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Bahri invariants for fractional Nirenberg-type flows

Received: 27 October 2016 / Accepted: 30 March 2017 / Published online: 18 April 2017

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Abstract This paper is concerned with prescribing the fractional Q -curvature on the unit sphere $\mathbb{S}^n$ endowed with its standard conformal structure g_0 , $n \geq 4$. Since the associated variational problem is noncompact, we approach this issue with techniques passed by Abbas Bahri, as the well known theory of critical points at infinity, as well as some lesser known topological invariants that appear here as criteria for existence results.

Mathematics Subject Classification 18G35 · 35A15 · 35A25 · 35J65 · 53C15 · 53C21 · 58E05

المخلص

تتعلق هذه الورقة بوصف الانحناء Q -الكسري على كرة الوحدة S^n باعتبار البناء المحافظ القياسي g_0 , $n \geq 4$. بما أن مسألة التغيرات المرتبطة غير متراسة، فإننا نقارب هذه المسألة باستخدام تقنيات طورها عباس بحري، مثل النظرية المعروفة جيداً الخاصة بالنقاط الحرجة عند اللانهاية، وبعض اللامتغيرات التوبولوجية الأقل شهرة التي تظهر هنا كمعايير لنتائج وجود.

1 Introduction

The armor is not for protecting the soul from spoilage.¹

In conformal geometry the Nirenberg problem is the well-known problem of prescribing the scalar (resp. Gauss) curvature on the sphere $\mathbb{S}^n$ for $n \geq 3$ (resp. $n = 2$). Given a prescribed function $K : \mathbb{S}^n \rightarrow \mathbb{R}$ this problem amounts to solve the following equations

$$-\Delta_{g_0} u + 1 = K e^u \quad \text{on } \mathbb{S}^2,$$

and

$$P_1^{g_0} u = K u^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n \quad \text{for } n \geq 3,$$

where $P_1^{g_0} = -\Delta_{g_0} + \frac{n-2}{4(n-1)} R_{g_0}$ is the conformal Laplacian and R_{g_0} the scalar curvature associated to g_0 . The operator $P_1^{g_0}$ is the first of a sequence of conformally covariant elliptic operators. The second operator was introduced by Paneitz in [29] (see also [19]) and is

$$P_2^{g_0} = (-\Delta_{g_0})^2 - \operatorname{div}_{g_0} (a_n R_{g_0} g_0 + b_n \operatorname{Ric}_{g_0}) d + \frac{(n-4)}{2} Q_n^{g_0}$$

¹ Translation of a quote from Ali Ibn Abi Talib, handwritten in arabic by Abbas Bahri in the first page of the manuscript of his paper [2].

where Ric_{g_0} and $Q_n^{g_0}$ are respectively the Ricci and the standard Q -curvature of g_0 , a_n and b_n are constants depending on n . In [21] Graham, Jenne, Masion and Sparling gave the first construction of a sequence of conformally covariant elliptic operators $(P_k^{g_0})_k$, which exist for all $k \in \mathbb{N}$ when the dimension n is odd, but only for $k \in \{1, 2, \dots, \frac{n}{2}\}$ when n is even.

In [30], Peterson constructed an intrinsically defined, arbitrary real number order, conformally covariant pseudo-differential operator. In [22], Graham and Zworski showed that the operators $P_k^{g_0}$ can be realized as the residues at $s = k$ of a meromorphic family of scattering operators. Using this point of view, a family of conformally covariant pseudo-differential operators $(P_s^{g_0})$ for noninteger s was given.

In recent years we have seen several works on the properties of the fractional Laplacians as nonlocal operators and their applications to various problems (see e.g. [9–13]). It is a known fact that $(-\Delta_{g_0})^s$ on $\mathbb{R}^n$ with $s \in (0, 1)$ is a nonlocal operator. In [10], Caffarelli and Silvestre expressed $(-\Delta_{g_0})^s$ as a generalized Dirichlet–Neumann map for an elliptic boundary value problem with local differential operators defined on $\mathbb{R}_+^{n+1}$. This work was extended in [14] where Chang and Gonzalez characterized $(P_s^{g_0})$ as such a Dirichlet-to-Neumann operator on a conformally compact Einstein manifold.

Observe now that when $\gamma \in (0, \frac{n}{4})$, under the conformal change of metric $g = u^{\frac{4}{n-2s}} g_0$, the new operator P_s^g associated to the new metric g is connected with the initial operator $P_s^{g_0}$ according to the following conformal transformation law

$$P_s^{g_0}(uf) = u^{\frac{n+2s}{n-2s}} P_s^g(f)$$

for any smooth function f . The formula for the scalar curvature or the Paneitz Branson Q -curvature can be extended, and the fractional Q -curvature for g of order s is defined as

$$Q_s^g = P_s^g(1).$$

We will in the sequel omit the superscript g_0 in $P_s^{g_0}$ and put $P_s^{g_0} = P_s$. We recall that P_s is the $2s$ order conformal Laplacian on $\mathbb{S}^n$, which is an intertwining operator having the expression

$$P_s = \frac{\Gamma(B + \frac{1}{2} + s)}{\Gamma(B + \frac{1}{2} - s)}$$

where Γ is the Gamma function and

$$B = \sqrt{-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2}.$$

In this paper we focus on the fractional Nirenberg problem on the standard sphere $\mathbb{S}^n$, $n \geq 4$, that is the prescribing fractional Q -curvature problem: find a conformal change of metric $g = u^{\frac{4}{n-2s}} g_0$ such that $Q_s^g = K$ where $K : \mathbb{S}^n \rightarrow \mathbb{R}$ is a prescribed function. This problem amounts to solve the following system

$$\begin{cases} P_s u = K u^{\frac{n+2s}{n-2s}} & \text{on } \mathbb{S}^n, \\ u > 0 & \text{on } \mathbb{S}^n. \end{cases} \quad (1.1)$$

Several authors studied problem (1.1) on $\mathbb{S}^n$, see e.g. [15, 16, 23, 24, 26], but their methods differ from the our. Note however that in [1] and [17] the authors used, as we do, the theory of critical points at infinity, but they conclude in different ways from us. More particularly in [17] Chen, Liu and Zheng addressed the problem (1.1) using the same framework that ours; they established an Euler–Hopf-type formula, and obtained some existence results under Bahri–Coron-type assumptions. However our results are original since they are obtained in a completely different way: by using some topological invariants which are associated to variational flows. These invariants are actually intersection numbers who have been introduced the first time by Bahri in [2], where in order to treat the scalar curvature problem on high dimensional spheres in the Riemannian framework, the author has taken the basic elements of the variational theory, that is to say the deformation along the variational flows and the analysis of changes of topology which they induce, even if they drive to infinity. As we shall see, these intersection numbers are proving to be criteria of existence of solutions.

Our paper will be organized as follows: In Sect. 2 we introduce some notations and constructions and we give our main results. In Sect. 3 we recall the variational framework. Section 4 and 5 are devoted to prove our results.

2 Notations, assumptions and main results

In this first section we will introduce three invariants which will be denoted by γ , τ and μ , and we will state our existence results. For this purpose, we need to introduce several notations, definitions and constructions.

