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Rational cohomologies of classifying spaces for homogeneous spaces of small rank

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Abstract Let a simply-connected homogeneous space X satisfy the condition of $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = 2$ and $\dim \pi_{\text{odd}}(X) \otimes \mathbb{Q} = 3$ (then, we say it is of $(2, 3)$ type), which is the smallest rank in non-formal pure spaces. Then, we compute the Sullivan minimal model of the Dold–Lashof classifying space $\text{Baut}_1 X$ according to Nishinobu and Yamaguchi (Topol Appl 196:290–307, 2015) and observe whether or not its rational cohomology is a polynomial algebra, which is a necessary condition for the Serre spectral sequence of any fibration over a sphere with fibre X to degenerate at term E_2 .

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المخلص

ليكن X فضاء متجانسا مترابطا بشكل بسيط يحقق الشرط $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = 2$ و $\dim \pi_{\text{odd}}(X) \otimes \mathbb{Q} = 3$ (نقول إنه من النوع $(2, 3)$ ، وهذه أقل رتبة في الفضاءات النقية غير الرسمية. ثم نحسب نموذج سليفان الأدنى لفضاء دولد-لاشوف التصنيفي $\text{Baut}_1 X$ بحسب [14]، ونبحث فيما إذا كانت كوهومولوجيته النسبية جبرية كثيرة حدود، وهو شرط ضروري لتحلل متتابعة سري الطيفية لأي عملية تكون للأطياف على كرة لها ليف X عند الحد- E_2 .

1 Introduction

A simply connected CW complex X is said to be rationally *elliptic* if the dimensions of rational cohomology and homotopy are both finite [4]. Furthermore, a rationally elliptic space X is said to be an F_0 space (or *positively elliptic*) if $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = \dim \pi_{\text{odd}}(X) \otimes \mathbb{Q}$ and $H^{\text{odd}}(X; \mathbb{Q}) = 0$. It is equivalent to

$$H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n),$$

in which $|x_i|$, the degree of x_i , are even and $f_1, \dots, f_n$ form a regular sequence in a graded $\mathbb{Q}$ -algebra $\mathbb{Q}[x_1, \dots, x_n]$. A rationally elliptic space X is said to be *pure* if the Sullivan model [20] (see Sect. 2) is given as $M(X) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), d)$, where $|x_i|$ are even and $|y_i|$ are odd with $dx_i = 0$ and $dy_i = f_i \in \mathbb{Q}[x_1, \dots, x_n]$ for $m \leq n$ [4]. A pure space X is an F_0 space if and only if $m = n$.

In 1977, Halperin [7] conjectured that the Serre spectral sequences of all fibrations $X \xrightarrow{i} E \rightarrow Y$ of simply-connected CW complexes degenerate at the E_2 term for any F_0 space X , which is equivalent to that $i^* : H^*(E; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ is an epimorphism. Let $\text{Baut}_1 X$ be the Dold–Lashof classifying space for a space

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X with $\text{aut}_1 X = \text{map}(X, X; id_X)$. Any orientable fibration with fibre X over a space Y is the pull-back of a universal fibration over $\text{Baut}_1 X$ by a map $Y \rightarrow \text{Baut}_1 X$ [3, 19]. Halperin's conjecture is equivalent to

Halperin Conjecture *When X is an F_0 -space, the rational cohomology algebra of $\text{Baut}_1 X$ is a polynomial algebra, that is*

$$H^*(\text{Baut}_1 X; \mathbb{Q}) \cong \mathbb{Q}[v_1, \dots, v_m]$$

for some elements of even degree $v_1, \dots, v_m$ ([12]).

Theorem 1.1 [16] *Let $X = G/H$ be a homogeneous space of $\text{rank } G = \text{rank } H$. Then, the Halperin conjecture is true for X .*

Of course, $X = G/H$ is an F_0 space when $\text{rank } G = \text{rank } H$. Therefore, the Serre spectral sequence of any fibration with fibre X degenerates at E_2 term. Unfortunately, for a general pure space X , it does not degenerate at E_2 term even if $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra. For example, when X is the $2n+1$ -dimensional sphere S^{2n+1} , $H^*(\text{Baut}_1 X; \mathbb{Q}) \cong \mathbb{Q}[v]$ with $|v| = 2n+2$. Then, there is a fibration $X \xrightarrow{i} \mathbb{C}P^n \rightarrow K(\mathbb{Z}, 2)$, where $i^*: 0 = H^{2n+1}(\mathbb{C}P^n; \mathbb{Q}) \rightarrow H^{2n+1}(X; \mathbb{Q})$ is not an epimorphism. Here, $\mathbb{C}P^n$ is the n -dimensional complex projective space. On the other hand, when $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra for a space X in general, any fibration $\xi: X \rightarrow E \rightarrow S^{2n+1}$ with fibre X over an odd-sphere S^{2n+1} is rationally trivial, i.e., $E_{\mathbb{Q}} \simeq (X \times S^{2n+1})_{\mathbb{Q}}$. Because the set of rational homotopy classes of classifying maps

