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Weakly submaximal spaces and compactifications

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Abstract In this paper, we characterize spaces such that their one-point compactification (resp., Herrlich compactification) is weakly submaximal. We also establish a necessary and sufficient condition on T_0 -spaces in order to get their one-point compactification (resp., Herrlich compactification) T_D -spaces.

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المخلص

في هذه المقالة، سوف نقوم بتمييز الفضاءات بحيث أن ترصيصها بنقطة واحدة (ترصيص هيرليتش على التوالي) يكون تحت أعظمية ضعيفة. كما نقدم شرط لازم وكافي على فضاءات T_0 بغرض جعل ترصيصها بنقطة واحدة (ترصيص هيرليتش على التوالي) فضاءات T_D .

1 Introduction

Submaximal spaces (that is, spaces such that every dense subset is open) have been studied by several authors (see for instance, [1, 4, 7, 9]). In [2], Belaid et al. have studied spaces such that their compactifications are submaximal spaces and have given a characterization of spaces such that their Wallman compactifications are submaximal.

A topological space X is said to be a T_D -space if $\overline{\{x\}} - \{x\}$ is closed, for each $x \in X$. A particular case of T_D -spaces is given by weakly submaximal spaces. A topological space X is said to be weakly submaximal if each finite subset F of X is a locally closed set (that is, for each finite subset F of X , $\overline{F} - F$ is a closed set of X) [6].

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Classically we have the following implications:

$$\begin{array}{c} T_1 \\ \Downarrow \\ \text{Submaximal} \implies \text{Weakly submaximal} \implies T_D \implies T_0 \end{array}$$

In 1993, Herrlich constructed with any T_0 -space X a minimal compactification $\beta_\omega X$ [5]. To honor Herrlich, $\beta_\omega X$ is called the Herrlich compactification. Herrlich has established that for T_1 -spaces, the extension $\beta_\omega X$ coincides with the Wallman compactification wX of X . We propose in this paper to give some properties of spaces such that their compactifications are weakly submaximal (resp., T_D -spaces). Particular cases of the one-point compactification and Herrlich are studied.

The second section of this paper contains some remarks and properties of weakly submaximal spaces and spaces such that their compactification are weakly submaximal.

In the third section we give a characterization of spaces such that their one-point compactification represents T_D -spaces (resp., weakly submaximal spaces)

The fourth section deals with Herrlich compactification, T_D -spaces and weakly submaximal spaces. We give a necessary and sufficient condition on the T_0 -space X in order to get its Herrlich compactification a T_D -space (resp., a weakly submaximal space).

2 Weakly submaximal spaces, compact spaces and compactification

Let us recall the definition of the Krull dimension of a T_0 -space. Let $(X, \mathcal{T})$ be a T_0 -space. Then X has a partial order $\leq$, induced by $\mathcal{T}$ by taking $x \leq y$ if and only if $y \in \overline{\{x\}}$. Hence $\overline{\{x\}} = \{y \mid x \leq y\}$. The notion of the Krull dimension defined on the prime spectrum of a ring has been generalized to T_0 -spaces [3]. The chain $x_0 < x_1 < \dots < x_n$ of elements of X is said to be a chain of length n . The supremum of the lengths of chains arriving to x_n is called the height of x_n . The Krull dimension of $(X, \mathcal{T})$ is the supremum of heights of elements of X and is denoted by $\dim_K(X, \mathcal{T})$.

Proposition 2.1 *Let $(X, \mathcal{T})$ be a T_0 -space. If X is weakly submaximal, then $\dim_K(X, \mathcal{T}) \leq 1$.*

Proof Suppose that there exist $x, y, z \in X$ such that $x < y < z$. Since $\overline{\{x, z\}} - \{x, z\}$ is a closed set of X , $\{y\} - \{z\}$ is closed, contradiction. $\square$

Proposition 2.2 *Let $(X, \mathcal{T})$ be a T_0 -space such that $\dim_K(X, \mathcal{T}) \leq 1$. If for each $x \in X$, $\overline{\{x\}}$ is finite, then X is weakly submaximal.*

Proof Let F be a finite subset of X . Since F is finite, $\overline{F} = \bigcup \{\overline{\{x\}} : x \in F\}$. Hence $\overline{F} - F = \bigcup \{\{y\} : y \in \overline{F} - F\}$. Since $\dim_K(X, \mathcal{T}) \leq 1$ and height of y is 1 for each $y \in \overline{F} - F$, $\{y\}$ is a closed set of X . Thus $\overline{F} - F$ is a closed set of X . $\square$

Remark 2.3 The set of natural numbers $\mathbb{N}$ equipped with the left topology is a T_D -space with Krull dimension upper to 1.

