(q; q)_k} \right) - \frac{z^{n+1}}{(q; q)_{n+1}} I_n(z, q) - \frac{z^{2n+1}}{(q; q)_n(q; q)_{n+1}} \\ &= \sum_{k=n+2}^{\infty} \frac{z^{k+n}}{(q; q)_n(q; q)_k} - \sum_{k=n+1}^{\infty} \frac{z^{k+n+1}}{(q; q)_k(q; q)_{n+1}} \\ &= \sum_{k=n+2}^{\infty} \frac{z^{k+n}}{(q; q)_n(q; q)_k} - \sum_{k=n+2}^{\infty} \frac{z^{k+n}}{(q; q)_{k-1}(q; q)_{n+1}} \\ &= \sum_{k=n+2}^{\infty} \left[\frac{(1 - q^{n+1}) - (1 - q^k)}{(q; q)_{n+1}(q; q)_k} \right] z^{k+n} \\ &= \sum_{k=n+2}^{\infty} \left[\frac{q^k - q^{n+1}}{(q; q)_{n+1}(q; q)_k} \right] z^{k+n} < 0, \end{aligned} \quad (9)$$

from which follows the rightmost inequality of (6) for $0 < z < 1$ and $q \in (0, 1)$.

Now, from (7) and (8), the leftmost inequality of (6) is equivalent to

$$\left[I_n(q; z) + \frac{z^n}{(q; q)_n} \right] \left[I_n(q; z) - \frac{z^{n+1}}{(q; q)_{n+1}} \right] > \frac{1 - q^{n+1}}{1 - q^{n+2}} (I_n(q; z))^2 \quad (10)$$

which is equivalent to

$$\frac{q^{n+1} - q^{n+2}}{1 - q^{n+2}} (I_n(q; z))^2 > \frac{z^{2n+1}}{(q; q)_n(q; q)_{n+1}} + I_n(q; z) \left(\frac{z^{n+1}}{(q; q)_{n+1}} - \frac{z^n}{(q; q)_n} \right). \quad (11)$$

On the other hand, we get

$$\begin{aligned} \frac{q^{n+1} - q^{n+2}}{1 - q^{n+2}} (I_n(q; z))^2 &= \left(\frac{q^{n+1} - q^{n+2}}{1 - q^{n+2}} \right) \sum_{k=0}^{\infty} z^{2n+2+k} \left[\sum_{j=0}^k \frac{1}{(q; q)_{n+1+j}(q; q)_{n+1+k-j}} \right] \\ &> \left(\frac{q^{n+1} - q^{n+2}}{1 - q^{n+2}} \right) \sum_{k=0}^{\infty} \frac{k+1}{(q; q)_{n+1}(q; q)_{n+k+1}} z^{2n+2+k} \\ &> (q^{n+1} - q^{n+2}) \sum_{k=0}^{\infty} \frac{k+1}{(q; q)_{n+1} [(1 - q^{n+2+k})(q; q)_{n+k+1}]} z^{2n+2+k} \\ &= (q^{n+1} - q^{n+2}) \sum_{k=0}^{\infty} \frac{k+1}{(q; q)_{n+1}(q; q)_{n+k+2}} z^{2n+2+k}. \end{aligned} \quad (12)$$

Using the equation

$$\frac{q^{n+1} - q^{n+k+2}}{q^{n+1} - q^{n+2}} = \frac{1 - q^{k+1}}{1 - q} \leq k + 1, \quad \text{for all } k \in \mathbb{N} \text{ and } q \in (0, 1) \quad (13)$$

and (12), we obtain

$$\frac{q^{n+1} - q^{n+2}}{1 - q^{n+2}} (I_n(q; z))^2 > \sum_{k=0}^{\infty} \frac{[q^{n+1} - q^{n+k+2}]}{(q; q)_{n+1} (q; q)_{n+k+2}} z^{2n+2+k}. \quad (14)$$