$$[S^{2n+1}, \text{Baut}_1 X_{\mathbb{Q}}] = [M(\text{Baut}_1 X), M(S^{2n+1})] = \text{Hom}_{\mathbb{Q}}(\mathbb{Q}[v_1, \dots, v_m], \Lambda v)$$

is a one point of the constant map (zero map in graded $\mathbb{Q}$ -algebras) when $|v| = 2n+1$. In particular, the Serre spectral sequence of ξ degenerates at E_2 term. Recall an example of a rationally elliptic (non-pure) space X in [23], such that the Serre spectral sequence of any fibration with fibre X degenerates at E_2 term. Notice that $H^*(\text{Baut}_1 X; \mathbb{Q})$ is then a polynomial algebra.

Question 1.2 *If the Serre spectral sequence of any fibration with a rationally elliptic fibre X degenerates at E_2 term, then is $H^*(\text{Baut}_1 X; \mathbb{Q})$ a polynomial algebra? In particular, does it hold when X is pure?*

In this paper, we call a pure space X to be of “ (m, n) type” when $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = m$ and $\dim \pi_{\text{odd}}(X) \otimes \mathbb{Q} = n$ ($m \leq n$). For example, S^n is a pure space of $(0, 1)$ type when n is odd and of $(1, 1)$ type when n is even. Let X be a pure space of $(2, 3)$ type. Then

$$M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$$

where $|x_i|$ are even with $|x_1| \leq |x_2|$, $|y_i|$ are odd with $|y_1| \leq |y_2| \leq |y_3|$, $dx_1 = dx_2 = 0$ and $dy_i = f_i \in \mathbb{Q}[x_1, x_2]$ for $i = 1, 2, 3$.

Theorem 1.3 *Let X be a pure space of $(2, 3)$ type. If the Serre spectral sequence of any fibration over a sphere with fibre X degenerates at E_2 term, then $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra.*

A simply connected space Y of finite type is said to be *formal* if there is a quasi-isomorphism $M(Y) \rightarrow (H^*(Y; \mathbb{Q}), 0)$. It is known that F_0 spaces or the product $Y \times S^n$ of an F_0 -space Y and an odd-sphere S^n as in Theorem 1.4 of below are formal. Conversely, when a pure space X of $(2, 3)$ type is formal, there is an F_0 -space Y and an odd-integer n , such that $X_{\mathbb{Q}} \simeq (Y \times S^n)_{\mathbb{Q}}$. When X is formal, we can check whether or not $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra.

Theorem 1.4 [17] *Let Y be an F_0 space with $H^*(Y; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(f_1, f_2)$ and $X = Y \times S^n$ for an odd-integer n . Then, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra if and only if $\max\{|f_1|, |f_2|\} - 1 \leq n$.*

In addition, if a pure space X is of $(2, 2)$ type or of $(3, 3)$ type, then they are F_0 spaces and the Halperin conjecture is true due to Lupton [10].

Example 1.5 The following spaces X :

- (1) $SU(2) \times SU(2) \times SU(2)/T^2$ for an inclusion of $T^2 = S^1 \times S^1$ ([15])
- (2) $U(5)/U(2) \times U(2)$ ([6, 13])
- (3) $S^{2l} \times S^{2m} \times S^{2n+1}$ with $2l \leq 2m \leq n+1$
- (4) $\mathbb{C}P^l \times \mathbb{C}P^m \times S^{2n+1}$ with $l \leq m \leq n$

are formal spaces of (2, 3) type and $H^*(\text{Baut}_1 X; \mathbb{Q})$ are polynomial algebras.

For example, we know that

$$H^*(\text{Baut}_1 X; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2, u_3, u_4, u_5, u_6, u_7]$$

with $|u_1| = |u_2| = |u_3| = |u_4| = 2$ and $|u_5| = |u_6| = |u_7| = 4$ for the case (1) [14, Example 1.15].

However, pure spaces X of (2, 3) type are not formal spaces as in Theorem 1.4 in general and the minimal model structure of $\text{Baut}_1 X$ is not clear. Indeed, we can see some examples in [14].

