Abdelkarem Berkaoui

On representations of the set of supermartingale measures and applications in discrete time

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Abstract We investigate some new results concerning the m -stability property. We show in particular under the martingale representation property with respect to a bounded martingale S that an m -stable set of probability measures is the set of supermartingale measures for a family of discrete integral processes with respect to S .

المخلص

نبحث في بعض النتائج الجديدة المتعلقة بخاصية استقرار m . نثبت بشكل خاص، في ظل خاصية تمثيل الحزمة بالنسبة لحزمة محدودة S ، أن مجموعة مستقرة m من مقاييس الاحتمالية هي مجموعة المقاييس فوق الحزمة لعائلة من العمليات الصحيحة المنفصلة بالنسبة لـ S .

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1 Introduction

The m -stability property plays a primordial role within the theory of dynamic risk measures and in Financial Mathematics in general. In dynamic setting it is crucial that a dynamic risk measure satisfies the recursiveness property when evaluating risk for a financial position, by taking into account the new incoming information in a consistent way. It should also be the case for a pricing mechanism when pricing a financial claim in incomplete markets, avoiding so the creation of arbitrage opportunities in a given time axis. In decision making and in econometrics the m -stability property known by rectangularity is an essential assumption in modeling preferences and utility functions and in constructing set of priors. We refer to [5, 6] for more details.

Let us consider a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0\dots T}, \mathbb{P})$ and a set $\mathcal{Q}$ of probability measures in $\mathbf{P}$ containing at least one equivalent to $\mathbb{P}$, where $\mathbf{P}$ (resp., $\mathbf{P}^e$) is the set of $\mathbb{P}$ -absolutely continuous (resp., $\mathbb{P}$ -equivalent) probability measures. In [2] it was shown that the m -stability assumption on $\mathcal{Q}$ is a corner stone in generalizing a number of results from one-period case to multi-period case. We recall in particular that for an m -stable set $\mathcal{Q}$, we get the following:

- (i) Suppose $\mathcal{Q} = \{\mathbb{Q} \in \mathbf{P} : \mathbb{E}^{\mathbb{Q}}(X) = 0\} =: \mathcal{M}(0, X)$ for a vector-valued random variable $X = (X^1, \dots, X^k)$. Then $\mathcal{Q}$ is the set of martingale measures for the $\mathbb{R}^k$ -valued adapted process M defined by $M_t^i = \mathbb{E}_t^{\mathbb{Q}}(X^i) := \mathbb{E}^{\mathbb{Q}}(X^i | \mathcal{F}_t)$ for some (or any) $\mathbb{Q} \in \mathcal{Q}^e$ and $i = 1, \dots, k$, where $\mathcal{Q}^e$ is given by $\mathcal{Q}^e = \mathcal{Q} \cap \mathbf{P}^e$. In this case we can write a $\mathcal{Q}$ -supermartingale as the sum of a local martingale and a decreasing process, by applying the well-known theorem of Föllmer and Kabanov [7].
- (ii) The set $\mathcal{Q}'$, to be redefined later, is the set of martingale measures for a family $\mathcal{Y}$ of adapted processes. Such family $\mathcal{Y}$ can be replaced by a finite one if we suppose further that $\mathcal{Q}$ is optionally m -stable with respect to a vector-valued bounded random variable V .

A. Berkaoui (✉)
Department of Mathematics and Statistics, College of Science, Al-Imam Mohammed Ibn Saud Islamic University (IMSIU),
P.O. Box 90950, Riyadh 11623, Saudi Arabia
E-mail: aamberkaoui@imamu.edu.sa

Our goal in this paper is to prove the $\mathcal{Q}$ -supermartingale decomposition under minimal assumptions on $\mathcal{Q}$. To do that, we will start by proving in Sect. 3 some interesting properties of the set $\mathcal{Q}^{st}$ (See Definition 2.2 below). We will state in particular two fundamental assertions: (1) $\mathcal{Q}$ -supermartingales are $\mathcal{Q}^{st}$ -supermartingales, and (2) $\mathcal{Q}^{st}$ is the set of supermartingale measures for a family $\mathcal{Y}$ of bounded adapted processes.

In Sect. 4 we suppose the martingale representation property of the filtration with respect to an adapted process S with values in $\mathbb{R}^d$. We shall state the existence of a convex cone C of vector-valued adapted processes such that $\mathcal{Q}^{st}$ is the set of supermartingale measures for a family of processes of the form $\alpha \bullet S := \sum_{s < \cdot} \alpha_s \cdot \Delta_s S$ with $\alpha \in C$ and $\Delta_s S = S_{s+1} - S_s$. We apply such result and deduce that any positive (or bounded) $\mathcal{Q}$ -supermartingale X can be written as $X_0 + \alpha \bullet S - B$ with B an increasing process and $\alpha \in C$.

In Sect. 5 and further under the assumption $\mathcal{Q} = \mathcal{Q}'$, we shall prove that $\mathcal{Q}^{st}$ is the set of martingale measures for a family $\mathcal{Y}$ of bounded adapted processes and the process $\alpha \bullet S$ appearing in the previous decomposition of X is a local $\mathcal{Q}$ -martingale, generalizing the theorem of Föllmer and Kabanov.

We prove also under the martingale representation property that $(\mathcal{Q}^{st})' = (\mathcal{Q}')^{st}$, which is the commutativity property of the two operators $\mathcal{Q} \rightarrow \mathcal{Q}^{st}$ and $\mathcal{Q} \rightarrow \mathcal{Q}'$.

2 Notation and review

In this section we recall the definition of the main properties which will be used along this paper, and also some established characterizations of these properties.

2.1 Notation

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a discrete time filtration $(\mathcal{F}_t)_{t \in \mathcal{I}}$ with $\mathcal{I} = \{0, \dots, T\}$ and $\mathcal{I}^* = \mathcal{I} \setminus \{T\}$. We denote by Π (resp., Π^c) the set of all (resp., $\mathbb{L}^1$ -closed convex) subsets in $\mathbf{P}$, with $\Pi^e = \{\mathcal{Q} \in \Pi : \mathcal{Q} \cap \mathbf{P}^e \neq \emptyset\}$ and $\Pi^{c,e} = \Pi^c \cap \Pi^e$. For a set $\mathcal{Q}$ of probability measures in $\mathbf{P}$, we define the dynamic expectation operator $\mathcal{E}$ by $\mathcal{E}_t(X) = \text{esssup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}(X)$ for $t \in \mathcal{I}$ and $X \in \mathbb{L}^\infty$ with $\mathbb{E}_t^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{Q}}(X | \mathcal{F}_t)$, the family of acceptance sets $\mathcal{A}_{t,u} = \{X \in \mathbb{L}^\infty(\mathcal{F}_u) : \mathcal{E}_t(X) \leq 0 \text{ a.s.}\}$ for $t < u$ with $\mathcal{A} := \mathcal{A}_{0,T}$, and the set of Radon–Nikodym densities $\mathcal{Z} = \{Z^{\mathbb{Q}} := d\mathbb{Q}/d\mathbb{P} : \mathbb{Q} \in \mathcal{Q}\}$, $\mathcal{Z}^e = \{Z \in \mathcal{Z} : Z > 0 \text{ a.s.}\}$ with $Z_\tau := \mathbb{E}_\tau(Z)$ for $Z \in \mathcal{Z}$ and a stopping time τ .