We assume that $K : \mathbb{S}^n \rightarrow \mathbb{R}$ is a C^2 function having a finite set of $h+1$ nondegenerate critical points $y_0, \dots, y_\ell, y_{\ell+1}, \dots, y_h$, ordered such that

$$I^+ = \{y_0, y_1, \dots, y_\ell\} \text{ is such that } \forall y_i \in I^+, -\Delta_{g_0} K(y_i) > 0 \quad (2.1)$$

and

$$I^- = \{y_{\ell+1}, \dots, y_h\} \text{ is such that } \forall y_i \in I^-, -\Delta_{g_0} K(y_i) < 0. \quad (2.2)$$

For $0 \leq i \leq h$, we denote the Morse index of K at y_i by

$$\text{ind}(K, y_i) = n - k_i, \quad (2.3)$$

and we assume that y_0 is an absolute maximum of K .

For a chosen integer $k \geq 1$, we introduce the set

$$I_k^+ = \{y \in I^+ \text{ such that } \text{ind}(K, y) = n - k\}. \quad (2.4)$$

Let Z be a pseudogradient for K such that (generically) the intersections of the unstable and the stable manifolds of the various critical points of K are transverse. We assume that

$$W_u(y_j) \cap W_s(y_i) = \emptyset \quad \forall y_j \in I^- \quad \forall y_i \in I^+ \quad (\mathbf{H}_1)$$

where $W_s(y)$ (resp. $W_u(y)$) is the stable (resp. unstable) manifold of y with respect to Z . For a subset I of I_k^+ which will be specified in due time, we define

$$X = \bigcup_{y \in I} \overline{W_s(y)} \quad (2.5)$$

X is then a k -dimensional compact manifold without boundary. For an integer $p \geq 2$, we define

$$B_p(X) = \left\{ \sum_{i=1}^p \alpha_i \delta_{a_i} \text{ such that } a_i \in X, \alpha_i \in [0, 1], \sum_{i=1}^p \alpha_i = 1 \right\} \quad (2.6)$$

where δ_{a_i} is the Dirac measure at a_i . Now, for λ large enough, we define the map

$$f_\lambda : B_p(X) \rightarrow \Sigma^+ \\ \sum_{i=1}^p \alpha_i \delta_{a_i} \mapsto \frac{\sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda)}}{\left\| \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda)} \right\|}$$

where Σ^+ is the positive unit half sphere of our functional space $H^s(\mathbb{S}^n)$ and $\varphi_{(a_i, \lambda)}$ given by (3.1) is a solution of Eq. (3.2), (see Sect. 3). Note that $f_\lambda(B_p(X))$ is a manifold of dimension $p(k+1) - 1$, (its singularities arise in dimension $p(k+1) - 3$ and lower). We let in the sequel

$$N = f_\lambda(B_p(X)) \quad (2.7)$$

and

$$m = \dim N = p(k+1) - 1. \quad (2.8)$$

Throughout this paper, w designates a Morse–Smale pseudogradient for J (see Definition 3.10). With respect to such a pseudogradient, stable and unstable manifolds of the critical points at infinity of J are well defined, as well as their Morse indexes (see Sect. 3). Furthermore, one can define the unstable manifold of N :

Definition 2.1 The unstable manifold of N , denoted by $W_u(N)$, is the set of flow lines of w emanating from N , (i.e. starting from a point of N).

In Sect. 4 we show that:

Proposition 2.2 Assuming that $W_u(N) \cap N = N$, $W_u(N)$ is a $(m+1)$ -dimensional manifold, with boundary. More precisely, in terms of homology, the boundary of $W_u(N)$ is the m -singular chain:

$$\text{Proposition 2.3} \quad \partial W_u(N) = N + \sum_{\phi \in A_m^0 \cup A_m^\infty} (N \cdot \phi) W_u(\phi).$$

Here we denote

$$A_m^0 = \{\phi \in A^0, \text{ind}(J, \phi) = m\}, \quad (2.9)$$

$$A_m^\infty = \{\phi \in A^\infty, \text{ind}(J, \phi) = m\}, \quad (2.10)$$

where

$$A^0 = \{\phi, \text{critical point of } J \text{ in } \Sigma^+\}, \quad (2.11)$$

$$A^\infty = \{\phi, \text{critical point at infinity of } J \text{ in } \Sigma^+\}, \quad (2.12)$$

$\text{ind}(J, \phi)$ is the Morse index of J at ϕ , and $N \cdot \phi$ is the intersection number $N \cdot W_s(\phi)$ (see [28] for the whole definition). This intersection number is well defined because w is transverse to $N = f_\lambda(B_p(X))$ outside $f_\lambda(B_{p-1}(X))$, which cannot dominate critical points of index m (for dimensional reasons). Furthermore, $\overline{W_s(\phi)}$ adds to $W_s(\phi)$ only stable manifolds of critical points of index larger than or equal to $m+1$, and, since N is of dimension m , these manifolds can be assumed to avoid it. Notice that here, ∂ denotes the boundary operator of the Floer–Milnor Homology associated to J , as it was defined in [32] (see also [20]). We recall that in this homology, singular chains are generated by the unstable manifolds of the critical points of J . Observe that since $\partial \circ \partial = 0$ and since N is a cycle in dimension m , that is $\partial N = 0$, it follows that:

Lemma 2.4

$$\partial \left(\sum_{\phi \in A_m^0 \cup A_m^\infty} N \cdot \phi W_u(\phi) \right) = 0.$$

From this lemma we derive the following lemma which proof is in Sect. 4:

Lemma 2.5 There exists a $(m+1)$ -chain σ , such that $W_u(N) - \sigma$ is a $(m+1)$ -cycle of Σ^+ , modulo N , that is to say

$$\partial(W_u(N) - \sigma) = N.$$

We continue our constructions. Let $\nu \rightarrow X$ be a tubular neighborhood of X in $\mathbb{S}^n$. We denote $\nu(z)$, for $z \in X$, the fiber at z of this tubular neighborhood. The fibers $\nu(z)$ has been built such that $K(z) = \max_{\nu(z)} K$ and z is the unique maximum of K on the fiber $\nu(z)$.

We now introduce for $\epsilon > 0$ and $z_1, \dots, z_p \in X$ such that $-\Delta K(z_i) > 0$ ($1 \leq i \leq p$), the set

$$\Gamma_\epsilon(z_1, \dots, z_p) = \left\{ \sum_{i=1}^p \frac{1}{K(z_i + h_i)^{\frac{n-2s}{4s}}} \varphi_{(z_i + h_i, \lambda_i)} + v \mid \text{s.t. } h_i \in \nu(z_i), \right. \\ \left. \lambda_i \geq \frac{1}{\epsilon}, \sum_{i=1}^p |h_i|^2 \leq \epsilon, \|v\| \leq \epsilon, v \text{ stisfying } (V_0) \right\},$$

where (V_0) designates the orthogonality relations (with respect to the inner product given by (3.7)):

$$\langle v, \varphi_{(x_i, \lambda_i)} \rangle = \left\langle v, \frac{\partial \varphi_{(x_i, \lambda_i)}}{\partial x_i} \right\rangle = \left\langle v, \frac{\partial \varphi_{(x_i, \lambda_i)}}{\partial \lambda_i} \right\rangle = 0, \quad \forall (x_i, \lambda_i) \in \bigcup_{i=1}^p (z_i + h_i, \lambda_i). \quad (V_0)$$

For $\delta > 0$, let us define the set

$$D = \Gamma_\epsilon(z_1, \dots, z_p) \cap J^{-1}\{c_\infty + \delta\}, \quad (2.13)$$

where $c_\infty = c_\infty(z_1, \dots, z_p) = S^{\frac{2s}{n}} \left(\sum_{i=1}^p \frac{1}{K(z_i)^{\frac{n-2s}{2s}}} \right)^{\frac{2s}{n}}$, and S the Sobolev constant (see Sect. 3). We then have:

Lemma 2.6 Since $-\Delta K(z_i) > 0$ ($1 \leq i \leq p$), then for ϵ sufficiently small and $\delta < \delta(\epsilon)$ small enough, D is a closed Fredholm (noncompact) manifold without boundary, of codimension $p(k+1) = m+1$.