Let $SU(n)$ be the special unitary group, $Sp(n)$ be the symplectic group, $SO(n)$ be the special orthogonal group, and $Q(n)$ be the quaternionic group [13], [6, p. 271]. The following X are non-formal homogeneous spaces of (2, 3) type.

- Theorem 1.6** (1) When $X = SU(2) \times SU(2) \times SU(2)/T^2$ for an inclusion of T^2 [different with it of Example 1.5(1)], $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra (isomorphic to it of Example 1.5(1)).
- (2) When $X = SU(n+3)/SU(n) \times SU(3)$ for $n \geq 3$, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra if and only if $n \neq 3$.
- (3) When $X = Sp(6)/SU(6)$, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra.
- (4) When $X = SO(12)/SU(6)$ or $Q(5)/SU(5)$, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is not free and $M(\text{Baut}_1 X)$ is isomorphic to $M(\text{Baut}_1 SU(6)/SU(3) \times SU(3))$ as (non-graded) $\mathbb{Q}$ algebra.

Note that the rational homotopy types of homogeneous spaces X in (1), (2), (3), and (4) of Theorem 1.6 are (2, 2, 3, 3, 3), (4, 6, $2n+1$, $2n+3$, $2n+5$), (6, 10, 15, 19, 23), and (6, 10, 11, 15, 19), respectively.

In this paper, we introduce the Sullivan model of $\text{Baut}_1 X$ for a rationally elliptic space X in Sect. 2. In Sect. 3, we consider a condition that $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra. In Sect. 4, we prove Theorems 1.3 and 1.6.

2 Preliminaries

We use the Sullivan minimal model $M(X)$ [20] of a simply connected space X of finite type. It is a free $\mathbb{Q}$ -commutative differential graded algebra (abbr., DGA) $(\Lambda V, d)$ with a $\mathbb{Q}$ -graded vector space $V = \bigoplus_{i \geq 1} V^i$, where $\dim V^i < \infty$ and a decomposable (minimality condition) differential, that is

$$d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1} \quad \text{and} \quad d \circ d = 0.$$

Here, $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. We often denote $(\Lambda V, d)$ simply by ΛV . Then, we have

$$xy = (-1)^{|x||y|}yx \quad \text{and} \quad d(xy) = d(x)y + (-1)^{|x|}x d(y).$$

We note that $M(X)$ determines the rational homotopy-type $X_{\mathbb{Q}}$ of X . In particular, there are the following isomorphisms:

$$\text{Hom}(V^i, \mathbb{Q}) \cong \pi_i(X) \otimes \mathbb{Q} (= \pi_i(X)_{\mathbb{Q}}) \quad \text{and} \quad H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}).$$

In particular, $d = 0$ if and only if $H^*(X; \mathbb{Q})$ is free. Furthermore, $V^{\text{odd}} = 0$ (then $d = 0$) if and only if $H^*(X; \mathbb{Q})$ is a polynomial algebra. For $d = d_2 + \dots$ with $d_i : V \rightarrow \Lambda^i V = V \wedge \dots \wedge V$ (i factors), the linear map $d_2 : V \rightarrow V \wedge V$ is the dual of the Lie bracket of the rational homotopy Lie algebra $\pi_*(\Omega X)_{\mathbb{Q}}$ of X , where $[x, y] = (-1)^{(|x|+1)(|y|+1)}[y, x]$ and $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$ for $x, y, z \in \pi_*(\Omega X)_{\mathbb{Q}}$. Thus, $d_2 = 0$ if and only if $\pi_*(\Omega X)_{\mathbb{Q}}$ is abelian (see [4] for a general introduction and the standard notations).

Let $\text{Der}_i M(X)$ be the set of $\mathbb{Q}$ derivations of $M(X)$ decreasing the degree by i with $\sigma(xy) = \sigma(x)y + (-1)^{i \cdot |x|} x\sigma(y)$ for $x, y \in M(X)$. The boundary operator $\partial : \text{Der}_i M(X) \rightarrow \text{Der}_{i-1} M(X)$ is defined by

$$\partial(\sigma) = d \circ \sigma - (-1)^i \sigma \circ d$$

for $\sigma \in \text{Der}_i M(X)$. We denote $\bigoplus_{i \geq 0} \text{Der}_i M(X)$ by $\text{Der} M(X)$ in which $\text{Der}_1 M(X)$ is ∂ cycles. Then, $\text{Der} M(X)$ is a DGL by the Lie bracket:

$$[\sigma, \tau] := \sigma \circ \tau - (-1)^{|\sigma||\tau|} \tau \circ \sigma.$$