2.2 On the m-stability property

Definition 2.1 We say that a set of probability measures $\mathcal{Q}$ is m-stable with respect to the filtration $(\mathcal{F}_t)_{t \in \mathcal{I}}$ if for any $Z^1, Z^2 \in \mathcal{Z}$ with $Z^2 > 0$ a.s. and a stopping time τ we have $Z := Z_\tau^1 Z_\tau^2 / Z_\tau^2 \in \mathcal{Z}$.

Definition 2.2 For any $\mathcal{Q} \in \Pi^{c,e}$, we define $\mathcal{Q}^{st}$ to be the intersection of all m-stable closed convex subsets in $\mathbf{P}$ containing $\mathcal{Q}$, and denote by $\mathcal{E}^{st}$ and $\mathcal{A}^{st}$, respectively, the dynamic expectation operator and the acceptance set associated to $\mathcal{Q}^{st}$.

We recall some interesting characterizations of the m-stability property, stated in Delbaen [5].

Proposition 2.3 Let $\mathcal{Q} \in \Pi^{c,e}$. Then the following assertions are equivalent:

- (1) $\mathcal{Q}$ is m-stable,
- (2) For any $t \in \mathcal{I}^*$, $F \in \mathcal{F}_t$ and $Z, Z^1, Z^2 \in \mathcal{Z}$ with $Z^1 > 0$ and $Z^2 > 0$ a.s., we have $Z_t \left(\mathbf{1}_F \frac{Z^1}{Z_t^1} + \mathbf{1}_{F^c} \frac{Z^2}{Z_t^2} \right) \in \mathcal{Z}$,
- (3) $\mathcal{E}$ satisfies the recursiveness property, i.e., for any $X \in \mathbb{L}^\infty$ and for any stopping times $\tau \leq \sigma$ we have $\mathcal{E}_\tau(\mathcal{E}_\sigma(X)) = \mathcal{E}_\tau(X)$,
- (4) $\mathcal{E}$ is time consistent, i.e., for any $X, Y \in \mathbb{L}^\infty$ and for stopping times $\tau \leq \sigma$ we have $\mathcal{E}_\sigma(X) \geq \mathcal{E}_\sigma(Y)$ implies $\mathcal{E}_\tau(X) \geq \mathcal{E}_\tau(Y)$,
- (5) for any $X \in \mathbb{L}^\infty$, the process $\mathcal{E}_\cdot(X)$ is a $\mathcal{Q}$ -supermartingale,
- (6) for any stopping time $\tau > 0$ and $\mathcal{A} := \mathcal{A}_{0,T}$ we have $\mathcal{A} = \mathcal{A}_{0,\tau} + \mathcal{A}_{\tau,T}$.

2.3 On the optional m-stability property

This subsection is devoted to recalling the definition of the optional m-stability. This concept was first introduced by Jacka et al. in [8]. We shall say that an $\mathbb{R}^d$ -valued random variable V is viable if for each component v of V satisfies $v \in \mathbb{L}_+^\infty$ and $1/v \in \mathbb{L}_+^\infty$. For a random variable $Y \in \mathbb{L}_+^1$ with $\mathbb{E}(Y) = 1$, we define the probability measure $\mathbb{Q}^Y$ by $\mathbb{Q}^Y(F) = \mathbb{E}(Y\mathbf{1}_F)$ for $F \in \mathcal{F}$.

Definition 2.4 (See Jacka and Berkaoui [8]) We say that $\mathcal{Q}$ is optionally m-stable with respect to a viable random vector V if for all $t = 0 \dots T - 1$, whenever $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ are such that there exists $\mathbb{Q} \in \mathcal{Q}$, $F \in \mathcal{F}_t$, $\alpha^i \in \mathbb{L}_+^0(\mathcal{F}_{t+1})$ with each $\alpha^i Z^i \in \mathbb{L}^1$ where Z^i and Z are respective densities of $\mathbb{Q}^i$ and $\mathbb{Q}$ for $i = 1, 2$ and $Y = \mathbf{1}_F \alpha^1 Z^1 + \mathbf{1}_{F^c} \alpha^2 Z^2$ satisfies $\mathbb{Q}_t(V) = \mathbb{Q}_t^Y(V)$, then we have $\mathbb{Q}^Y \in \mathcal{Q}$.

Characterizations of this property can be found in [1, 8].

2.4 On the smallest set of martingale measures

Here we recall the definition of the set $\mathcal{Q}'$ associated with $\mathcal{Q}$, and also Theorem 2.1 in [3], stating some properties of $\mathcal{Q}'$. For a set $\mathcal{Q}$ of probability measures and its acceptance set $\mathcal{A}$, we define

$$\mathcal{Q}' = \{\mathbb{Q} \in \mathbf{P} : \mathbb{E}^{\mathbb{Q}}(X) = 0 \text{ for all } X \in \text{lin}(\mathcal{A})\},$$

where $\text{lin}(\mathcal{A}) = \mathcal{A} \cap -\mathcal{A}$. We prove easily that the acceptance set $\mathcal{A}'$ of $\mathcal{Q}'$ is given by the closure in weak star sense of $\text{lin}(\mathcal{A}) + \mathbb{L}_+^\infty$. By supposing that $\mathcal{Q}$ is m-stable, the next theorem will state mainly that $\mathcal{Q}'$ is m-stable and that $\mathcal{Q}'$ is the set of martingale measures for a family of bounded adapted processes.

Theorem 2.5 Suppose $\mathcal{Q}$ is m-stable. Then

- (1) for all stopping times τ we have $\text{lin}(\mathcal{A}'_{\tau, T}) = \text{lin}(\mathcal{A}_{\tau, T})$;
- (2) for all stopping times τ we have $\text{lin}(\mathcal{A}) = \text{lin}(\mathcal{A}_{0, \tau}) \oplus \text{lin}(\mathcal{A}_{\tau, T})$;
- (3) $\mathcal{Q}'$ is m-stable;
- (4) $\mathcal{Q}' = \mathcal{M}(\mathcal{Y})$ where $\mathcal{Y} = \{\mathcal{E}_\cdot(X) : X \in \text{lin}(\mathcal{A})\}$;
- (5) $\mathcal{Q}'$ is the smallest set of martingale measures containing $\mathcal{Q}$;
- (6) $\mathcal{Q}$ -martingales are $\mathcal{Q}'$ -martingales.

3 Intermediate results

Now we investigate some properties of the mapping $\mathcal{Q} \rightarrow \mathcal{Q}^{st}$.

3.1 Properties

First we express the triplet $(\mathcal{Q}^{st}, \mathcal{A}^{st}, \mathcal{E}^{st})$ in terms of the triplet $(\mathcal{Q}, \mathcal{A}, \mathcal{E})$.