In Sect. 4 we sketch the proof of this lemma.

2.1 The intersection numbers γ and τ

In [2] Bahri introduced the intersection number γ (see also [32]) which can be defined by the formula:

$$\gamma(w) = (W_u(N) - \sigma) \cdot D. \quad (2.14)$$

In Sect. 4 below, we will sketch the proof of the following Proposition:

Proposition 2.7 $\gamma(w)$ is well defined, independently of the variational flow w belonging to $\mathcal{P}$ (see Definition 3.12), it depends only on the submanifolds N and D . We denote by $\gamma = \gamma(N, D)$ the value of this invariant.

Observe that $\sum^+$ is not a manifold, but there exists a neighborhood $\mathcal{V}$ of $\sum^+$, which is open in $H^s(\mathbb{S}^n)$, and such that $\sum^+$ is a retract of $\mathcal{V}$. We are thinking of intersection theory in that sense. On the other hand, another invariant was introduced by Bahri in [2]. This is the quantity τ , which is the intersection number

$$\tau = W_u(N) \cdot D. \quad (2.15)$$

Definition 2.8 A critical point at infinity ϕ is said to be dominated by N if $W_u(N) \cap W_s(\phi) \neq \emptyset$.

We introduce the following assumption

(H₂) There are no critical points at infinity of index m , which are dominated by N .

Our first existence result is based on a comparison between γ and τ :

Theorem 2.9 Under the assumptions **(H₁)** and **(H₂)**, if $\tau \neq \gamma$, then, problem (1.1) has a solution of Morse index m , and thus, K is the fractional Q -curvature of some metric g , conformally equivalent to g_0 .

Observe that the value of the invariant γ can be computed. In Sect. 4 we prove that:

Proposition 2.10 We have $\gamma = \gamma(N, D) = 0$.

Thus, Theorem 2.9 becomes

Corollary 2.11 Assuming **(H₁)** and **(H₂)**, if $\tau \neq 0$, then (1.1) has a solution of Morse index m .

Let

$$m_\infty = \ell + \sum_{j=1}^{j=\ell} k_j.$$

Then we have:

Corollary 2.12 Under assumption **(H₁)**, suppose that $m_\infty < m$. If $\tau \neq 0$, then (1.1) has a solution of Morse index m .

A simple application of the preceding corollary is given by the following result:

Corollary 2.13 Assuming **(H₁)**, let $\ell = 1$ and $p = 2$. If $\tau \neq 0$, then (1.1) has a solution of Morse index $2k_1 + 1$, where $k_1 = n - \text{ind}(K, y_1)$.

2.2 A min–max characterization

From another side, one can provide a min–max characterization of some critical points of J . Indeed, let

$$\Omega(N) = \{\text{cycles contained in } \overline{W_u(N)} \text{ homologous to } N\},$$

and define

$$d = \min_{C \in \Omega(N)} \max_{u \in C} J(u).$$

Then one has

Theorem 2.14 Under the assumptions **(H₁)** and **(H₂)**, if $\tau \neq 0$, then d is a critical value of J , and there is a related critical point of Morse index m .

2.3 The intersection number μ

In contrast to the precedent Theorems, we have the following results based on another intersection number, also introduced in [2] by Bahri, and which is denoted by μ . The advantage with this local invariant is that one can dispense with the hypothesis $(\mathbf{H}_2)$. We will consider two cases.

2.3.1 First case

In this case, using (2.5) with $I = \{y_{i_0}\}$, we take

$$X = \overline{W_s(y_{i_0})} \quad (2.16)$$

where y_{i_0} satisfies

$$K(y_{i_0}) = \max_{y \in I_k^+} K(y). \quad (2.17)$$

X is then a k -dimensional sphere.

Let us define,

$$C_{y_0}(X) = \{\alpha \delta_{y_0} + (1 - \alpha) \delta_x / \alpha \in [0, 1], x \in X\}.$$

$C_{y_0}(X)$ is a manifold in dimension $k + 1$. We define, for λ large enough, the intersection number

$$\mu(y_0) = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_{i_0})_\infty \quad (2.18)$$

where $W_s(y_0, y_{i_0})_\infty$ is the stable manifold of the critical point at infinity $(y_0, y_{i_0})_\infty$ for the pseudogradient w . This intersection number is well defined by the same arguments as above. We then have

Theorem 2.15 *Under assumption $(\mathbf{H}_1)$, if $\mu(y_0) = 0$, then (1.1) has a solution of index k or $k + 1$.*

2.3.2 Second case

In this case, we consider a more general situation. Using (2.5) with $I = I_k^+ = \{y_{i_j}, 1 \leq j \leq d\}$, we take

$$X = \bigcup_{j=1}^d \overline{W_s(y_{i_j})}$$

where we denoted by d the cardinality of I_k^+ . X is a k -dimensional compact manifold without boundary.

For each $j = 1, \dots, d$, we define, for λ large,

$$\mu_j(y_0) = f_\lambda(C_{y_0}(X)) W_s(y_0, y_{i_j})_\infty. \quad (2.19)$$

where $W_s(y_0, y_{i_j})_\infty$ is the stable manifold of the critical point at infinity $(y_0, y_{i_j})_\infty$ for the pseudogradient w . By the same arguments as above, this intersection number is well defined, and we have

Theorem 2.16 *Under assumption $(\mathbf{H}_1)$, if $\mu_j(y_0) = 0$ for each $j = 1, \dots, d$, then (1.1) has a solution of index k or $k + 1$.*

3 Variational framework

Denote by $H^s(\mathbb{S}^n)$ the completion of $C^\infty(\mathbb{S}^n)$ with respect of the norm

$$\|u\|^2 = \int_{\mathbb{S}^n} P_s u \cdot u \, dv_{g_0}$$

where dv_{g_0} is the volume element of g_0 . Let in the sequel

$$\Sigma = \{u \in H^s(\mathbb{S}^n) / \|u\| = 1\}$$

and

$$\Sigma^+ = \{u \in \Sigma / u \geq 0\}.$$

The Euler functional associated to problem (1.1), denoted by J , is then defined on $H^s(\mathbb{S}^n)$ by

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^n} K |u|^{\frac{2n}{n-2s}} dv_{g_0}\right)^{\frac{n-2s}{n}}}.$$

One knows that if v is a critical point of J in Σ^+ , then the function $u = J(v)^{\frac{n-2s}{2}} v$ is a solution for (1.1), and therefore the new metric $g = u^{\frac{4}{n-2s}} g_0$ has its fractional Q-curvature $Q_s^g = K$.

Problem (1.1) therefore amounts to finding the critical points of the functional J . However, this problem is known to be delicate because it is part of the class of noncompact problems (e.g. Yang–Mill’s equations, nonlinear wave equations under nonrationally depending limit conditions, periodic orbits problem for contact flows on contact manifolds...). In addition, there are also topological obstructions of Kazden–Warner condition type to solve (1.1) (see e.g. [25]).