Furthermore, recall the definition (sign convention) of Tanré [21, p. 25]. Let (L, ∂) be a DGL of finite type with positive degree. Then, $C^*(L, \partial) = (\Lambda s^{-1} \sharp L, D = d_1 + d_2)$ with

$$\langle d_1 s^{-1} z; sx \rangle = -\langle z; \partial x \rangle \quad \text{and} \quad \langle d_2 s^{-1} z; sx_1, sx_2 \rangle = (-1)^{|x_1|} \langle z; [x_1, x_2] \rangle,$$

where $\langle s^{-1} z; sx \rangle = (-1)^{|z|} \langle z; x \rangle$ and $\sharp L$ is the dual space of L .

Theorem 2.1 [20, p. 314] *Let X be a rationally elliptic space. The DGA $C^*(\text{Der} M(X), \partial)$ is a Sullivan model $m(\text{Baut}_1 X)$ of $\text{Baut}_1 X$. In particular, $\pi_*(\Omega \text{Baut}_1 X) \otimes \mathbb{Q} \cong H_*(\text{Der} M(X), \partial)$ as graded Lie algebras.*

Here, the DGA $m(\text{Baut}_1 X)$ need not be minimal.

3 The condition to be a polynomial algebra

The following is a partial generalization of Smith's [17, Theorem 1].

Proposition 3.1 *Let X be a pure space. Suppose that it satisfies the conditions:*

- (i) $\max \pi_{\text{odd}}(X)_{\mathbb{Q}} \leq \min H^{\text{odd}}(X; \mathbb{Q})$ and
- (ii) $\text{Der} H^{\text{even}}(X; \mathbb{Q}) = 0$.

Then

- (1) $H_{\text{even}}(\text{Der} M(X)) = 0$, i.e., $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra.
- (2) Furthermore

$$\text{Baut}_1 X \simeq_{\mathbb{Q}} \prod_q K(\mathbb{Q}^{d_{2q}}, 2q)$$

where

$$d_{2q} := \sum_{k \geq q} \dim H^{2k-2q}(X; \mathbb{Q}) (\dim \pi_{2k-1}(X)_{\mathbb{Q}} - \dim \pi_{2k}(X)_{\mathbb{Q}}).$$

Here, $\max V := \max\{i \mid V^i \neq 0\}$ and $\min V := \min\{i \mid V^i \neq 0\}$ for a graded vector space $V = \bigoplus_i V^i$.

Convention For a Sullivan minimal model $(\Lambda V, d)$, the symbol (v, f) means the elementary derivation that takes a generator v of V to an element f of ΛV and the other generators to 0.

Proof of Proposition 3.1 Let $M(X) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), d)$, where $|x_i|$ are even and $|y_i|$ are odd with $dx_i = 0$ and $dy_i \in \mathbb{Q}[x_1, \dots, x_m]$.

- (1) follows since an element of $H_{\text{even}}(\text{Der} M(X))$ is given by a certain linear combination of

- (i) (y_i, a_i) for $a_i \in H^{\text{odd}}(X; \mathbb{Q})$ ($i = 1, \dots, n$) and
- (ii) $[(x_i, b_i) + \dots] \in \text{Der}_{\text{even}} M(X)$ for $b_i \in H^{\text{even}}(X; \mathbb{Q})$ ($i = 1, \dots, m$).

Note that (ii) induces the derivation $(x_i, b_i) \in \text{Der} H^{\text{even}}(X; \mathbb{Q})$.

- (2) follows since, when $H_{\text{even}}(\text{Der} M(X)) = 0$

$$\begin{aligned} H_*(\text{Der} M(X)) &= H_{\text{odd}}(\text{Der} M(X)) \cong \frac{\bigoplus_{1 \leq i \leq n} \mathbb{Q}\{(y_i, f_{is})\}_s}{\bigoplus_{1 \leq j \leq m} \mathbb{Q}\{\partial(x_j, g_{jt})\}_t} \\ &\cong \bigoplus_{k \geq q > 0} \frac{\text{Hom}(\pi_{2k-1}(X)_{\mathbb{Q}}, H^{2k-2q}(X; \mathbb{Q}))}{\text{Hom}(\pi_{2k}(X)_{\mathbb{Q}}, H^{2k-2q}(X; \mathbb{Q}))} \end{aligned}$$

where $\{f_{is}\}_s$ is the sub-basis of $H^{\text{even}}(X; \mathbb{Q})$ with $|y_i| - |f_{is}| = 2q - 1 > 0$ and $\{g_{jt}\}_t$ is the sub-basis of $H^{\text{even}}(X; \mathbb{Q})$ with $|x_j| - |g_{jt}| = 2q > 0$. Then, (2) holds from Theorem 2.1, since $\text{Baut}_1 X$ is a rational H-space from (1). $\square$

When X is of (2, 3) type, the conditions (i) and (ii) of Proposition 3.1, respectively, correspond to (I) and (II) of the following.