Proposition 2.4 *Let X be a topological space and $K(X)$ be a compactification of X . If $K(X)$ is weakly submaximal, then $\overline{\{x\}}^X$ is finite for each $x \in X$.*

Proof Let $z \in X$. First, let us prove that $\overline{\{x\}}^X$ is a compact closed set of X .

Let $z \in (\overline{\{x\}}^{K(X)} - \overline{\{x\}}^X) \cup \{x\}$. Since $K(X)$ is weakly submaximal, $\overline{\{x, z\}}^{K(X)} - \{x, z\}$ is a closed set of $K(X)$. Hence there exists an open set U of $K(X)$ such that $z \in U$ and $U \cap (\overline{\{x, z\}}^{K(X)} - \{x, z\}) = \emptyset$. Thus $(\overline{\{x\}}^{K(X)} - \{x\}) \cap X$ is a compact closed set of $K(X)$. Since $\overline{\{x\}}^X - \{x\} = (\overline{\{x\}}^{K(X)} - \{x\}) \cap X$, $\overline{\{x\}}^X - \{x\}$ is a compact closed set of X . Therefore, $\overline{\{x\}}^X$ is a compact closed set of X .

Let $y \in \overline{\{x\}}^X - \{x\}$. Since $K(X)$ is weakly submaximal, $\overline{\{x, y\}}^{K(X)} - \{x, y\}$ is a closed set of $K(X)$. Hence there exists an open neighborhood U_y of y such that $U_y \cap (\overline{\{x\}}^X - \{x\}) = \{y\}$. Thus $\overline{\{x\}}^X = \bigcup \{U_y : y \in \overline{\{x\}}^X - \{x\}\}$. Since $\overline{\{x\}}^X$ is compact, there exists a finite subset A of $\overline{\{x\}}^X - \{x\}$ such that $\overline{\{x\}}^X = \bigcup \{U_y : y \in A\}$. Thus $\overline{\{x\}}^X - \{x\}$ is finite, since $\overline{\{x\}}^X - \{x\}$ is compact. $\square$

3 One-point compactification, T_D -spaces and weakly submaximal spaces

Let us first recall the one-point compactification construction of a non compact topological space X : set $\tilde{X} = X \cup \{\infty\}$ with the topology $\tilde{T}$ whose members are the open subsets of X and all subsets U of X such that $X - U$ is a closed compact subset of X . Then $(\tilde{X}, \tilde{T})$ is a compactification of X called the one-point compactification or the Alexandroff compactification of X .

Proposition 3.1 *Let X be a non compact topological space. Then the one-point compactification $\tilde{X}$ of X is a T_D -space if and only if X is a T_D -space.*

Proof The necessary condition is straightforward. For the sufficient condition, let $x \in \tilde{X}$. We consider two cases:

Case 1 $x = \infty$. Then $\{x\}$ is a closed set of $\tilde{X}$ so that $\{x\}$ is a locally closed set of $\tilde{X}$.

Case 2 $x \in X$. Since X is a T_D -space, $\{x\}$ is a locally closed set of X . Hence there exists an open set O and a closed set C of X such that $\{x\} = C \cap O$. Since C is a closed set of X , $C \cup \{\infty\}$ is a closed set of $\tilde{X}$. Thus $\{x\} = (C \cup \{\infty\}) \cap O$. Therefore, $\{x\}$ is a locally closed set of $\tilde{X}$. $\square$

Proposition 3.2 *Let X be a non compact topological space. Then the one-point compactification $\tilde{X}$ of X is a weakly submaximal space if and only if for each finite set F of X , F is the intersection of a closed set of X and a co-compact open set of X .*

Proof For the necessary condition, let F be a finite set of X . Then $F \cup \{\infty\}$ is a finite set of $\tilde{X}$. Since $\tilde{X}$ is weakly submaximal, there exists an open set O of $\tilde{X}$ such that $F \cup \{\infty\} = O \cap \overline{F \cup \{\infty\}}^{\tilde{X}}$. Hence $O \cap X$ is a co-compact open set of X , since $\infty \in O$. Thus $F = (O \cap X) \cap C$ with $C = \overline{F \cup \{\infty\}}^{\tilde{X}} \cap X$, C is a closed set of X and $O \cap X$ is a co-compact open set of X .