3.1 Defect of compactness

The functional J does not satisfy the Palais–Smale condition [denoted (P.S.) for short], and there exist noncompact orbits for the decreasing gradient flow of J , that is to say there exist critical points at infinity (asymptotes) for the functional J .

Let us introduce for $a \in \mathbb{S}^n$ and $\lambda > 0$ the function

$$\varphi_{(a,\lambda)}(x) = c_n \left(\frac{2\lambda}{2 + (\lambda^2 - 1)(1 - \cos d(a, x))} \right)^{\frac{n-2s}{2}} \quad (3.1)$$

where $d = d_{g_0}$ is the geodesic distance on $\mathbb{S}^n$ and $c_n > 0$ is chosen such that the functions $\varphi_{(a,\lambda)}$ are solutions of

$$\begin{cases} P_s u = u^{\frac{n+2s}{n-2s}} & \text{on } \mathbb{S}^n, \\ u > 0 & \text{on } \mathbb{S}^n. \end{cases} \quad (3.2)$$

Notice that, under the stereographic projection, Eq. (3.2) is transformed to equation

$$\begin{cases} (-\Delta)^s u = u^{\frac{n+2s}{n-2s}} & \text{on } \mathbb{R}^n, \\ u > 0 & \text{on } \mathbb{R}^n, \end{cases}$$

whose solutions are functions given for $a \in \mathbb{R}^n$ and $\lambda > 0$ by

$$\omega_{(a,\lambda)}(x) = c_n \left(\frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{\frac{n-2s}{2}}.$$

Using Lemma A1 of [17], we introduce the sharp constant S for the continuous—but noncompact—Sobolev embedding $H^s(\mathbb{S}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{S}^n)$, which is reached by the functions $\varphi_{(a,\lambda)}$, namely:

$$S = \|\varphi_{(a,\lambda)}\|^2 = \int_{\mathbb{S}^n} P_s \varphi_{(a,\lambda)} \varphi_{(a,\lambda)} dv_{g_0}. \quad (3.3)$$

We introduce the set $\mathcal{E} = (0, +\infty)^p \times (\mathbb{S}^n)^p \times (0, +\infty)^p$ which is the space of the variables $(\alpha, a, \lambda) = (\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$ and we consider the following subset of $\mathcal{E}$:

$$B_\epsilon = \left\{ (\alpha, a, \lambda) \in \mathcal{E}, \text{ s.t. }, \varepsilon_{ij} < \epsilon, \lambda_i > \frac{1}{\epsilon} \right\},$$

where $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d(a_i, a_j)^2 \right)^{-\frac{n-2s}{2}}$.

Now, we define the set of potential critical points at infinity of J in Σ^+ to be:

$$V(p, \epsilon) = \left\{ u \in \Sigma^+ / \exists (\alpha, a, \lambda) \in B_\epsilon \text{ s.t. } \left\| u - \sum_{i=1}^p \alpha_i \varphi_i \right\| < \epsilon, \left| J(u)^{\frac{n}{n-2s}} \alpha_i^{\frac{4s}{n-2s}} K(a_i) - 1 \right| < \epsilon \right\}.$$

As we said above the functional J fails to satisfy (PS) condition (this failure was analyzed in several earlier works see e.g. [8, 27, 31]), which leads to the existence of noncompact sequences along which the functional is bounded and its gradient goes to zero. The characterization of the sequences failing the (P.S) condition for J in Σ^+ was achieved in [17] (lemma 2.2), such sequences are described by the following result:

Proposition 3.1 Assume that (1.1) has no solution. Let (v_k) be a sequence in Σ^+ such that $J'(v_k) \rightarrow 0$ and $J(v_k)$ is bounded. Then, there exists an integer $p \in \mathbb{N}^*$, a sequence (ϵ_k) , $\epsilon_k > 0$, $\epsilon_k \rightarrow 0$, such that, up to an extracted subsequence, $v_k \in V(p, \epsilon_k)$.

Notice that any function $u \in V(p, \epsilon)$ possesses an optimal representation. The following proposition follows from the corresponding results in [4, 5, 17]:

Proposition 3.2 Let ϵ small enough. For any u in $V(p, \epsilon)$, the minimization problem

$$\min_{(\alpha, a, \lambda) \in B_\epsilon} \left\| u - \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} \right\|. \quad (3.4)$$

has a unique solution (α, a, λ) in B_ϵ up to permutation.

Thus, for $\epsilon > 0$ sufficiently small, any function $u \in V(p, \epsilon)$ can be uniquely written as: $u = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v$, where $(\alpha, a, \lambda) \in B_\epsilon$ and $v \in H_\epsilon(a, \lambda)$, where

$$H_\epsilon(a, \lambda) = \{v \text{ satisfying (3.6), } \|v\| < \epsilon\} \quad (3.5)$$

where (3.6) designates the following orthogonality relations

$$\langle v, \varphi_{(a_i, \lambda_i)} \rangle = \left\langle v, \frac{\partial \varphi_{(a_i, \lambda_i)}}{\partial a_i} \right\rangle = \left\langle v, \frac{\partial \varphi_{(a_i, \lambda_i)}}{\partial \lambda_i} \right\rangle = 0, \quad \forall i \in \{1, \dots, p\} \quad (3.6)$$

with respect to the inner product associated to the norm $\|\cdot\|$:

$$\langle u, v \rangle = \int_{\mathbb{S}^n} P_s u \cdot v \, dv_{g_0} \quad (3.7)$$

Otherwise, from Lemma 3.5 of [17], one has

Lemma 3.3 There exists a C^1 -map $\bar{v} : B_\epsilon \rightarrow H_\epsilon(a, \lambda)$, $(\alpha, a, \lambda) \mapsto \bar{v}(\alpha, a, \lambda)$, such that for any $(\alpha, a, \lambda) \in B_\epsilon$, $\bar{v} = \bar{v}(\alpha, a, \lambda)$ is the unique minimum of the functional J on $H_\epsilon(a, \lambda)$:

$$J \left(\sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + \bar{v} \right) = \min_{v \in H_\epsilon(a, \lambda)} J \left(\sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v \right) \quad (3.8)$$

and we have the estimate

$$\|\bar{v}\| = O \left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2s}{2n}} (\log \varepsilon_{ij}^{-1})^{\frac{n-2s}{n}} \right). \quad (3.9)$$

For such a $\bar{v}$ satisfying (3.8) we set in the sequel

$$\bar{u} = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + \bar{v}. \quad (3.10)$$

We want to understand the behavior of the functional J near infinity, that is when a flow line of $-\nabla J$ the opposite of the gradient of J , penetrates into a set $V(p, \epsilon)$. We provide a local expansion of J in this set (see Lemma 3.6 of [17]):

Proposition 3.4 For $\epsilon > 0$ small enough, let $u = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v$ in $V(p, \epsilon)$, with $(\alpha, a, \lambda) \in B_\epsilon$ and $v \in H_\epsilon(a, \lambda)$. Then we have the following expansion of J :

$$J(u) = \gamma_0[1 + \gamma_1 + \gamma_2 + \gamma_3]$$

where

$$\gamma_0 = \frac{\sum_{i=1}^p \alpha_i^2 S}{\left(\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}} K(a_i) S \right)^{\frac{n-2s}{n}}}, \quad (3.11)$$

$$\gamma_1 = \frac{(n-2s)}{n} \frac{c_1}{\Gamma_1} \sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}} \frac{(-\Delta K(a_i))}{\lambda_i^2} \quad (3.12)$$

$$\gamma_2 = \sum_{i \neq j} c_0^{\frac{2n}{n-2s}} c_2 \omega_n \varepsilon_{ij} \left(\frac{\alpha_i \alpha_j}{\Gamma_2} - \frac{2\alpha_i^{\frac{n+2s}{n-2s}} K(a_i)}{\Gamma_1} \right) \quad (3.13)$$

$$\gamma_3 = Q(v - \bar{v}) + o \left(\|\bar{v}\|^2 + o \left(\sum_{i \neq j} \varepsilon_{ij} \right) \right) \quad (3.14)$$

$$\Gamma_1 = \sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}} K(a_i) S; \quad \Gamma_2 = \sum_{i=1}^p \alpha_i^2 S \quad (3.15)$$

where Q is a quadratic form, $\bar{v}$ is the minimizer given by (3.8), c_0, c_1 and c_2 are nonnegative constants.