Proposition 3.1 Let $\mathcal{Q} \in \Pi^{c,e}$. Then

- (1) $\mathcal{Q}^{st}$ is the closed convex hull in $\mathbb{L}^1$ of the set $B := \bigcap_{t \in \mathcal{I}^*} \{\mathbb{Q} \in \mathbf{P}^e : Z_{t+1}^{\mathbb{Q}}/Z_t^{\mathbb{Q}} = \mathbf{1}_F Z_{t+1}^1/Z_t^1 + \mathbf{1}_{F^c} Z_{t+1}^2/Z_t^2 \text{ for some } Z^1, Z^2 \in \mathcal{Z}^e, F \in \mathcal{F}_t\}$.
- (2) $\mathcal{A}_{s,t}^{st} = \mathcal{A}_{s,s+1} + \mathcal{A}_{s+1,t}^{st} = \bigoplus_{u=s \dots t-1} \mathcal{A}_{u,u+1}$ for all $s, t \in \mathcal{I}^*$ with $s < t$. In particular, we have $\mathcal{A}_{t,t+1}^{st} = \mathcal{A}_{t,t+1}$ for $t \in \mathcal{I}^*$.
- (3) $\mathcal{E}_t^{st} = \mathcal{E}_t \circ \mathcal{E}_{t+1}^{st} = \mathcal{E}_t \circ \dots \circ \mathcal{E}_{T-1}$ for $t \in \mathcal{I}^*$.

Proof (1) For the reverse inclusion, let $\mathbb{Q} \in B$ and $Z := Z^{\mathbb{Q}}$ which means that for each $t \in \mathcal{I}^*$ we have $Z_{t+1}/Z_t = \mathbf{1}_{F_t} Z_{t+1}^{t,1}/Z_t^{t,1} + \mathbf{1}_{F_t^c} Z_{t+1}^{t,2}/Z_t^{t,2}$ for some $Z^{t,1}, Z^{t,2} \in \mathcal{Z}^e$ and $F_t \in \mathcal{F}_t$. We define $R^0 = \mathbf{1}_{F_0} Z^{0,1} + \mathbf{1}_{F_0^c} Z^{0,2}$ and for $t = 1 \dots T - 1$, we define $R^t = R^{t-1} \left[\mathbf{1}_{F_t} Z^{t,1}/Z_t^{t,1} + \mathbf{1}_{F_t^c} Z^{t,2}/Z_t^{t,2} \right]$. Thanks to Assertion (2) in Proposition 2.3 we deduce by induction on $t = 0 \dots T - 1$ that all $R^t \in \mathcal{Z}^{st}$ and since $Z = R^{T-1}$ we deduce that $B \subseteq \mathcal{Q}^{st}$ and, therefore, $\overline{\text{co}}(B) \subseteq \mathcal{Q}^{st}$. For the direct inclusion we remark that B is m-stable and $\mathcal{Q} \subseteq B$, so $\mathcal{Q}^{st} \subseteq \overline{\text{co}}(B)$.

- (2) Fix $s < t$ and take $X \in \mathcal{A}_{s,t}^{st}$ with $\mathcal{E}_s^{st}(X) = 0$. We have $\mathcal{A}_{s,t}^{st} \subseteq \mathcal{A}^{st}$, then there exists some $X_u \in \mathcal{A}_{u,u+1}$ for $u \in \mathcal{I}^*$ such that $X = X_0 + \cdots + X_{T-1}$. We apply $\mathcal{E}_s^{st}$ on both parts of the equality and obtain that $0 = \mathcal{E}_s^{st}(X) = X_{0,s-1} + \mathcal{E}_s^{st}(X_{s,T-1})$ with $X_{r,u} = X_r + \cdots + X_u$. We get $X_{0,s-1} = -\mathcal{E}_s^{st}(X_{s,T-1}) \geq 0$ and, therefore, $X_{0,s-1} = 0$. Now we apply $\mathcal{E}_{t+1}^{st}$ and obtain that $X = \mathcal{E}_{t+1}^{st}(X) = X_{s,t} + \mathcal{E}_{t+1}^{st}(X_{t+1,T-1})$, which means that $X_{t+1,T-1} = \mathcal{E}_{t+1}^{st}(X_{t+1,T-1}) =: Y_{t+1} \in \mathbb{L}^\infty(\mathcal{F}_{t+1})$. So $X = X_{s,t} + Y_{t+1} \in \mathcal{A}_{s,s+1} + \cdots + \mathcal{A}_{t,t+1}$. Inversely we have $\mathcal{A}_{s,s+1} + \cdots + \mathcal{A}_{t,t+1} \subseteq \mathcal{A}^{st}$ and then $\mathcal{A}_{s,s+1} + \cdots + \mathcal{A}_{t,t+1} \subseteq \mathcal{A}_{s,t}^{st}$.
- (3) We have $\mathcal{E}_t^{st} = \mathcal{E}_t^{st} \circ \mathcal{E}_{t+1}^{st}$ thanks to Assertion (3) in Proposition 2.3. It suffices to prove that $\mathcal{E}_t(X) = \mathcal{E}_t^{st}(X)$ for all $X \in \mathbb{L}^\infty(\mathcal{F}_{t+1})$. Indeed we have $\mathcal{E}_t(X) \leq \mathcal{E}_t^{st}(X)$ since $\mathcal{Q} \subseteq \mathcal{Q}^{st}$ and for $X \in \mathbb{L}^\infty(\mathcal{F}_{t+1})$ we have $X - \mathcal{E}_t(X) \in \mathcal{A}_{t,t+1}$ and $\mathcal{A}_{t,t+1}^{st} = \mathcal{A}_{t,t+1}$. So $\mathcal{E}_t^{st}(X - \mathcal{E}_t(X)) \leq 0$ which means that $\mathcal{E}_t^{st}(X) \leq \mathcal{E}_t(X)$. Therefore, $\mathcal{E}_t^{st} = \mathcal{E}_t^{st} \circ \mathcal{E}_{t+1}^{st} = \mathcal{E}_t \circ \mathcal{E}_{t+1}^{st}$. $\square$

Theorem 3.2 Let $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2 \in \Pi^{c,e}$. Then we have the following:

- (1) $(\mathcal{Q}^{st})^{st} = \mathcal{Q}^{st}$.
- (2) $\mathcal{Q}_1^{st} \subseteq \mathcal{Q}_2^{st}$ if $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$.
- (3) $(\mathcal{Q}_1 \cap \mathcal{Q}_2)^{st} = (\mathcal{Q}_1)^{st} \cap (\mathcal{Q}_2)^{st}$.
- (4) For an increasing sequence $\mathcal{Q}^n \in \Pi^{c,e}$ with $\mathcal{Q} := \bigcup_{n \geq 1} \mathcal{Q}^n$, we have
 - (i) $\mathcal{Q}$ is m -stable if each $\mathcal{Q}^n$ is.
 - (ii) $\mathcal{Q}^{st} = \overline{\bigcup_{n \geq 1} (\mathcal{Q}^n)^{st}}$, with the closure taken in $\mathbb{L}^1$.