Theorem 3.2 [14, Theorem 1.13] *Let X be a pure space of (2, 3) type, where $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_i|$ is even with $|x_1| \leq |x_2|$ and $|y_i|$ is odd with $|y_1| \leq |y_2| \leq |y_3|$. Let $dy_i = f_i \in \mathbb{Q}[x_1, x_2]$ for $i = 1, 2, 3$. Then $H^*(\text{Baut}_1 X; \mathbb{Q})$ is not a polynomial algebra if and only if the following (I) or (II) holds:*

- (I) *There is an element of odd degree $a \in H^*(X; \mathbb{Q})$ with $|a| < |y_3|$.*
- (II) *When $|x_1| < |x_2|$*

$$f_1 = x_1^l \in \mathbb{Q}[x_1], \quad \partial f_2 / \partial x_2 \cdot x_1^k \in (f_1), \quad \partial f_3 / \partial x_2 \cdot x_1^k \in (f_1, f_2)$$

for some $1 < l$ and some $0 < k < \min\{l, |x_2|/|x_1|\}$. Here, $\partial f_i / \partial x_j$ is the partial differentiation of f_i by x_j and (S) is the ideal of $\mathbb{Q}[x_1, x_2]$ generated by a set S .

Recall that a $\mathbb{Q}$ -graded commutative algebra H is said to be *free* when

$$H \cong \mathbb{Q}[x_1, \dots, x_m] \otimes \Lambda(y_1, \dots, y_n)$$

with $|x_i|$ even and $|y_i|$ odd for some m and n .

Theorem 3.3 *When X is a pure space of (2, 3) type, only even degree generators appear in the free case of $H^*(\text{Baut}_1 X; \mathbb{Q})$.*

Proof Let $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$, where $|x_i|$ is even with $|x_1| \leq |x_2|$ and $|y_i|$ is odd with $|y_1| \leq |y_2| \leq |y_3|$.

Suppose that $M(\text{Baut}_1 X)$ does not contain an element of odd degree. Then, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a (finitely generated) polynomial algebra, since $\pi_*(\text{Baut}_1 X)_{\mathbb{Q}} < \infty$, and the differential of $M(\text{Baut}_1 X)$ is zero.

Suppose that $M(\text{Baut}_1 X)$ contains an element of odd degree. Then, $H^*(\text{Baut}_1 X; \mathbb{Q})$ cannot be a polynomial algebra. Thus, it is sufficient to consider the two cases (I) and (II) of Theorem 3.2:

- (I) Let (y_3, a) be a cycle of $\text{Der}_{\text{even}} M(X)$ with a non-exact cocycle $a = g_1 y_1 + g_2 y_2$ of $M(X)$ for $g_i \in \mathbb{Q}[x_1, x_2]$. Since $\mathbb{Q}[x_1, x_2]$ is a unique factorization domain (UFD) and $f_2 \notin (f_1)$, $f_1 (= dy_1)$, and $f_2 (= dy_2)$ have a common factor in $\mathbb{Q}[x_1, x_2]$, i.e., there is an element $h \in (x_1, x_2)$, such that $f_1 = f'_1 h$ and $f_2 = f'_2 h$. Here, we assume that the degree of a is minimal in such elements. It induces that $g_1 = k f'_2$ and $g_2 = -k f'_1$ for some $k \in \mathbb{Q} - 0$. Then, the Lie bracket in $\text{Der} M(X)$

$$[(y_3, a), (y_i, 1)] = (y_3, \partial a / \partial y_i) = (y_3, g_i)$$

is a non-zero non-exact ∂ cycle, since g_i is a non-zero non-exact d cocycle.

- (II) Let $\sigma = (x_2, x_1^k) + (y_2, f) + (y_3, g)$ be a non-exact cycle of $\text{Der}_{\text{even}} M(X)$ with $f = ay_1$ and $g = by_1 + cy_2$ for $a, b, c \in \mathbb{Q}[x_1, x_2]$. Then, the Lie bracket in $\text{Der} M(X)$

$$[(y_3, x_2), \sigma] = (y_3, x_1^k)$$

is a non-zero non-exact cycle, since $|dy_2| \geq |dy_1| = |x^l| > |x^k|$.