By lemma 3.4 in [17] Q is definite positive on $H_\epsilon(a, \lambda)$, furthermore it is bounded.

Lemma 3.5 Assume the ε_{ij} are small enough. Then there exists $\alpha_0 = \alpha_0(p) > 0$ such that for all v in $H_\epsilon(a, \lambda) : Q(v) \geq \alpha_0 \|v\|^2$.

For a proof, one can operate as in the proof of Proposition 3.1 in [3]. □

3.2 Critical points at infinity

Definition 3.6 A critical point at infinity of J in Σ^+ is a limit of a flow line $u(t)$ of the equation

$$\begin{cases} \frac{\partial u(t)}{\partial t} = -J(u(t)) \\ u(0) = u_0 \end{cases}$$

such that $u(t)$ remains in a set $V(p, \epsilon(t))$ for all $t \geq t_0$, where $\epsilon(t) \rightarrow 0$ when $t \rightarrow +\infty$, and where u_0 is an initial condition.

By Proposition 3.2, $u(t)$ can be written as $u(t) = \sum_{i=1}^p \alpha_i(t) \varphi_{(a_i(t), \lambda_i(t))} + v(t)$. If we let $\alpha_i = \lim_{t \rightarrow \infty} \alpha_i(t)$ and $a_i = \lim_{t \rightarrow \infty} a_i(t)$, then such a critical point at infinity is denoted by $\sum_{i=1}^p \alpha_i \varphi_{(a_i, +\infty)}$ or $(a_1, \dots, a_p)_\infty$. Let us introduce

$$F_\infty^+ = \bigcup_{1 \leq p \leq \ell+1} F_p^+, \quad \text{where } F_p^+ = \{(y_{i_1}, \dots, y_{i_p}) \in (I^+)^p / y_{i_j} \neq y_{i_k} \text{ if } j \neq k\}.$$

Recall that A^∞ is the set of critical points at infinity of J in Σ^+ . It was shown in section 5 of [17], (or in [33]), that A^∞ is in bijective correspondence with F_∞^+ .

Proposition 3.7 $\phi \in A^\infty$ iff there exists $p \in \{1, \dots, \ell + 1\}$ and $(y_{i_1}, \dots, y_{i_p}) \in F_p^+$, such that $\phi = (y_{i_1}, \dots, y_{i_p})_\infty$.

It was also proven [17, 33] that:

Proposition 3.8 *The Morse index of J at the critical point at infinity $(y_{i_1}, \dots, y_{i_p})_\infty$ is given by the formula:*

$$\text{ind}(J, (y_{i_1}, \dots, y_{i_p})_\infty) = p - 1 + \sum_{j=1}^p k_{i_j}$$

where $k_{i_j} = n - \text{ind}(K, y_{i_j})$ and $\text{ind}(K, y_{i_j})$ is the Morse index of K at y_{i_j} .

The next lemma provides a vector field said *pseudogradient of the Morse Lemma at infinity*, which has a very nice behavior near critical points at infinity (see Lemma 4.3 in [17]):

Lemma 3.9 *There exists on $V(p, \epsilon)$ a vector field w for which there exists constants $c, c', C > 0$ independent of $\sum_i \alpha_i \varphi_i$ in $V(p, \epsilon)$ such that $(\bar{v}$ resp. $\bar{u}$ is given by (3.8) resp. (3.10)):*

1. $-\nabla J(\bar{u})(w + \frac{\partial \bar{v}}{\partial W}) \geq c \left(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij} \right)$
2. $-\nabla J(\sum_i \alpha_i \varphi_i) w \geq c \left(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \epsilon_{ij} \right)$
3. $\|w\| \leq C$.
4. $|\text{d}\lambda_i(w)| \leq c' \lambda_i, \forall i \in \{1, \dots, p\}$.
5. *The only region where the λ_i are not bounded along the decreasing flow lines of w , is where $(a_1, \dots, a_p)$ is close to some $(y_{i_1}, \dots, y_{i_p}) \in F_p^+$, and the λ_i are comparable.*

In particular such a pseudogradient of the Morse Lemma at infinity satisfies (PS) condition outside the critical points at infinity.

Given that within our framework critical points at infinity may come into play, we need such appropriate type of pseudogradients. Following [2], we introduce the notion of a *Morse–Smale pseudogradient for J* and the more general notion of *flow belonging to $\mathcal{P}$* :

Definition 3.10 (*Morse–Smale pseudogradient for a functional*) We say that a vector field or a flow V on $\sum^+$ is a *Morse–Smale pseudogradient for J* , if V has the expected critical points at infinity of J and it behaves around them as the pseudogradient w of the Morse Lemma at infinity (given by Lemma 3.9 above), and if its flow lines are transverse to N , where N is defined by (2.7).

Of course such Morse–Smale pseudogradients for J exist as soon as the pseudogradient w of the Morse Lemma at infinity exists. For example we could assume that the Morse–Smale pseudogradients for J are equal to w in an arbitrary neighborhood of the critical points at infinity.

Definition 3.11 (*Admissible functional on $\sum^+$*) Let I be a $\mathcal{C}^2$ functional on $\sum^+$.

- When I has critical points at infinity, I is said *admissible* if it admits a pseudogradient of the Morse Lemma at infinity, hence if there exists a Morse–Smale pseudogradient for I .
 - When I does not have any critical point at infinity, it is said *admissible* if there exists a pseudogradient for I (in the classical sense) which is transverse to N .
- Such a pseudogradient transverse to N is then called a Morse–Smale pseudogradient for I .

Definition 3.12 (*Flows belonging to $\mathcal{P}$*) We say that a flow V on $\sum^+$ belongs to $\mathcal{P}$ if it is a Morse–Smale pseudogradient for an admissible functional on $\sum^+$.

Remark 3.13 Note that if the functional I is admissible we can carry out the same program that we have realized for our functional J .

Remark 3.14 If the functional I does not have any critical point at infinity and if N is a set of regular points of I , then I is admissible since in this case any pseudogradient of I , (in the classical sense), is transverse to N .

3.3 Stable and unstable manifolds of a critical point at infinity

Denote by ϕ the critical point at infinity $(y_{i_1}, \dots, y_{i_p})_\infty$. We can define its stable manifold $W_s(\phi)$ quite simply as the set of all points of $\sum^+$ which are attracted by the asymptote ϕ through the decreasing flow lines of a Morse–Smale pseudogradient w for J .