Proof Assertions (1) and (2) are trivial. (3) We shall show that $(\mathcal{A}^1 + \mathcal{A}^2)_{t,t+1} = \mathcal{A}_{t,t+1}^1 + \mathcal{A}_{t,t+1}^2$ for all $t \in \mathcal{I}^*$, with the closure taken in weak star sense in $\mathbb{L}^\infty$. The inverse inclusion is trivial. Let us prove the direct one. We know that the dynamic expectation operator $\mathcal{E}$ of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is given by $\mathcal{E}_t(X) = \text{essinf}_{i=1,2} \mathcal{E}_t^i(X)$. So for $X \in (\mathcal{A}^1 + \mathcal{A}^2)_{t,t+1}$ we have $X = \mathcal{E}_t(X) + \mathbf{1}_F(X - \mathcal{E}_t^1(X)) + \mathbf{1}_{F^c}(X - \mathcal{E}_t^2(X)) =: \mathcal{E}_t(X) + X^1 + X^2 \leq X^1 + X^2$ with $F = \{\mathcal{E}_t^1(X) \leq \mathcal{E}_t^2(X)\}$. We remark that $X^i \in \mathcal{A}_{t,t+1}^i$ for $i = 1, 2$ and deduce the result.

(4,i) Let $Z^1, Z^2 \in \mathcal{Z}$ with $Z^2 > 0$ a.s. and a stopping time τ . Then there exists an integer n such that $Z^1, Z^2 \in \mathcal{Z}^n$ and since $\mathcal{Q}^n$ is m -stable we deduce that $Z := Z_\tau^1 Z_\tau^2 / Z_\tau^2 \in \mathcal{Z}^n \subseteq \mathcal{Z}$.

(4,ii) We have from one side $\mathcal{Q}^n \subseteq (\mathcal{Q}^n)^{st}$. Hence, $\bigcup_{n \geq 1} \mathcal{Q}^n \subseteq \overline{\bigcup_{n \geq 1} (\mathcal{Q}^n)^{st}} =: \tilde{\mathcal{Q}}$ and since $\tilde{\mathcal{Q}}$ is an m -stable closed convex set we deduce that $\mathcal{Q}^{st} \subseteq \tilde{\mathcal{Q}}$. For the other inclusion we have $\mathcal{Q}^n \subseteq \mathcal{Q}$ which implies $\tilde{\mathcal{Q}} \subseteq \mathcal{Q}^{st}$. $\square$

3.2 Link with the concept of supermartingale

We investigate the relationship of the m -stability property with the concept of supermartingale. We start by giving the definition of a supermartingale measure.

Definition 3.3 We say that a probability measure $\mathbb{Q} \in \mathbf{P}$ is a supermartingale measure for a family $\mathcal{Y}$ of adapted processes if each element $Y \in \mathcal{Y}$ is a $\mathbb{Q}$ -supermartingale. We denote by $\mathcal{M}_{sp}(\mathcal{Y})$ the set of all supermartingale measures for the family $\mathcal{Y}$, and we will say that $\mathcal{Q}$ is a set of supermartingale measures if it is the set of supermartingale measures for a family $\mathcal{Y}$ of bounded adapted processes.

Now we state results related to that.

Theorem 3.4 Let $\mathcal{Q} \in \Pi^{c,e}$. Then we have the following:

- (1) $\mathcal{Q}^{st}$ is the set of supermartingale measures for the family $\mathcal{Y} = \{\mathcal{E}^{st}(X) : X \in \mathcal{A}^{st}\}$.
- (2) $\mathcal{Q}^{st}$ is the smallest set of supermartingale measures containing $\mathcal{Q}$.

Proof (1) we know from Assertion (4) in Proposition 2.3 that the process $\mathcal{E}^{st}(X)$ is a $\mathcal{Q}^{st}$ -supermartingale for all $X \in \mathbb{L}^\infty$, so in particular $\mathcal{Q}^{st} \subseteq \mathcal{M}_{sp}(\mathcal{Y})$. Inversely let $\mathbb{Q} \in \mathcal{M}_{sp}(\mathcal{Y})$ then $\mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{Q}}(\mathcal{E}_T^{st}(X)) \leq \mathcal{E}_0^{st}(X) \leq 0$ for all $X \in \mathcal{A}^{st}$. So $\mathbb{Q} \in \mathcal{Q}^{st}$.

- (2) Let $\tilde{\mathcal{Q}} = \mathcal{M}_{sp}(\mathcal{Y})$ for a family $\mathcal{Y}'$ of bounded adapted processes with $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$. Since $\tilde{\mathcal{Q}}$ is an m -stable closed convex set, we deduce that $\mathcal{Q}^{st} \subseteq \tilde{\mathcal{Q}}$. $\square$

For $\mathcal{Q} \in \Pi$ we denote by $m(\mathcal{Q})$ (resp., $\text{spm}(\mathcal{Q})$) the set of all bounded $\mathcal{Q}$ -martingales (resp., $\mathcal{Q}$ -supermartingales).

Theorem 3.5 *Let $\mathcal{Q} \in \Pi^{c,e}$. Then we have the following:*

- (1) $\text{spm}(\mathcal{Q}) = \{Y = (Y_t)_{t \in \mathcal{I}} : Y_{t+1} - Y_t \in \mathcal{A}_{t,t+1} \text{ for all } t \in \mathcal{I}^*\}$.
- (2) $\mathcal{A}_{t,t+1} = \{Y_{t+1} - Y_t : Y \in \text{spm}(\mathcal{Q})\} =: B_t \text{ for all } t \in \mathcal{I}^*$.
- (3) $\text{spm}(\mathcal{Q}) = \text{spm}(\mathcal{Q}^{st})$.
- (4) $\mathcal{Q}^{st}$ is the greatest subset in $\mathbf{P}$ that satisfies $\text{spm}(\mathcal{Q}) = \text{spm}(\mathcal{Q}^{st})$.
- (5) $m(\mathcal{Q}) = m(\mathcal{Q}^{st})$.

Proof (1) It is straightforward from the definitions of $\text{spm}(\mathcal{Q})$ and $\mathcal{A}_{t,t+1}$ for $t \in \mathcal{I}^*$.

(2) The inclusion $B_t \subseteq \mathcal{A}_{t,t+1}$ is straightforward. For the direct inclusion let $X \in \mathcal{A}_{t,t+1}$ and define the process Y by $Y_s = X$ for $s > t$ and $Y_s = 0$ for $s \leq t$. Then $Y \in \text{spm}(\mathcal{Q})$ by (1) and $X = Y_{t+1} - Y_t$.

(3) We apply Assertion (1) and the fact that $\mathcal{A}_{t,t+1} = \mathcal{A}_{t,t+1}^{st}$ for all $t \in \mathcal{I}^*$.

(4) Let $\tilde{\mathcal{Q}} \subseteq \mathbf{P}$ satisfying $\text{spm}(\mathcal{Q}) = \text{spm}(\tilde{\mathcal{Q}})$. Then from Assertion (2) we get $\mathcal{A}_{t,t+1} = \tilde{\mathcal{A}}_{t,t+1}$ for all $t \in \mathcal{I}^*$ and, therefore, $\mathcal{A}^{st} = \tilde{\mathcal{A}}^{st}$. We conclude that $\tilde{\mathcal{Q}} \subseteq \tilde{\mathcal{Q}}^{st} = \mathcal{Q}^{st}$.