Therefore, $H^*(\text{Baut}_1 X; \mathbb{Q})$ cannot be free in any case of (I) and (II) from Theorem 2.1. For example, the bracket in the case of (I) induces $d_2(s^{-1}(y_3, g_i)^{\sharp}) = s^{-1}(y_3, a)^{\sharp} \cdot s^{-1}(y_i, 1)^{\sharp} + \dots$ in $C^*(\text{Der} M(X))$ from the construction of the functor $C^*: \mathbf{DGL} \rightarrow \mathbf{DGA}$ of Sect. 2. $\square$

A rationally elliptic space X is said to be *two stage* if $dV_1 = 0$ and $dV_2 \subset \Lambda V_1$ for $M(X) = (\Lambda V, d)$ with $V = V_1 \oplus V_2$. In particular, a pure space is two stage.

Example 3.4 Let $M(X) = (\Lambda(v_1, v_2, v_3, v_4), d)$ with $|v_i|$ odd, $dv_1 = dv_2 = dv_4 = 0$ and $dv_3 = v_1 v_2$. Suppose $|v_1| = |v_2| < |v_3| \leq |v_4|$ and $|v_4| < |v_1| + |v_3|$. Then

$$H^*(\text{Baut}_1 X; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2] \otimes \Lambda(y_1, y_2)$$

where $|x_1| = |v_3| + 1$, $|x_2| = |v_4| + 1$, $|y_1| = |v_4| - |v_2| + 1$, and $|y_2| = |v_4| - |v_1| + 1$ [14, page 302]. Thus, when X is two stage but not pure, $H^*(\text{Baut}_1 X; \mathbb{Q})$ may be free but not a polynomial algebra.

Example 3.5 When X is a pure space of $(2, 3)$ type, arbitrarily, many elements of odd degree can exist in $\pi_*(\text{Baut}_1 X)_{\mathbb{Q}}$ ([11]). For example, let $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = 2$, $|x_2| = 2k$, $|y_1| = 3$, $|y_2| = 2k + 1$, $|y_3| = 2kn - 1$, $dy_1 = x_1^2$, $dy_2 = x_1x_2$, and $dy_3 = x_2^n$ for $k > 2$. Then, there are the non-exact cocycles $w_i := x_2^i y_1 - x_1 x_2^{i-1} y_2$ in $M(X)$ for $i = 1, \dots, n - 1$. Then, $(y_3, w_i) \in \text{Der} M(X)$ are $2k(n - i) - 4$ -dimensional non-exact ∂ cycles.

Remark 3.6 The authors do not know whether or not the element of odd degree of $M(\text{Baut}_1 X)$ in the proof of Theorem 3.3 is a cocycle, i.e., makes an element of $H^*(\text{Baut}_1 X; \mathbb{Q})$. If X is a pure space of $(2, 4)$ type, the minimal model of $\text{Baut}_1 X$ may have an element of odd degree with differential non-zero. For example, let $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3, y_4), d)$ with $|x_1| = 2$, $|x_2| = 6$, $|y_1| = 5$, $|y_2| = 9$, $|y_3| = 13$, $|y_4| = 17$, $dx_1 = dx_2 = 0$, $dy_1 = x_1^3$, $dy_2 = x_1^2 x_2$, $dy_3 = x_1 x_2^2$, and $dy_4 = x_2^3$. Then, there is the Lie bracket in $\text{Der}_4 M(X)$:

$$[(y_4, x_1 y_3 - x_2 y_2), (y_3, x_2 y_1 - x_1 y_2)] = (y_4, x_1 x_2 y_1 - x_1^2 y_2),$$

which is a non-exact ∂ cycle, i.e., $dw \neq 0$ for a 5-dimensional element w of $M(\text{Baut}_1 X)$ from the construction of the functor C^* .

4 Proofs

Recall that the model of a fibration $X \rightarrow E \rightarrow Y$ of simply connected spaces is given as

$$M(Y) \rightarrow (M(Y) \otimes \Lambda V, D) \rightarrow (\Lambda V, d) = M(X)$$

where $D|_{M(Y)} = d_Y$ for $M(Y) = (\Lambda V_Y, d_Y)$ and $d = \overline{D}$ [4]. Here, $(M(Y) \otimes \Lambda V, D)$ may not be minimal.

Proof of Theorem 1.3 Suppose that $H^*(\text{Baut}_1 X; \mathbb{Q})$ is not a polynomial algebra. Then, (I) or (II) of Theorem 3.2 holds.