For the sufficient condition, let F be a finite set of $\tilde{X}$. We consider two cases:

Case 1 $\infty \notin F$. Then F is a finite set of X so that there exists an open set O of X such that $F = O \cap \overline{F}$. Hence $F = O \cap (\overline{F} \cup \{\infty\})$. Thus F is a locally closed set of $\tilde{X}$, since $\overline{F} \cup \{\infty\}$ is a closed subset of $\tilde{X}$ and O is an open subset of $\tilde{X}$.

Case 2 $\infty \in F$. Set $G = F - \{\infty\}$. Since G is a finite set of X , there exists a co-compact open set O of X and a closed set C of X such that $G = O \cap C$. It is immediate that $C \cup \{\infty\}$ is a closed set of $\tilde{X}$. Since O is co-compact, $O \cup \{\infty\}$ is an open set of $\tilde{X}$. Hence $F = G \cup \{\infty\} = (O \cup \{\infty\}) \cap (C \cup \{\infty\})$. Thus F is locally closed. $\square$

Example 3.3 Let X be an infinite set equipped with the co-finite topology (that is, closed sets of X are the finite subsets). Then every open set is co-compact so that the one-point compactification $\tilde{X}$ of X is weakly submaximal.

4 Herrlich compactification, weakly submaximal spaces and T_D -spaces

First, let us recall the construction of Herrlich compactification of T_0 -spaces [5]: let (X, T_X) be a T_0 -space. Consider the set $\Gamma(X)$ of all filters $\mathcal{F}$ on X that satisfy the following two conditions:

- (a) $\mathcal{F}$ does not converge in X .
- (b) Every finite open cover of X contains some members of $\mathcal{F}$.

Let $\Omega(X)$ be the set of minimal elements of $\Gamma(X)$ and define:

- (i) $X_\omega^* = X \cup \Omega(X)$.
- (ii) $A_\omega^* = A \cup \{\mathcal{F} \mid \mathcal{F} \in \Omega(X) \text{ and } A \in \mathcal{F}\}$ for $A \subseteq X$.

Then $\beta_\omega = \{A_\omega^* \mid A \in T_X\}$ is a base for a topology $\mathcal{T}_\omega^*$ on X_ω^* . The space $(X_\omega^*, \mathcal{T}_\omega^*)$ is compact and X is dense in X_ω^* . Hence the space with underlying set X_ω^* and topology $\mathcal{T}_\omega^*$ is a compactification of X , and it will be called the Herrlich compactification of X or the T_0 -compactification of X , and will be denoted by $\beta_\omega X$. In the case of T_1 -spaces the Herrlich compactification coincides with the Wallman compactification.

Under previous notations, the following properties hold [5]:

Proposition 4.1 *Let X be a T_0 -space. Then the following properties hold:*

1. Let A and B two subsets of X . If $A \subseteq B$, then $A_\omega^* \subseteq B_\omega^*$.
2. For each open set U of X , the set U_ω^* is the largest open set $\mathcal{U}$ of $\beta_\omega X$ such that $U = \mathcal{U} \cap X$.
3. Each finite subset of $\Omega(X)$ is a closed set in $\beta_\omega X$.
4. Every $\mathcal{F} \in \Omega(X)$ has a base consisting of open sets.

Proposition 4.2 *Let X be a T_0 -space. Then the following statements are equivalent:*

- (i) $\dim_K(\beta_\omega X) \leq 1$.
- (ii) $\dim_K(X) \leq 1$.

Proof (i) $\implies$ (ii) Straightforward.

(ii) $\implies$ (i) Suppose that there exists $\mathcal{F} \in \beta_\omega X - X$ such that the height of $\mathcal{F}$ is 2. Since $\beta_\omega X - X$ is a T_1 -space, there exists $x \in X$ such that $\mathcal{F} \in \overline{\{x\}}^{\beta_\omega X}$ and height of x is 1. Then $\{x\}$ is a closed set of X . Since $\mathcal{F}$ does not converge to $x \in X$, there exists an open set O of X such that $x \in O$ and $O \notin \mathcal{F}$. Thus $\{X - \{x\}, O\}$ is a finite open cover of X . Since $O \notin \mathcal{F}$, $X - \{x\} \in \mathcal{F}$, contradicting the fact that $\mathcal{F} \in \overline{\{x\}}^{\beta_\omega X}$. Since this fact contradicts that $\mathcal{F} \in \{x\}$, our assertion follows. $\square$

Proposition 4.3 *Let X be a topological space. Then the following statements are equivalent:*

- (1) $\beta_\omega X$ is a submaximal space.
- (2) X has the following properties:
 - (i) X is a weakly submaximal space.
 - (ii) $\dim_K X \leq 1$.
 - (iii) $\overline{\{x\}}^X$ is finite, for each $x \in X$.