(5) We remark that $X \in m(\mathcal{Q})$ if and only if $\pm X \in \text{spm}(\mathcal{Q})$. We use Assertion (3) and deduce the result. $\square$

An immediate consequence of Theorem 3.5 is as follows:

Corollary 3.6 *Let $\mathcal{Q}^1, \mathcal{Q}^2 \in \Pi^{c,e}$. Then the following three assertions are equivalent:*

- (i) $\text{spm}(\mathcal{Q}^1) \subseteq \text{spm}(\mathcal{Q}^2)$,
- (ii) $\mathcal{A}_{t,t+1}^1 \subseteq \mathcal{A}_{t,t+1}^2$ for all $t \in \mathcal{I}^*$,
- (iii) $(\mathcal{Q}^2)^{st} \subseteq (\mathcal{Q}^1)^{st}$.

For a set $B \subseteq \mathbb{L}^0$ we denote $\text{lin}(B) = B \cap -B$ and for $\mathcal{Q} \subseteq \mathbf{P}$ we denote $\mathcal{Q}' = \{\mathbb{Q} \in \mathbf{P} : \mathbb{E}^{\mathbb{Q}}(X) = 0 \text{ for all } X \in \text{lin}(\mathcal{A})\}$. The set $\mathcal{Q}'$ was first introduced by Berkouli in [2] and defined properly in [3]. We refer to Theorem 2.1 in [3] for more details on this set.

Theorem 3.7 *Let $\mathcal{Q} \in \Pi^{c,e}$. Then we have the following:*

- (1) $m(\mathcal{Q}) = \{Y = (Y_t)_{t \in \mathcal{I}} : Y_{t+1} - Y_t \in \text{lin}(\mathcal{A}_{t,t+1}) \text{ for all } t \in \mathcal{I}^*\}$.
- (2) $\text{lin}(\mathcal{A}_{t,t+1}) = \{Y_{t+1} - Y_t : Y \in m(\mathcal{Q})\} =: D_t \text{ for all } t \in \mathcal{I}^*$.
- (3) $m(\mathcal{Q}) = m((\mathcal{Q}^{st})')$.
- (4) $(\mathcal{Q}^{st})'$ is the greatest subset in $\mathbf{P}$ that satisfies $m(\mathcal{Q}) = m((\mathcal{Q}^{st})')$.

Proof (1) It is straightforward from the definitions of $m(\mathcal{Q})$ and $\text{lin}(\mathcal{A}_{t,t+1})$ for $t \in \mathcal{I}^*$.

(2) The inclusion $D_t \subseteq \mathcal{A}_{t,t+1}$ is straightforward. For the direct inclusion let $X \in \text{lin}(\mathcal{A}_{t,t+1})$ and define the process Y by $Y_s = X$ for $s > t$ and $Y_s = 0$ for $s \leq t$. Then $Y \in m(\mathcal{Q})$ and $X = Y_{t+1} - Y_t$.

(3) Since $m(\mathcal{Q}) = m(\mathcal{Q}^{st})$ it suffices to show that $m(\mathcal{Q}) = m(\mathcal{Q}')$. We apply Theorem 2.5 to deduce that $\text{lin}(\mathcal{A}_{t,t+1}) = \text{lin}(\mathcal{A}_{t,t+1}')$ for all $t \in \mathcal{I}^*$. The result is concluded thanks to Assertion (1).

(4) Let $\tilde{\mathcal{Q}} \in \Pi$ such that $m(\mathcal{Q}) = m(\tilde{\mathcal{Q}})$, then from Assertion (2) we get $\text{lin}(\mathcal{A}_{t,t+1}) = \text{lin}(\tilde{\mathcal{A}}_{t,t+1})$ for all $t \in \mathcal{I}^*$ and, therefore, $(\mathcal{A}^{st})' = (\tilde{\mathcal{A}}^{st})'$ thanks to Theorem 2.5. We conclude that $\tilde{\mathcal{Q}} \subseteq (\mathcal{Q}^{st})'$. $\square$

An immediate consequence of Theorem 3.7 is as follows:

Corollary 3.8 *Let $\mathcal{Q}^1, \mathcal{Q}^2 \in \Pi^{c,e}$. Then the following three assertions are equivalent:*

- (i) $m(\mathcal{Q}^1) \subseteq m(\mathcal{Q}^2)$,
- (ii) $\text{lin}(\mathcal{A}_{t,t+1}^1) \subseteq \text{lin}(\mathcal{A}_{t,t+1}^2)$ for all $t \in \mathcal{I}^*$,
- (iii) $((\mathcal{Q}^2)^{st})' \subseteq ((\mathcal{Q}^1)^{st})'$.

4 Main results

In this section we precise further the results of Theorem 3.4. In what follows, we suppose the assumption **MRP**(S): the filtration $(\mathcal{F}_t)_{t \in \mathcal{I}}$ satisfies the martingale representation property with respect to a bounded martingale S with values in $\mathbb{R}^d$, which means that any square integrable martingale X can be written as the sum $\mathbb{E}(X) + \alpha \bullet S$ for some vector-valued adapted process α .

Theorem 4.1 Let $\mathcal{Q} \in \Pi^{c,e}$. Then

- (1) $\mathcal{Q}^{st} = \mathcal{M}_{sp}(\mathbf{C} \bullet S)$ with $\mathbf{C} \bullet S = \bigoplus_{t \in \mathcal{I}^*} \mathbf{C}_t \cdot \Delta_t S$ and each $\mathbf{C}_t$ is a convex cone in $\mathbb{L}^\infty(\mathcal{F}_t; \mathbb{R}^d)$ given by $\mathbf{C}_t = \{\alpha_t \in \mathbb{L}^\infty(\mathcal{F}_t; \mathbb{R}^d) : \alpha_t \cdot \Delta_t S \in \mathcal{A}_{t,t+1}\}$.
- (2) Suppose $\mathcal{Q}$ has a finite number of extreme points, then $\mathcal{Q}^{st} = \mathcal{M}_{sp}(g \bullet S)$ for a matrix-valued adapted process g .

We state first in the next theorem a similar version of Theorem 4.1 in the one-period model. We say that an element $\mathbb{Q} \in \mathcal{Q}$ is an $\mathcal{F}_t$ -extreme point of $\mathcal{Q}$ if there are no elements $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ and $\alpha_t \in \mathbb{L}^\infty(\mathcal{F}_t; [0, 1])$ such that $\mathbb{E}_t^{\mathbb{Q}}(X) = \alpha_t \mathbb{E}_t^{\mathbb{Q}^1}(X) + (1 - \alpha_t) \mathbb{E}_t^{\mathbb{Q}^2}(X)$ for all $X \in \mathbb{L}^\infty(\mathcal{F}_{t+1})$.

Theorem 4.2 Let $\mathcal{Q} \in \Pi^{c,e}$ and suppose $\mathcal{I} = \{0, 1\}$ and a non-necessarily trivial $\mathcal{F}_0$. Then

- (1) $\mathcal{Q}^{st} = \mathcal{M}_{sp}(0; \mathbf{C}_0 \cdot (S_1 - S_0))$ where $\mathbf{C}_0$ is a convex cone in $\mathbb{L}^\infty(\mathcal{F}_0; \mathbb{R}^d)$ given by $\mathbf{C}_0 = \{\alpha_0 \in \mathbb{L}^\infty(\mathcal{F}_0; \mathbb{R}^d) : \alpha_0 \cdot (S_1 - S_0) \in \mathcal{A}\}$.
- (2) Suppose $\mathcal{Q}$ has a finite number of $\mathcal{F}_0$ -extreme points, then $\mathcal{Q}^{st} = \mathcal{M}_{sp}(0; g_0 \cdot (S_1 - S_0))$ for a matrix-valued $\mathcal{F}_0$ -measurable random variable g_0 .