When (I) holds, there is a d cocycle of odd degree $g_1 y_1 + g_2 y_2$ with non-exact elements $g_1, g_2 \in \mathbb{Q}[x_1, x_2]$ as in the proof of Theorem 3.3. Let $Y = S^{|y_i|+1}$ with $H^*(Y; \mathbb{Q}) = \mathbb{Q}[z]/(z^2)$ for $i = 1$ or 2 . Then, we can construct a model of a fibration $X \rightarrow E \rightarrow Y$ as

$$(\Lambda(z, v), d_Y) \rightarrow (\Lambda(z, v) \otimes \Lambda(x_1, x_2, y_1, y_2, y_3), D) \rightarrow (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$$

with $|z| = |y_i| + 1$, $|v| = 2|y_i| + 1$, $d_Y z = 0$, $d_Y v = z^2$, $Dx_1 = Dx_2 = 0$, $Dy_i = dy_i + z$, and $Dy_j = dy_j$ for $j \neq i$ and $Dy_3 = dy_3$. Then, the element $[g_1 y_1 + g_2 y_2]$ of $H^*(X; \mathbb{Q})$ cannot not be extended to an element $[g_1 y_1 + g_2 y_2 + b]$ in $H^*(E; \mathbb{Q})$ for any element $b \in (z_1, z_2)$.

When (II) holds, let $Y = S^{|x_i|-|a_i|+1}$ with $H^*(Y; \mathbb{Q}) = \Lambda z$. Then, we can construct a model of a fibration $X \rightarrow E \rightarrow Y$ as

$$(\Lambda z, 0) \rightarrow (\Lambda(z, x_1, x_2, y_1, y_2, y_3), D) \rightarrow (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$$

with $|z| = |x_2| - k|x_1| + 1$, $Dx_1 = 0$, and $Dx_2 = zx_1^k$ for some $k > 0$, $Dy_1 = dy_1$, $Dy_2 = dy_2 + zay_1$, and $Dy_3 = dy_3 + zby_1 + zcy_2$ for some $a, b, c \in \mathbb{Q}[x_1, x_2]$. Then, the element $[x_2]$ of $H^*(X; \mathbb{Q})$ is not in $H^*(E; \mathbb{Q})$.

In any case of (I) and (II), $i^* : H^*(E; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ are not epimorphisms for rational fibrations $X \rightarrow E \rightarrow Y$ given by the models. That is, the Serre spectral sequences of them do not degenerate at E_2 term. $\square$

Remark 4.1 The Serre spectral sequence of any fibration with fibre X and base S^{n+1} degenerates at E_2 term if and only if the natural induced map $\rho : H_n(\text{Der} M(X)) \rightarrow \text{Der}_n H^*(X; \mathbb{Q})$ [5, page 389] is the zero map.

In the followings, we see the Sullivan minimal models $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ ($dx_1 = dx_2 = 0$) of Example 1.5 as

- (1) $|x_1| = |x_2| = 2$ and $|y_1| = |y_2| = |y_3| = 3$ with $dy_1 = x_1^2$, $dy_2 = x_2^2$, and $dy_3 = 0$.
- (2) $|x_1| = 2$, $|x_2| = 4$, $|y_1| = 5$, $|y_2| = 7$, and $|y_3| = 9$ with $dy_1 = 2x_1x_2 - x_1^3$, $dy_2 = x_2^2 - 3x_1^2x_2 + x_1^4$, and $dy_3 = 0$.
- (3) $|x_1| = 2l$, $|x_2| = 2m$, $|y_1| = 4l - 1$, $|y_2| = 4m - 1$, and $|y_3| = 2n + 1$ with $dy_1 = x_1^2$, $dy_2 = x_2^2$, and $dy_3 = 0$.

- (4) $|x_1| = |x_2| = 2$, $|y_1| = 2l + 1$, $|y_2| = 2m + 1$, and $|y_3| = 2n + 1$ with $dy_1 = x_1^{l+1}$, $dy_2 = x_2^{m+1}$, and $dy_3 = 0$.

For example, for the case (2), by the direct computation of Proposition 3.1, we can compute that

$$H^*(\text{Baut}_1 X; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2, u_3, u_4, u_5, \dots, u_{12}]$$

with $|u_1| = |u_2| = 2$, $|u_3| = |u_4| = |u_5| = 4$, $|u_6| = \dots = |u_9| = 6$, $|u_{10}| = |u_{11}| = 8$, and $|u_{12}| = 10$.