In addition, using the equations

$$\frac{z^{n+1}}{(q; q)_{n+1}} I_n(q; z) = \sum_{k=0}^{\infty} \frac{z^{2n+2+k}}{(q; q)_{n+1} (q; q)_{k+n+1}} \quad (15)$$

and

$$\begin{aligned} \frac{z^n}{(q; q)_n} I_n(q; z) &= \sum_{k=0}^{\infty} \frac{z^{2n+1+k}}{(q; q)_n (q; q)_{k+n+1}} \\ &= \frac{z^{2n+1}}{(q; q)_n (q; q)_{n+1}} + \sum_{k=0}^{\infty} \frac{z^{2n+2+k}}{(q; q)_n (q; q)_{k+n+2}}, \end{aligned} \quad (16)$$

we obtain

$$\begin{aligned} I_n(q; z) &\left(\frac{z^{n+1}}{(q; q)_{n+1}} - \frac{z^n}{(q; q)_n} \right) + \frac{z^{2n+1}}{(q; q)_n (q; q)_{n+1}} \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{(q; q)_{n+1} (q; q)_{n+k+1}} - \frac{1}{(q; q)_n (q; q)_{n+k+2}} \right] z^{2n+2+k} \\ &= \sum_{k=0}^{\infty} \left[\frac{(1 - q^{n+k+2}) - (1 - q^{n+1})}{(q; q)_{n+1} (q; q)_{n+k+2}} \right] z^{2n+2+k} \\ &= \sum_{k=0}^{\infty} \left[\frac{q^{n+1} - q^{n+k+2}}{(q; q)_{n+1} (q; q)_{n+k+2}} \right] z^{2n+2+k}. \end{aligned} \quad (17)$$

Combining (14) and (17), we get the leftmost inequality of (6). Finally, since

$$\lim_{z \rightarrow 0} \frac{I_{n-1}(q; z) I_{n+1}(q; z)}{(I_n(z; q))^2} = \frac{1 - q^{n+1}}{1 - q^{n+2}},$$

we conclude that in the first inequality in (6), the value $\frac{1 - q^{n+1}}{1 - q^{n+2}}$ is the best possible constant. Therefore, the proof of Theorem 3.1 is complete. $\square$

Theorem 3.2 For all $q \in (0, 1)$ and for every $n \in \mathbb{N}$, the following Turán type inequalities:

$$\left[\frac{q - q^{n+2}}{1 - q^{n+2}} \right] (\mathcal{I}_n(q; z))^2 \leq \mathcal{I}_{n-1}(q; z) \mathcal{I}_{n+1}(q; z) < (\mathcal{I}_n(q; z))^2, \quad (18)$$

hold for all $z > 0$. The value $\left[\frac{q - q^{n+2}}{1 - q^{n+2}} \right]$ is the best possible constant.

Proof Let $q \in (0, 1)$, $n \in \mathbb{N}$, and $z > 0$. Using the equations

$$\mathcal{I}_{n-1}(q; z) = \mathcal{I}_n(q; z) + \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n \quad (19)$$

and

$$\mathcal{I}_{n+1}(q; z) = \mathcal{I}_n(q; z) - \frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} z^{n+1}, \quad (20)$$

we obtain

$$\begin{aligned} \mathcal{I}_{n+1}(q; z) \mathcal{I}_{n-1}(q; z) - (\mathcal{I}_n(q; z))^2 &= \mathcal{I}_n(q; z) \left[\frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n - \frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} z^{n+1} \right] - \frac{q^{n^2} z^{2n+1}}{(q; q)_n (q; q)_{n+1}} \\ &= \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} \sum_{j=n+2}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{(q; q)_j} z^{j+n} - \frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} \sum_{j=n+1}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{(q; q)_j} z^{j+n+1} \\ &= \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} \sum_{j=n+2}^{\infty} \frac{q^{\frac{j(j-1)}{2}}}{(q; q)_j} z^{j+n} - \frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} \sum_{j=n+2}^{\infty} \frac{q^{\frac{(j-2)(j-1)}{2}}}{(q; q)_{j-1}} z^{j+n} \\ &= \sum_{j=n+2}^{\infty} \frac{q^{\frac{n(n-1)+(j-1)(j-2)}{2}} [q^{j-1} - q^n]}{(q; q)_{n+1} (q; q)_j} z^{j+n} < 0. \end{aligned} \quad (21)$$

From (21), we obtain the second inequality of (18) for all $q \in (0, 1)$ and $z > 0$.