Proof (1) We will show that $\mathcal{Q}$ is optionally m-stable with respect to $\Delta S := S_1 - S_0$. Let $\mathbb{Q} \in \mathbf{P}$ such that $\mathbb{E}_0^{\mathbb{Q}}(\Delta S) = \mathbb{E}_0^{\mathbb{Q}'}(\Delta S)$ for some $\mathbb{Q}' \in \mathcal{Q}$. Then for all $X \in \mathcal{A}$ and thanks to the assumption **MRP**(S), there exists some vector-valued $\mathcal{F}_0$ -measurable random variable α_0 such that $X = \mathbb{E}_0(X) + \alpha_0 \cdot \Delta S$ and, therefore, $\mathbb{E}_0^{\mathbb{Q}}(X) = \mathbb{E}_0(X) + \alpha_0 \cdot \mathbb{E}_0^{\mathbb{Q}}(\Delta S) = \mathbb{E}_0(X) + \alpha_0 \cdot \mathbb{E}_0^{\mathbb{Q}'}(\Delta S) = \mathbb{E}_0^{\mathbb{Q}'}(X) \leq 0$, so $\mathbb{Q} \in \mathcal{Q}$. By applying Theorem 2.17 in [8] we deduce that $\mathcal{Q} = \mathcal{M}_{sp}(0; \mathbf{C}_0 \cdot \Delta S)$ with $\mathbf{C}_0 = \{\alpha_0 \in \mathbb{L}^\infty(\mathcal{F}_0; \mathbb{R}^d) : \alpha_0 \cdot \Delta S \in \mathcal{A}\}$.

(2) We define the set K to be the closure of $\mathbf{C}_0$ in $\mathbb{L}^0(\mathbb{R}^d)$ with respect to the topology of convergence in measure and verify that $K = \{\alpha_0 \in \mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d) : \alpha_0 \cdot Y^i \leq 0 \text{ for all } i = 1 \dots k\}$ where $Y^i = \mathbb{E}_0^{\mathbb{Q}^i}(\Delta S)$ and $(\mathbb{Q}^1, \dots, \mathbb{Q}^k)$ are the $\mathcal{F}_0$ -extreme points of $\mathcal{Q}$. The set K is an $\mathcal{F}_0$ -stable closed convex cone in $\mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d)$. Thanks to Theorem 4.6 in [9] there exists a random closed convex cone W in $\mathbb{R}^d$ such that $\alpha_0 \in K$ if and only if $\alpha_0 \in \mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d)$ and $\alpha_0 \in W$ a.s. which means that $\alpha_0 \cdot Y^i \leq 0$ a.s. for $i = 1 \dots k$. We denote by $g_0 = (g_0^1, \dots, g_0^d)$ the generating family of W , i.e., for all $\alpha_0 \in W$, there exists positive $\mathcal{F}_0$ -measurable scalar random variables $\lambda^1, \dots, \lambda^d$ such that $\alpha_0 = \lambda \cdot g_0 = \sum_{i=1}^d \lambda^i g_0^i$. We shall prove that $\mathcal{Q} = \mathcal{M}_{sp}(0; g_0 \cdot \Delta S)$. For the direct inclusion we have $\mathbb{E}_0^{\mathbb{Q}}(g_0 \cdot \Delta S) \leq 0$ for all $\mathbb{Q} \in \mathcal{Q}$ since $g_0 \in K$. For the inverse inclusion let $\mathbb{Q} \in \mathcal{M}_{sp}(0; g_0 \cdot \Delta S)$ and $X \in \mathcal{A}$, then there exists a sequence $\alpha_0^n \in \mathbf{C}_0$ such that $X = \lim_{n \rightarrow \infty} \alpha_0^n \cdot \Delta S$ with the limit taken in weak star topology. There exists then a sequence $\lambda^n \geq 0$ such that $\alpha_0^n = \lambda^n \cdot g_0$; therefore, $\alpha_0^n \cdot \Delta S = \lambda^n \cdot g_0 \cdot \Delta S$ and then $\mathbb{E}_0^{\mathbb{Q}}(X) = \lim_{n \rightarrow \infty} \mathbb{E}_0^{\mathbb{Q}}(\lambda^n \cdot (g_0 \cdot \Delta S)) = \lim_{n \rightarrow \infty} \lambda^n \cdot \mathbb{E}_0^{\mathbb{Q}}(g_0 \cdot \Delta S) \leq 0$. So $\mathbb{Q} \in \mathcal{Q}$. $\square$

Now we prove Theorem 4.1.

Proof We shall show first that for each $t \in \mathcal{I}^*$, the assumption **MRP**(S_t, S_{t+1}) is satisfied on the one-period model $\{t, t+1\}$ with $\Delta_t S := S_{t+1} - S_t$. Indeed for a process $X = (X_t, X_{t+1})$ with $\mathbb{E}_t(X_{t+1}) = X_t$ we define the process Y by $Y_s = \mathbb{E}_s(X_{t+1})$ for $s \in \mathcal{I}$ and remark that Y is a martingale. So there exists a process α such that $Y = Y_0 + \alpha \bullet S$. In particular, we get $X_{t+1} - X_t = Y_{t+1} - Y_t = \alpha_t \Delta_t S$.

We denote by $\mathcal{Q}^t$ for $t \in \mathcal{I}^*$, the set of probability measures $\mathbb{Q} \in \mathbf{P}$, defined on $(\Omega, \mathcal{F}_{t+1})$ such that $\mathbb{E}_t^{\mathbb{Q}}(X) \leq 0$ for all $X \in \mathcal{A}_{t,t+1}$. For Assertion (1) we apply Assertion (1) in Theorem 4.2 and obtain that $\mathcal{Q}^t = \mathcal{M}_{sp}(0; \mathbf{C}_t \cdot \Delta_t S)$. Now to prove that $\mathcal{Q}^{st} = \mathcal{M}_{sp}(\mathbf{C} \bullet S)$, we remark that the direct inclusion is trivial from the definition of $\mathbf{C}$. For the inverse inclusion let $\mathbb{Q} \in \mathcal{M}_{sp}(\mathbf{C} \bullet S)$ and $X \in \mathcal{A}^{st}$. So $X = X_0 + \dots + X_{T-1}$ with each $X_t \in \mathcal{A}_{t,t+1}$, then $\mathbb{E}_t^{\mathbb{Q}}(X_t) = \mathbb{E}_t^{\mathbb{Q}^t}(X_t)$ where $\mathbb{Q}^t$ is the restriction of $\mathbb{Q}$ on $\mathcal{F}_{t+1}$. So $\mathbb{Q}^t \in \mathcal{Q}^t$ and, therefore, $\mathbb{E}_t^{\mathbb{Q}}(X_t) = \mathbb{E}_t^{\mathbb{Q}^t}(X_t) \leq 0$.