Before talking about the unstable manifold, which is a more subtle notion than the stable manifold, note that near the critical point at infinity ϕ , for a function $u = \sum_{i=1}^p \alpha_i \varphi_{(a_i, \lambda_i)} + v$ in $V(p, \epsilon)$, the expansion of the functional J after further computations and suitable change of variables becomes:

$$J(u) = \gamma_0(\alpha, a) \left[1 - \sum_{j=1}^p \frac{\Delta K(y_{i_j})}{\lambda_j^2} \right],$$

and then we see that the flow w splits the variable λ from the other variables (α, a) .

The unstable manifold is more delicate to define because it exists only as the limit of $W_u(u_\lambda)$ the unstable manifold of the critical point u_λ of the reduced problem in a section relative to λ . Since there is a Morse lemma in this section and since the flow w splits the variable λ from the other variables near ϕ , one can therefore think of $W_u(\phi)$ to be represented by some $W_u(u_\lambda)$ for λ very large. Ultimately, when $\lambda_j = +\infty$ for $j = 1, \dots, p$, the unstable manifold of ϕ can be defined as:

$$W_u(\phi) = W_u(\bar{\alpha}, y) \times \{+\infty\}$$

where $W_u(\bar{\alpha}, y)$ is the unstable manifold of the critical point $(\bar{\alpha}, y)$ of the functional

$$\gamma_0 = \gamma_0(\alpha, a) = \frac{\sum_{i=1}^p \alpha_i^2 S}{\left(\sum_{i=1}^p \alpha_i^{\frac{2n}{n-2s}} K(a_i) S \right)^{\frac{n-2s}{n}}}$$

which critical point is given by $y = (y_{i_1}, \dots, y_{i_p})$ and $\bar{\alpha} = \left(\frac{1}{K(a_1)^{\frac{n-2s}{4s}}}, \dots, \frac{1}{K(a_p)^{\frac{n-2s}{4s}}} \right)$.

(For anymore details we send back the curious reader at [17] or [33]).

4 Proofs of Propositions and Lemmas

4.1 Proof of Proposition 2.2

We begin by noting that the condition $W_u(N) \cap N = N$ can be satisfied. Just assume that $2m + 1 < n$, then we can assume generically on $-\nabla J$ (the opposite of the gradient of J) or any Morse–Smale pseudogradient w that $W_u(N) \cap N = N$. Denoting by η_t the one parameter group generated by the flow w , one has

$$W_u(N) = \bigcup_{t \geq 0} \bigcup_{x \in N} \eta_t(x) = \bigcup_{t \geq 0} \eta_t(N) = [0, +\infty[\times N.$$

One can easily see that the boundary of $W_u(N)$ is $\partial W_u(N) = \{0\} \times N \cup \{+\infty\} \times N$.

4.2 Proof of Proposition 2.3

From the algebraic topological point of view $W_u(N)$ is a chain of dimension $m + 1$ and its boundary $\partial W_u(N)$ is a chain of dimension m . Observe that the upper boundary $\{0\} \times N$ is represented by the chain N . As to the lower boundary $\{+\infty\} \times N$, we know by using Proposition 7.24 and Theorem 8.2 of [6], that the corresponding chain is made of unstable manifolds of critical points or critical points at infinity of J which are dominated by N and whose Morse indexes are equal to m , that is the chain $\sum_{\phi \in A_m^0 \cup A_m^\infty} (N, \phi) W_u(\phi)$. We recall that N, ϕ is the intersection number $N, W_s(\phi)$. Hence we have:

$$\partial W_u(N) = N + \sum_{\phi \in A_m^0 \cup A_m^\infty} (N, \phi) W_u(\phi).$$

4.3 Proof of Lemma 2.5

$\sum^+$ is contractible, then in particular the homology group $H_m(\sum^+) = 0$, and so there exists a $(m+1)$ -chain σ , such that

$$\partial\sigma = \sum_{\phi \in A_m^0 \cup A_m^\infty} (N \cdot \phi) W_u(\phi)$$

and thus, $W_u(N) - \sigma$ is a $(m+1)$ -cycle of $\sum^+$, modulo N , i.e:

$$\partial(W_u(N) - \sigma) = N.$$

Observe that the boundary $\partial\sigma$ splits as it follows: $\partial\sigma = \varsigma_0 + \varsigma_\infty$, where

$$\varsigma_0 = \sum_{\phi \in A_m^0} (N \cdot \phi) W_u(\phi) \quad \text{and} \quad \varsigma_\infty = \sum_{\phi \in A_m^\infty} (N \cdot \phi) W_u(\phi). \quad (4.1)$$

ς_0 (resp. ς_∞) is the contribution of the critical points (resp. of the critical points at infinity) of J of Morse index m , into the boundary $\partial\sigma$.

4.4 Proof of Lemma 2.6

The proof of Lemma 2.6 is very technical, we will recall its key steps. For anymore details we send back to [2] or [7] (in particular see the Remark p. 394 in [7]). Let

$$u = \sum_{i=1}^p \frac{1}{K(z_i + h_i)^{\frac{n-2s}{4s}}} \varphi_{(z_i+h_i, \lambda_i)} + v \in \Gamma_\epsilon(z_1, \dots, z_p).$$

Considering the expansion of $J(u)$ given by Proposition 3.4 and observing that α_i is equal to its optimal value (see Section 5 in [17])

$$\bar{\alpha}_i = \frac{1}{K(z_i + h_i)^{\frac{n-2s}{4s}}},$$

then, after further calculation using in particular the estimates given by (3.9) and Lemma 3.5 we obtain the following expansion of the functional:

$$\begin{aligned} J(u) = & c_\infty(z_1 + h_1, \dots, z_p + h_p) \left[1 - \tilde{c} \sum_{i=1}^p \frac{\Delta K(z_i + h_i)}{K(z_i + h_i)^{\frac{n}{2s}} \lambda_i^2} - \sum_{i \neq j} c_{ij} \varepsilon_{ij} (1 + o(1)) \right. \\ & \left. + o\left(\sum_{i=1}^p \frac{|\nabla K(z_i + h_i)|^2}{\lambda_i^2} + \frac{1}{\lambda_i^4} \right) + Q(v - \bar{v}) \right] \end{aligned}$$

where $\tilde{c}$, c_{ij} are positive constants,

$$c_\infty(z_1 + h_1, \dots, z_p + h_p) = S^{\frac{2s}{n}} \left(\sum_{i=1}^p \frac{1}{K(z_i + h_i)^{\frac{n-2s}{2s}}} \right)^{\frac{2s}{n}},$$

Q is a definite positive quadratic form on $v - \bar{v}$, $\bar{v}$ is a uniquely determined function satisfying (V_0) and

$$\varepsilon_{ij} = \frac{1}{\left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |(z_i + h_i) - (z_j + h_j)|^2 \right)^{\frac{n-2s}{2}}}.$$