Proof (1) $\implies$ (2) Follows immediately from Proposition 2.1, Proposition 2.4 and Proposition 4.2.

(2) $\implies$ (1) Since $\dim_K X \leq 1$, $\dim_K \beta_\omega X \leq 1$ by Proposition 4.2.

Let $x \in \beta_\omega X$. We discuss two cases:

Case 1 $x \in \beta_\omega X - X$. By Proposition 4.1, $\{x\}$ is a closed set of $\beta_\omega X$.

Case 2 $x \in X$. Then $\overline{\{x\}}^X$ is finite. Suppose that there exist two distinct elements $\mathcal{F}, \mathcal{G} \in \beta_\omega X - X$ such that $\mathcal{F}, \mathcal{G} \in \overline{\{x\}}^{\beta_\omega X}$. Let O be an open set of X such that $O \in \mathcal{F}$. Then $x \in O$. Since X is a T_D -space, $U = X - (\overline{\{x\}}^X - \{x\})$ is an open set of X . Hence $\{U, O\}$ is a finite open cover of X . Since $U \notin \mathcal{G}$, $O \in \mathcal{G}$. Since this contradicts the claim that $\mathcal{F}$ and $\mathcal{G}$ are distinct, $\overline{\{x\}}^X - X$ has at most one element. Then $\overline{\{x\}}^{\beta_\omega X}$ is finite.

Finally, Proposition 2.2 implies that $\beta_\omega X$ is weakly submaximal. $\square$

Example 4.4 The digital line (that is, $\mathbb{Z}$ equipped with the Khalimsky topology generated by $\mathcal{G} = \{2n - 1, 2n, 2n + 1\} \mid n \in \mathbb{Z}\}$) satisfies conditions of Proposition 4.3. Then its Herrlich compactification $\beta_\omega \mathbb{Z}$ is a weakly submaximal space.

Definition 4.5 Let X be a topological space and $x \in X$.

(1) An open cover $\mathcal{U}$ of X is called a bad covering of X (b-covering, for short) if it has not a finite subcover. Call $\mathcal{U}$ a good covering of X (g-covering, for short) if $\mathcal{U}$ is not a b-covering.

(2) An open set O of X is said to be a x -open set if $x \in O$ and there exists a collection $\mathcal{U}$ of open sets satisfying the following conditions:

- (i) for each $U \in \mathcal{U}$, $x \notin U$ and $U \subset O$;
- (ii) for each b-covering $\mathcal{O}$ of X such that $\mathcal{O} \cup \{O\}$ is a g-covering of X , $\mathcal{O} \cup \mathcal{U}$ is a g-covering of X .

Lemma 4.6 *Let X be a T_0 -space and $\mathcal{U}$ be an open cover of X . Then $\mathcal{U}$ is a g-covering of X if and only if $\{U_\omega^* \mid U \in \mathcal{U}\}$ is an open cover of $\beta_\omega X$.*

Proof For the necessary condition, let $\mathcal{U}$ be a g-covering of X . Then exists a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\{U \mid U \in \mathcal{U}'\}$ is a cover of X . Hence, for each $\mathcal{F} \in \Omega(X)$, there exists $U \in \mathcal{U}'$ such that $\mathcal{F} \in U_\omega^*$. Thus $\{U_\omega^* \mid U \in \mathcal{U}'\}$ is an open cover of $\beta_\omega X$. Therefore, $\{U_\omega^* \mid U \in \mathcal{U}\}$ is an open cover of $\beta_\omega X$.