For Assertion (2) we shall show that $\mathcal{Q}^t$ has a finite number of $\mathcal{F}_t$ -extreme points. Let $(\mathbb{Q}^1, \dots, \mathbb{Q}^k)$ be the extreme points of $\mathcal{Q}$ with their respective densities $(Z^1, \dots, Z^k)$. Then the probability measures $(\mathbb{Q}^{t,1}, \dots, \mathbb{Q}^{t,k})$, defined on $\mathcal{F}_{t+1}$ by their respective densities $(Z^{t,1}, \dots, Z^{t,k}) := (Z_{t+1}^1, \dots, Z_{t+1}^k)$ are the $\mathcal{F}_t$ -extreme points of $\mathcal{Q}^t$. Indeed we have $\mathcal{A}_{t,t+1}$ as the dual cone of $\mathcal{Q}^t$ in $\mathbb{L}^\infty(\mathcal{F}_{t+1})$ and $\mathcal{A}_{t,t+1} := \{X \in \mathbb{L}^\infty(\mathcal{F}_{t+1}) : \mathbb{E}_t^{\mathbb{Q}}(X) \leq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\} = \{X \in \mathbb{L}^\infty(\mathcal{F}_{t+1}) : \mathbb{E}_t^{\mathbb{Q}^{t,i}}(X) \leq 0 \text{ for all } i = 1 \dots k\}$. $\square$

Remark 4.3 For the finite sample space case, all assertions of Theorem 4.1 are satisfied since the singleton $\{\mathbb{P}\}$ is the unique martingale measure for a vector-valued adapted process S , and then the assumption **MRP**(S) is satisfied.

Example 4.4 We consider the one-period model with $\Omega = \{\omega_1, \omega_2\}$. We verify easily that $\mathcal{Q} := [(1/2, 1/2), (1/3, 2/3)] = \mathcal{M}_{sp}(0; X)$ since $\mathcal{Q}$ has only two extreme points with $X = (X^1, X^2)$, $X^1 = (1, -1)$ and $X^2 = (-2, 1)$.

5 Applications

We suppose again that the assumption **MRP**(S) is satisfied along this section and investigate the decomposition of $\mathcal{Q}$ -supermartingales.

Theorem 5.1 *Let $\mathcal{Q} \in \Pi^{c,e}$. Then any bounded $\mathcal{Q}$ -supermartingale X can be decomposed as follows: $X = X_0 + \alpha \bullet S - C$ with C an increasing process and the process $\alpha \bullet S$ is a $\mathcal{Q}$ -supermartingale.*

Proof For $t \in \mathcal{I}^*$ we denote by $\mathcal{Q}^t$ for $t \in \mathcal{I}^*$, the set of probability measures $\mathbb{Q} \in \mathbf{P}$, defined on $(\Omega, \mathcal{F}_{t+1})$ such that $\mathbb{E}_{\mathbb{Q}}^{\mathbb{Q}}(X) \leq 0$ for all $X \in \mathcal{A}_{t,t+1}$. It has been proved in the proofs of Theorems 4.1 and 4.2 that the assumption **MRP**(S_t, S_{t+1}) is satisfied on the one-period model $\{t, t+1\}$ with $\Delta_t S := S_{t+1} - S_t$ and that $\mathcal{Q}^t$ is optionally m-stable with respect to $\Delta_t S$. We apply then Proposition 3.3 in [2] and obtain that $\mathcal{A}_{t,t+1} \subseteq K_t \cdot \Delta_t S + \mathbb{L}_-^0(\mathcal{F}_{t+1})$ where $K_t = \{\alpha_t \in \mathbb{L}^0(\mathcal{F}_{t+1}; \mathbb{R}^d) : \alpha_t \cdot \Delta_t S \in \mathcal{A}_{t,t+1}\}$. So for a bounded $\mathcal{Q}$ -supermartingale X we have $X_{t+1} - X_t \in \mathcal{A}_{t,t+1}$ for each $t \in \mathcal{I}^*$, and then there exists some $\alpha_t \in K_t$ and $B_t \in \mathbb{L}_+^0(\mathcal{F}_{t+1})$ such that $X_{t+1} - X_t = \alpha_t \cdot \Delta_t S - B_t$. We deduce that $X = X_0 + \alpha \bullet S - C$ with $C_0 = 0$ and $C_t = B_0 + \dots + B_{t-1}$ for $t \in \{1, \dots, T\}$. $\square$

Corollary 5.2 *Let $\mathcal{Q} \in \Pi^{c,e}$. Then any bounded random variable Y can be written as follows: $Y = \mathcal{E}_0^{st}(Y) + (\alpha \bullet S)_T - B$ with B a positive random variable and the process $\alpha \bullet S$ is a $\mathcal{Q}$ -supermartingale.*

Proof We apply Theorem 5.1 for the $\mathcal{Q}$ -supermartingale $\mathcal{E}^{st}(Y)$. $\square$

An immediate consequence of Theorem 5.1 is as follows:

Corollary 5.3 *Let $\mathcal{Q} \in \Pi^{c,e}$. Then any bounded $\mathcal{Q}$ -martingale X can be decomposed as follows: $X = X_0 + \alpha \bullet S$.*

Next we investigate the case where the process $\alpha \bullet S$ appearing in Theorem 5.1 is a local $\mathcal{Q}$ -martingale. We generalize then the result of Föllmer and Kabanov [7].

Theorem 5.4 *Let $\mathcal{Q} \in \Pi^{c,e}$ such that $\mathcal{Q} = \mathcal{Q}'$. Then any bounded $\mathcal{Q}$ -supermartingale X can be decomposed as follows: $X = X_0 + \alpha \bullet S - C$ with C an increasing process and the process $\alpha \bullet S$ is a local $\mathcal{Q}$ -martingale.*

To prove Theorem 5.4, we state first some preliminary Lemmas.

Lemma 5.5 *Let $\mathcal{Q} \in \Pi^{c,e}$. Then $\mathcal{A}' \subseteq K' + \mathbb{L}_-^0$ where K' is the closure in $\mathbb{L}^0$ of $\text{lin}(\mathcal{A})$.*

Proof We shall prove that the set $K' + \mathbb{L}_-^0$ is closed in $\mathbb{L}^0$. We prove it first for the one-period model. We suppose then that $T = 1$ and $\mathcal{F}_0$ is not necessarily trivial. Let us define $D := \{\alpha \in \mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d) : \alpha \cdot \Delta S \in K'\}$ and verify easily that D is an $\mathcal{F}_0$ -stable closed vector space in $\mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d)$. So thanks to Lemma A.4 in [11] and Lemma 2.5 in [4] there exists a generating family $g = (g^1, \dots, g^d)$ of the set D . We deduce that $K' = \{\alpha \cdot Y : \alpha \in \mathbb{L}^0(\mathcal{F}_0; \mathbb{R}^d)\}$ with $Y = g \cdot \Delta S$. We apply Lemma 2.1 in [10] to conclude that $K' + \mathbb{L}_-^0$ is closed in $\mathbb{L}^0$. Now for the multi-period case we proceed as follows: for all $t \in \mathcal{I}^*$ we define $D_t = \{\alpha_t \in \mathbb{L}^0(\mathcal{F}_t; \mathbb{R}^d) : \alpha_t \cdot \Delta_t S \in K'\}$, $g_t = (g_t^1, \dots, g_t^d)$ the generating family of D_t , $Y_t = g_t \cdot \Delta_t S$ and $B_t = \{\alpha_t \cdot Y_t : \alpha_t \in \mathbb{L}^0(\mathcal{F}_t; \mathbb{R}^d)\}$. We shall prove by induction on $t = T-1, \dots, 0$ that the set $B^t := B_t + \dots + B_{T-1} + \mathbb{L}_-^0$ is closed in $\mathbb{L}^0$. For $t = T-1$ we apply the one-period case. Now we suppose that the sets $B^{t+1}, \dots, B^{T-1}$ are closed, we shall prove that B^t is closed. We remark that $B^t = B_t + B^{t+1}$ and follow the proof of Proposition 3.3 in [2]. Define the set

$$\mathcal{N} := \{\alpha_t \in \mathbb{L}^0(\mathcal{F}_t; \mathbb{R}^d) : -\alpha_t \cdot Y_t \in B^{t+1} \text{ a.s.}\}.$$