We now use the fact that $K(z_i)$ is the maximum of the function K on the fiber $\nu(z_i)$ to observe that $c_\infty(z_1 + h_1, \dots, z_p + h_p) \geq c_\infty$, where

$$c_\infty = c_\infty(z_1, \dots, z_p) = S^{\frac{2s}{n}} \left(\sum_{i=1}^p \frac{1}{K(z_i)^{\frac{n-2s}{2s}}} \right)^{\frac{2s}{n}},$$

and it is close to it as we will by making ϵ small enough. Then using the Morse Lemma for the function K we derive that

$$J(u) \geq c_\infty \left[1 + \sum_{i=1}^p |h_i|^2 - \tilde{c} \sum_{i=1}^p \frac{\Delta K(z_i + h_i)}{K(z_i + h_i)^{\frac{n}{2s}} \lambda_i^2} + \alpha_0 \|v - \bar{v}\|^2 \right]$$

where α_0 is a positive constant. Observe that the boundary of $\Gamma_\epsilon(z_1, \dots, z_p)$ is defined by:

$$\left\{ \|v\| = \epsilon, \text{ or } \lambda_i = \frac{1}{\epsilon}, \text{ or } \sum_{i=1}^p |h_i|^2 = \epsilon \right\},$$

therefore we can choose $\delta = \delta(\epsilon)$ small enough such that the boundary of $\Gamma_\epsilon(z_1, \dots, z_p)$ does not intersect the level $J^{-1}\{c_\infty + \delta\}$, and then

$$D = \Gamma_\epsilon(z_1, \dots, z_p) \cap J^{-1}\{c_\infty + \delta\},$$

is a Fredholm manifold of codimension $p(k+1) = m+1$.

4.5 Proof of Proposition 2.7

The proof of the fact that the invariant γ is independent of the variational flow was first performed by Bahri in [2] (lemma 7), in the case where $\ell = 1$ and $p = 2$. The proof was extended to $\ell \geq 1$ and $p \geq 2$ in [7] (proposition 3.1). But this proof could induce the misconception that this outcome is only valid for scalar curvature type flows, whereas it depends on a more general variational principle, which was mentioned in [32]. Let us sketch the main idea of this proof.

Taking two flows V_1 and V_2 in the set $\mathcal{P}$, (see Definition 3.12), we let for $i = 1, 2$, $\gamma(V_i) = c(V_i).D$ with $c(V_i) = (W_u^i(N) - \sigma^i)$ where $W_u^i(N)$ given by Definition 2.1 and σ^i given by Lemma 2.5 are defined with respect of the flow V_i . $c(V_i)$ is then a chain of dimension $m+1$ with boundary $\partial c(V_i) = \partial(W_u^i(N) - \sigma^i) = N$. Denote by c the $(m+1)$ -chain $c = c(V_1) - c(V_2)$. Since $\partial c = N - N = 0$, c is a $(m+1)$ -cycle of Σ^+ . Note that Σ^+ is contractible which implies in particular that $H_{m+1}(\Sigma^+) = 0$ and thus there exists a chain $C \in H_{m+2}(\Sigma^+)$ such that $\partial C = c$. Observe now that since D is of codimension $(m+1)$ and without boundary, we may assume up to perturbation, that generically $C \cap D$ is a one dimensional chain and thus with zero boundary (modulo 2). We derive that

$$0 = \partial(C \cap D) = (\partial C).D = c.D = (c(V_1) - c(V_2)).D$$

hence $c(V_1).D = c(V_2).D$ that is $\gamma(V_1) = \gamma(V_2)$.

4.6 Proof of Proposition 2.10

We know by Proposition 2.7 that the invariant γ does not depend on the variational flow of our functional J , but it relies on a more general variational principle. To compute the value of γ , one can use the variational flow on Σ^+ of the functional denoted J' (here the prime ' is not the derivative):

$$\left\{ \begin{array}{l} J'(u) = \frac{\int_{\mathbb{S}^n} P_s u u + u^2}{\left(\int_{\mathbb{S}^n} u^{p+1} \right)^{\frac{2}{p+1}}} \\ 1 < p < \frac{n+2s}{n-2s}. \end{array} \right. \quad (4.2)$$

Problem (4.2) is a compact variational problem, and J' satisfies (P.S) condition. We denote by $W'_u(N)$ the unstable manifold of N with respect of a pseudogradient w' for J' , (for example $w' = -\nabla J'$ is here an appropriate pseudogradient for J'). We claim that:

Lemma 4.1 (i) *There are no critical points of index m for J' in Σ^+ .*
(ii) *J' is an admissible functional in the sense of Definition 3.11.*
(iii) $W'_u(N).D = 0$.

Proof of (i). The Euler–Lagrange equation of the functional J' is

$$\begin{cases} P_s u + u = u^p, & \text{on } \mathbb{S}^n; \\ u > 0, & \text{on } \mathbb{S}^n \end{cases} \quad (4.3)$$

which is transformed under the stereographic projection to the equation

$$\begin{cases} (-\Delta)^s u + u = u^p, & \text{on } \mathbb{R}^n; \\ u > 0, & \text{on } \mathbb{R}^n. \end{cases} \quad (4.4)$$

There is a unique solution to Eq. (4.4) on $\mathbb{R}^n$, which is positive and spherically symmetric (see e.g. [18]), and which by the stereographic projection, gives rise to a unique minimum $\hat{u}$ of the functional J' in Σ^+ . The Morse index of J' at $\hat{u}$ is then zero, and claim (i) is proved.

Proof of (ii). Note that claim (i) implies that N is a set of regular points of the functional J' , and thus J' is an admissible functional (see Remark 3.14).

Proof of (iii). Observe that since $\overline{W'_u(N)}$ is compact, using Proposition 7.24 of [6] we have that:

$$\overline{W'_u(N)} - W'_u(N) = \{\hat{u}\}.$$

Then, taking ϵ and δ small enough in the construction of D , we can make $W'_u(N)$ avoiding D , that is $W'_u(N).D = 0$. Hence claim (iii), and Lemma 4.1 is proved.

Denote by ς'_0 and ς'_∞ the chains defined as in (4.1), but with respect to the pseudogradient w' (see Remark 3.13). Since there are no critical points of J' of index m and since there are no critical points at infinity of any index (this problem being compact), we derive that $\varsigma'_0 = \varsigma'_\infty = 0$. Considering definition (2.14) and Proposition 2.7, we can compute the value of the invariant γ by using the variational flow of J' , and thus we have:

$$\gamma(w') = (W'_u(N) - \sigma').D$$

where σ' is the $(m+1)$ -chain such that $\partial\sigma' = \varsigma'_0 + \varsigma'_\infty$. Since $\partial\sigma' = 0$, the chain σ' can be taken to be zero so one gets

$$\gamma(w') = W'_u(N).D = 0.$$

We can now use Proposition 2.7, with $V_1 = w$ and $V_2 = w'$, to conclude that

$$\gamma(w) = \gamma(w')$$

since the pseudogradients w and w' clearly belong to $\mathcal{P}$. Hence $\gamma = \gamma(N, D) = 0$ which completes the proof of Proposition 2.10.