For the sufficient condition, let $\mathcal{U}$ be an open cover of X such that $\{U_\omega^* \mid U \in \mathcal{U}\}$ is an open cover of $\beta_\omega X$. Since $\beta_\omega X$ is compact, there exists a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\{U_\omega^* \mid U \in \mathcal{U}'\}$ is an open cover of $\beta_\omega X$. That $\mathcal{U}$ is a g-covering of X follows immediately from the fact that $\mathcal{U}'$ is a finite cover of X . $\square$

Proposition 4.7 *Let X be a T_0 -space. Then the following statements are equivalent:*

- (1) $\beta_\omega X$ is a T_D -space.
- (2) For each $x \in X$ there exists a x -open set O of X such that $O \cap \overline{\{x\}}^X = \{x\}$.

Proof (1) $\implies$ (2) Let $x \in X$. Since $\beta_\omega X$ is a weakly submaximal space, $\overline{\{x\}}^{\beta_\omega X} - \{x\}$ is a closed set of $\beta_\omega X$. Then there exists an open set O of X such that $x \in O$ and $O_\omega^* \cap \overline{\{x\}}^{\beta_\omega X} = \{x\}$. For $\mathcal{F} \in O_\omega^* \cap \Omega(X)$, there exists an open set $U_\mathcal{F}$ of X such that $x \notin U_\mathcal{F}$, $U_\mathcal{F}^* \subset O_\omega^*$ and $U_\mathcal{F} \in \mathcal{F}$, since $\mathcal{F} \notin \overline{\{x\}}^{\beta_\omega X}$. Set $\mathcal{C} = \{U_\mathcal{F} \mid \mathcal{F} \in O_\omega^* \cap \Omega(X)\}$. Let $\mathcal{O}$ be a b-covering of X such that $\mathcal{O} \cup \{O\}$ is a g-covering of X . Since for each $\mathcal{F} \in O_\omega^* \cap \Omega(X)$, there exists $U \in \mathcal{C}$ such that $\mathcal{F} \in U_\omega^*$. Hence $\{U_\omega^* \mid U_\mathcal{F} \in \mathcal{C}\} \cup \{O_\omega^* \mid O \in \mathcal{O}\}$ is an open cover of $\beta_\omega X$. Since $\beta_\omega X$ is compact, $\mathcal{O} \cup \mathcal{U}$ is a g-covering of X . Thus O is a x -open set of X such that $O \cap \overline{\{x\}}^X = \{x\}$.

(2) $\implies$ (1) Let $x \in X$ and O be a x -open set of X such that $O \cap \overline{\{x\}}^X = \{x\}$. Let $\mathcal{U}$ be a collection of open sets of X satisfying the following conditions:

- (i) for each $U \in \mathcal{U}$, $x \notin U$ and $U \subset O$;
- (ii) for each b-covering $\mathcal{O}$ of X such that $\mathcal{O} \cup \{O\}$ is a g-covering of X , $\mathcal{O} \cup \mathcal{U}$ is a g-covering of X .

Let $\mathcal{F} \in O_\omega^* \cap \Omega(X)$. Since $\{\mathcal{F}\}$ is a closed set of $\beta_\omega X$, $\beta_\omega X - \{\mathcal{F}\}$ is an open set of $\beta_\omega X$. Let $\mathcal{V}$ be a collection of open sets of X such that $\beta_\omega X - \{\mathcal{F}\} = \cup(V_\omega^* : V \in \mathcal{V})$. That $\mathcal{V}$ is a b-covering of X and $\mathcal{V} \cup \{O\}$ is a g-covering of X follows immediately from Lemma 4.6. Then $\mathcal{V} \cup \mathcal{U}$ is a g-covering of X . Hence there exists $U \in \mathcal{U}$ such that $\mathcal{F} \in U_\omega^*$. Thus $\mathcal{F} \notin \overline{\{x\}}^{\beta_\omega X}$ so that $O_\omega^* \cap \overline{\{x\}}^{\beta_\omega X} = \{x\}$. Therefore, $\overline{\{x\}}^{\beta_\omega X} - \{x\}$ is a closed set of $\beta_\omega X$. $\square$

Example 4.8 The Herrlich compactification of the digital line (defined in Example 4.4) is a T_D -space. In fact, for $x \in \mathbb{Z}$, let O be an open set of $\mathbb{Z}$ defined as follows: if $x = 2n$ then $O = \mathbb{Z}$, if $x = 2n + 1$ then $O = \mathbb{Z} - \{2n, 2n + 2\}$. Hence O is a x -open and $O \cap \overline{\{x\}} = \{x\}$. Thus $\beta_\omega \mathbb{Z}$ is a T_D -space.

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