The inequality

$$\left[\frac{q - q^{n+2}}{1 - q^{n+2}} \right] (\mathcal{I}_n(q; z))^2 \leq \mathcal{I}_{n-1}(q; z) \mathcal{I}_{n+1}(q; z) \quad (22)$$

is equivalent to

$$\left(\frac{1 - q}{1 - q^{n+2}} \right) (\mathcal{I}_n(q; z))^2 \geq \frac{q^{n^2} z^{2n+1}}{(q; q)_n (q; q)_{n+1}} + \mathcal{I}_n(q; z) \left(\frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} z^{n+1} - \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n \right). \quad (23)$$

On the other hand, we get

$$\frac{q^{n^2} z^{2n+1}}{(q; q)_n (q; q)_{n+1}} + \mathcal{I}_n(q; z) \left(\frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} z^{n+1} - \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n \right) = \sum_{k=0}^{\infty} \frac{q^{\frac{n(n+1)+(n+k)(n+k+1)}{2}}}{(q; q)_{n+1} (q; q)_{n+k+2}} (1 - q^{k+1}) z^{2n+2+k}. \quad (24)$$

Using the fact

$$(n + j + 1)(n + j) + (n + k - j + 1)(n + k - j) \leq n(n + 1) + (n + k)(n + k + 1)$$

where $n \in \mathbb{N}$ and $0 \leq j \leq k$, we get

$$\begin{aligned} \left(\frac{1 - q}{1 - q^{n+2}} \right) (\mathcal{I}_n(q; z))^2 &= \left(\frac{1 - q}{1 - q^{n+2}} \right) \sum_{k=0}^{\infty} z^{2n+2+k} \sum_{j=0}^k \frac{q^{\frac{(n+j)(n+j+1)+(n+k-j)(n+k-j+1)}{2}}}{(q; q)_{n+1+j} (q; q)_{n+k-j+1}} \\ &\geq \left(\frac{1 - q}{1 - q^{n+2}} \right) \sum_{k=0}^{\infty} \frac{z^{2n+2+k}}{(q; q)_{n+1} (q; q)_{n+k+1}} \sum_{j=0}^k q^{\frac{(n+j)(n+j+1)+(n+k-j)(n+k-j+1)}{2}} \\ &\geq \left(\frac{1 - q}{1 - q^{n+2}} \right) \sum_{k=0}^{\infty} \frac{z^{2n+2+k} (k + 1) q^{\frac{n(n+1)+(n+k)(n+k+1)}{2}}}{(q; q)_{n+1} (q; q)_{n+k+1}} \\ &\geq (1 - q) \sum_{k=0}^{\infty} \frac{z^{2n+2+k} (k + 1) q^{\frac{n(n+1)+(n+k)(n+k+1)}{2}}}{(q; q)_{n+1} (q; q)_{n+k+2}}. \end{aligned} \quad (25)$$

Now, from the previous inequality, (13) and (24), we have

$$\begin{aligned} \left(\frac{1-q}{1-q^{n+2}} \right) (\mathcal{I}_n(q; z))^2 &\geq \sum_{k=0}^{\infty} \frac{q^{\frac{n(n+1)+(n+k)(n+k+1)}{2}}}{(q; q)_{n+1} (q; q)_{n+k+2}} (1-q^{k+1}) z^{2n+2+k} \\ &= \frac{q^{n^2} z^{2n+1}}{(q; q)_n (q; q)_{n+1}} + \mathcal{I}_n(q; z) \left(\frac{q^{\frac{n(n+1)}{2}}}{(q; q)_{n+1}} z^{n+1} - \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n \right). \end{aligned} \quad (26)$$

Therefore, the inequality (23) holds, from which follows the leftmost inequality of (18) for all $q \in (0, 1)$ and $z > 0$.

Furthermore,

$$\lim_{z \rightarrow 0} \frac{\mathcal{I}_{n-1}(q; z) \mathcal{I}_{n+1}(q; z)}{(\mathcal{I}_n(z; q))^2} = \frac{q - q^{n+2}}{1 - q^{n+2}}.$$

The proof of Theorem 3.2 is completed. $\square$

Remark Using the equations (9) and (21), we deduce that the functions $F_1(q; z) = \mathcal{I}_n^2(q; z) - \mathcal{I}_{n-1}(q; z) \mathcal{I}_{n+1}(q; z)$ and $F_2(q; z) = \mathcal{I}_n^2(q; z) - \mathcal{I}_{n-1}(q; z) \mathcal{I}_{n+1}(q; z)$ are absolutely monotonic functions on $(0, \infty)$.

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