5 Proofs of Theorems and Corollaries

5.1 Proof of Theorem 2.9

Proof Notice that assumption (H_2) is written $\forall \phi \in A_m^\infty$, $W_u(N) \cap W_s(\phi) = \emptyset$. Using (H_2) , we derive that the intersection number $N.\phi = N.W_s(\phi) = 0$, and then

$$\varsigma_\infty = \sum_{\phi \in A_m^\infty} N.\phi W_u(\phi) = 0.$$

On the other hand, assuming that there is no solution for (1.1) we have

$$\varsigma_0 = \sum_{\phi \in A_m^0} N.\phi W_u(\phi) = 0$$

and then $\partial\sigma = 0$. Thus, σ can be taken to be zero, and so $\partial W_u(N) = N$. Using (2.14) and (2.15) we obtain $\gamma = \tau$. A contradiction! Thus, $\varsigma_0 \neq 0$, i.e., $\partial W_u(N)$ contains some unstable manifold $W_u(w)$ of some solution w of (1.1), which is dominated by N , that is of Morse index equal to $\dim N = m$. $\square$

5.2 Proof of Corollary 2.12

Proof The most massive critical point at infinity is $\phi^\ell = (y_0, y_1, \dots, y_\ell)_\infty$, and its Morse index (see Proposition 3.8) is precisely $m_\infty = \ell + \sum_{j=1}^{\ell} k_j$, ($k_0 = 0$). Since any other critical point at infinity $\phi^{\ell'} = (y_{i_1}, y_{i_2}, \dots, y_{i_{\ell'}})_\infty$, with $\ell' \leq \ell$, has a Morse index smaller than the one of ϕ^ℓ , and since $m_\infty < m$, we derive that $A_m^\infty = \emptyset$, so condition **(H₂)** is satisfied, and then, since $\tau \neq 0 = \gamma$, the result follows by Theorem 2.9. $\square$

5.3 Proof of Corollary 2.13

Proof In this case, $I^+ = \{y_0, y_1\}$, $I_{k_1}^+ = \{y_1\}$, $X = \overline{W_s(y_1)}$, and $N = f_\lambda(B_2(X))$ with dimension $2k_1 + 1$. We also take $(z_1, z_2) = (y_0, y_1)$ in the construction of D provided by (2.13), that is $D = \Gamma_\epsilon(y_0, y_1) \cap J^{-1}\{c_\infty + \delta\}$ where

$$c_\infty = c_\infty(y_0, y_1) = S^{\frac{2s}{n}} \left(\frac{1}{K(y_0)^{\frac{n-2s}{2s}}} + \frac{1}{K(y_1)^{\frac{n-2s}{2s}}} \right)^{\frac{2s}{n}}.$$

Observe that, in this case, there are only three critical points at infinity: $(y_0)_\infty$, $(y_1)_\infty$ and $(y_0, y_1)_\infty$. As $k_0 = 0$, the greatest Morse index is equal to $m_\infty = k_1 + 1$, and since $m = 2k_1 + 1$ we have $m_\infty < m$. The result follows then by Corollary 2.12. $\square$

5.4 Proof of Theorem 2.14

For $c \in \mathbb{R}$ we let $J^c = \{u \in \Sigma^+, J(u) \leq c\}$. First, using the characterization of d , we know that we can find a cycle homologous to N in $\overline{W_u(N)} \cap J^{d+\varepsilon}$, for $\varepsilon > 0$ sufficiently small. Next, we can write the exact homology sequence of the pair $(\overline{W_u(N)} \cap J^{d+\varepsilon}, \overline{W_u(N)} \cap J^{d-\varepsilon})$. Arguing by contradiction, we assume that there is no critical point of index m at the level d . Observe that since Σ^+ is contractible, we have in particular that $H_{m+1}(\Sigma^+) = H_m(\Sigma^+) = 0$. Hence, under the assumption **(H₂)**, using Proposition 7.24 of [6], we derive that

$$H_m(\overline{W_u(N)} \cap J^{d+\varepsilon}, \overline{W_u(N)} \cap J^{d-\varepsilon}) = 0.$$

Therefore the homomorphism $H_m(\overline{W_u(N)} \cap J^{d+\varepsilon}) \rightarrow H_m(\overline{W_u(N)} \cap J^{d-\varepsilon})$ is onto, and so there is a cycle homologous to N in $\overline{W_u(N)} \cap J^{d-\varepsilon}$. This contradicts the definition of d . The result follows. $\square$

5.5 Proof of Theorem 2.15

Proof Assuming that (1.1) has no solution, let

$$u = \frac{\alpha \varphi_{(y_0, \lambda)} + (1 - \alpha) \varphi_{(x, \lambda)}}{\|\alpha \varphi_{(y_0, \lambda)} + (1 - \alpha) \varphi_{(x, \lambda)}\|} \in f_\lambda(C_{y_0}(X)).$$

The flow of w acts essentially on α . The action of bringing α to zero or to one depends on whether $\alpha < \frac{1}{2}$, in which case u goes to $\overline{W_u(y_{i_0})}_\infty = X$, or $\alpha > \frac{1}{2}$, in which case u goes to $\overline{W_u(y_0)}_\infty = \{y_0\}$. On the other hand, we have another action on $x \in X = \overline{W_s(y_{i_0})}$, when $\alpha = 1 - \alpha = \frac{1}{2}$. Observe that we then have

$$u = \frac{1}{2} \varphi_{(y_0, \lambda)} + \frac{1}{2} \varphi_{(x, \lambda)}.$$

Since only x can move, then y_0 remains one of the concentration points of u , and x goes to $W_s(y_j)$, where y_j is a critical point of K dominated by y_0 . Thus, u goes to $W_u(y_0, y_{i_0}) \cup Z$ where

$$Z = \bigcup_{y_j \in X \setminus \{y_{i_0}\}} W_u(y_0, y_j)$$

is a manifold in dimension at most $k-1$. Therefore, $X \cup W_u(y_0, y_{i_0}) \cup Z$ contains a strong retract of $f_\lambda(C_{y_0}(X))$. Since $\mu(y_0) = 0$, one can be more precise. This strong retract does not intersect $W_u(y_0, y_{i_0})$, and thus it is contained in $X \cup Z$. It can be written as $X \cup S$, where $S \subset Z$ is a stratified set (see Proposition 7.24 in [6]), and thus is of dimension at most $k-1$. Therefore $H_q(X \cup S) = 0$ for all $q \in \mathbb{N}^*$, since $f_\lambda(C_{y_0}(X))$ is a contractible set. Using the exact homology sequence of the pair $(X \cup S, X)$, we have

$$\cdots \longrightarrow H_{k+1}(X \cup S) \xrightarrow{\pi} H_{k+1}(X \cup S, X) \xrightarrow{\partial} H_k(X) \xrightarrow{i} H_k(X \cup S) \longrightarrow \cdots$$

Since $H_q(X \cup S) = 0$ for each $q \in \mathbb{N}^*$, then we derive that $H_k(X) = H_{k+1}(X \cup S, X)$. Observe that $(X \cup S, X)$ is a stratified set of dimension at most k , so we have $H_{k+1}(X \cup S, X) = 0$, and therefore, $H_k(X) = 0$ which is a contradiction. Whence the result. $\square$

5.6 Proof of Theorem 2.16

Proof By the same argument as above,

$$X \cup \bigcup_{j=1}^d W_u(y_0, y_{i_j}) \cup S$$

is a strong retract of $f_\lambda(C_{y_0}(X))$, where $S \subset Z$ is a stratified set, and

$$Z = \bigcup_{y_j \in X \setminus \{y_{i_0}\}} W_u(y_0, y_j)$$

is of dimension at most k . Now, since $\mu_j(y_0) = 0$ for each $j = 1, \dots, d$, then we derive that $X \cup S$ is a strong retract of $f_\lambda(C_{y_0}(X))$, and, because $f_\lambda(C_{y_0}(X))$ is a contractible set, then $H_q(X \cup S) = 0$ for all $q \in \mathbb{N}^*$. Using the exact homology sequence of the pair $(X \cup S, X)$, we derive that $H_{k+1}(X \cup S, X) = H_k(X) = 0$, a contradiction. The result then follows. $\square$

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