We prove easily that $\mathcal{N}$ is a closed $\mathcal{F}_t$ -stable convex cone in $\mathbb{L}^0(\mathcal{F}_t; \mathbb{R}^d)$. We prove now that it is a vector space. Let $\alpha_t \in \mathcal{N}$ so there exists $\alpha_s \in \mathbb{L}^0(\mathcal{F}_s; \mathbb{R}^d)$ for $s = t+1 \dots T-1$ and $z \in \mathbb{L}_+^0$ such that $\alpha_t \cdot Y_t + \dots + \alpha_{T-1} \cdot Y_{T-1} = z \geq 0$; therefore, $z = 0$ and $\alpha_t \cdot Y_t = -\alpha_{t+1} \cdot Y_{t+1} - \dots - \alpha_{T-1} \cdot Y_{T-1} \in B^{t+1}$ which means that $-\alpha_t \in \mathcal{N}$. We denote by $\mathcal{N}^\perp$ the orthogonal vector space of $\mathcal{N}$. The decomposition of B^t becomes

$B^t = \mathcal{N}_t^\perp \cdot Y_t + B^{t+1}$. Let $X^n \in B^t$ be a sequence converging in measure to some X . Then, there exists a subsequence, denoted also by X^n , converging a.s. to X . So there exists some $\alpha_t^n \in \mathcal{N}_t^\perp$ and $Y^n \in B^{t+1}$ such that $X^n = \alpha_t^n \cdot Y_t + Y^n$. We claim that $\liminf_{n \rightarrow \infty} |\alpha_t^n| < \infty$ a.s. In fact, if we suppose the opposite, then there exists an integer-valued random sequence n_k such that $\lim_{k \rightarrow \infty} |\alpha_t^{n_k}| = \infty$. Therefore,

$$\frac{X^{n_k}}{|\alpha_t^{n_k}|} = \frac{\alpha_t^{n_k}}{|\alpha_t^{n_k}|} \cdot Y_t + \frac{Y^{n_k}}{|\alpha_t^{n_k}|} =: \lambda_t^{n_k} \cdot Y_t + J^{n_k},$$

where $\lambda_t^{n_k} \in \mathcal{N}_t^\perp$ is bounded and $J^{n_k} \in B^{t+1}$. Then, there exists a subsequence $\lambda_t^{n'_k}$, converging a.s. to some $\lambda_t \in \mathcal{N}_t^\perp$. Therefore, the sequence $J^{n'_k}$ converges a.s. to some $J \in B^{t+1}$ and then $\lambda_t \cdot Y_t + J = 0$, which means that $\lambda_t \in \mathcal{N}$ and since $\lambda_t \in \mathcal{N}^\perp$ then $\lambda_t = 0$ which contradicts the fact that $|\lambda_t| = \lim_{k \rightarrow \infty} |\lambda_t^{n'_k}| = 1$. Now, since $\liminf_{n \rightarrow \infty} |\alpha_t^n| < \infty$ a.s., there exists an integer-valued random sequence n_k such that $\alpha_t^{n_k} \in \mathbb{L}^0(\mathcal{F}_t; \mathbb{R}^d)$ converges a.s. to some $\alpha_t \in \mathbb{L}^0(\mathcal{F}_t; \mathbb{R}^d)$ and then $Y^{n_k} \in B^{t+1}$ converges a.s. to some $Y \in B^{t+1}$ since B^{t+1} is closed by induction hypothesis. We conclude that B^t is closed and then $B^0 = K' + \mathbb{L}_-^0$ is closed. $\square$

Lemma 5.6 Let $\mathcal{Q} \in \Pi^{c,e}$. Then $(\mathcal{Q}^{st})' = (\mathcal{Q}')^{st}$

Proof For the direct inclusion we have $(\mathcal{Q}')^{st} \subseteq ((\mathcal{Q}^{st})')^{st}$ and since $(\mathcal{Q}^{st})'$ is m-stable thanks to Theorem 2.5, then $(\mathcal{Q}')^{st} \subseteq (\mathcal{Q}^{st})'$. For the reverse inclusion we shall show that $(\mathcal{A}')^{st} \subseteq (\mathcal{A}^{st})'$. It suffices to show that $\mathcal{A}'_{t,t+1} \subseteq (\mathcal{A}^{st})'$ for all $t \in \mathcal{I}^*$. Let $X \in \mathcal{A}'_{t,t+1}$, then by Theorem 5.1 there exists some α_t such that $X \leq \alpha_t \cdot \Delta_t S =: Y_t$ and $\mathbb{E}_t^{\mathbb{Q}}(Y_t) \leq 0$ for all $\mathbb{Q} \in \mathcal{Q}$. Let $\alpha_t^N = \mathbf{1}_{(|\alpha_t| \leq N)} \alpha_t$ then $Y_t^N := \alpha_t^N \cdot \Delta_t S \in \mathcal{A}'$ so thanks to Lemma 5.5 we deduce that $Y_t^N \leq X_t^N$ for some element X_t^N in the closure in $\mathbb{L}^0$ of $\text{lin}(\mathcal{A})$. By truncating and by taking the expectation we obtain that $Y^N = X^N \in \text{lin}(\mathcal{A}_{t,t+1})$. We take the limit and obtain that $Y_t \in \text{lin}(\mathcal{A}_{t,t+1})$. So for any $\mathbb{Q} \in (\mathcal{Q}^{st})'$ we have $\mathbb{E}_t^{\mathbb{Q}}(X) \leq \mathbb{E}_t^{\mathbb{Q}}(Y_t) \leq 0$. Therefore, $X \in (\mathcal{A}^{st})'$. $\square$

Lemma 5.7 Let $\mathcal{Q} \in \Pi^{c,e}$ such that $\mathcal{Q} = \mathcal{Q}'$. Then $\mathcal{Q}^{st} = \mathcal{M}(g \bullet S)$ for some matrix-valued adapted process g .

Proof Thanks to Lemma 5.6 we have $\mathcal{Q}^{st} = (\mathcal{Q}^{st})'$ and by applying Lemma 5.5 in [2] and Theorem 4.1 we deduce the result. $\square$

Now we prove Theorem 5.4.

Proof of Theorem 5.4 The theorem is an immediate consequence of Lemma 5.7 and Theorem 2 in [7]. $\square$

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