Theorem 4.2 [1] When $X = \frac{SU(3+n)}{SU(3) \times SU(n)}$ with blockwise inclusion for $n \geq 3$, the (non-minimal) Sullivan model is given as

$$m(X) = (\Lambda(c_2^1, c_3^1, c_2^2, \dots, c_n^2, v_2, v_3, \dots, v_{n+1}, v_{n+2}, v_{n+3}), D)$$

where $|c_i^t| = 2i$, $|v_i| = 2i - 1$, $D(c_i^t) = 0$, $D(v_2) = c_2^1 + c_2^2$, $D(v_3) = c_3^1 + c_3^2$, $D(v_j) = c_2^1 c_{j-2}^2 + c_3^1 c_{j-3}^2 + c_j^2$ ($4 \leq j \leq n$), $D(v_{n+1}) = c_2^1 c_{n-1}^2 + c_3^1 c_{n-2}^2$, $D(v_{n+2}) = c_2^1 c_n^2 + c_3^1 c_{n-1}^2$, and $D(v_{n+3}) = c_3^1 c_n^2$.

Then, we have the following from direct computation.

Theorem 4.3 The minimal model of $X = \frac{SU(3+n)}{SU(3) \times SU(n)}$ ($n \geq 3$) is given as

- (i) $n = 2m + 1$ ($m \geq 1$)

$$\begin{aligned} M(X) &= (\Lambda(x_1, x_2, y_1, y_2, y_3), d) \quad \text{where} \\ |x_1| &= 4, \quad |x_2| = 6, \quad |y_1| = 4m + 3, \quad |y_2| = 4m + 5, \quad |y_3| = 4m + 7 \\ d(x_1) &= d(x_2) = 0 \\ d(y_1) &= \sum_{i=0}^{\left[\frac{m+1}{3}\right]} (-1)^{m-i} \binom{m+1-i}{2i} x_1^{m+1-3i} x_2^{2i} \\ d(y_2) &= \sum_{i=0}^{\left[\frac{m}{3}\right]} (-1)^{m-i} \binom{m+1-i}{1+2i} x_1^{m-3i} x_2^{1+2i} \\ d(y_3) &= \sum_{i=1}^{\left[\frac{m+2}{3}\right]} (-1)^{m+1-i} \binom{m+1-i}{-1+2i} x_1^{m+2-3i} x_2^{2i}. \end{aligned}$$

- (ii) $n = 2m + 2$ ($m \geq 1$)

$$\begin{aligned} M(X) &= (\Lambda(x_1, x_2, y_1, y_2, y_3), d) \quad \text{where} \\ |x_1| &= 4, \quad |x_2| = 6, \quad |y_1| = 4m + 5, \quad |y_2| = 4m + 7, \quad |y_3| = 4m + 9 \\ d(x_1) &= d(x_2) = 0 \\ d(y_1) &= \sum_{i=0}^{\left[\frac{m}{3}\right]} (-1)^{m-i} \binom{m+1-i}{1+2i} x_1^{m-3i} x_2^{1+2i} \\ d(y_2) &= \sum_{i=0}^{\left[\frac{m+2}{3}\right]} (-1)^{m+1-i} \binom{m+2-i}{2i} x_1^{m+2-3i} x_2^{2i} \\ d(y_3) &= \sum_{i=0}^{\left[\frac{m+1}{3}\right]} (-1)^{m+1-i} \binom{m+1-i}{2i} x_1^{m+1-3i} x_2^{1+2i}. \end{aligned}$$

Lemma 4.4 Let $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$, where $|x_1| = a$, $|x_2| = b$, $|y_1| = 2a - 1$, $|y_2| = a + b - 1$, $|y_3| = 2b - 1$, $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_1x_2$, $dy_3 = x_2^2$ with a and b both even. Then, $H_*(\text{Der } M(X))$, i.e., $\pi_*(\Omega\text{Baut}_1 X) \otimes \mathbb{Q}$ have the same (non-graded) Lie algebra structures for all $a < b < 2a$.

Proof There are two types for (a, b) .

Case 1. Suppose X_1 satisfies $a < b \leq \frac{3}{2}a$. Then, $\text{Der } M(X_1)$ is given by

Degree	Generators	$\text{Im } \partial$
$2b - 1$	$(y_3, 1)$	0
$a + b - 1$	$(y_2, 1)$	0
$2b - a - 1$	(y_3, x_1)	0
$2a - 1$	$(y_1, 1)$	0
b	$(x_2, 1)$	$-(y_2, x_1) - 2(y_3, x_2)$
$b - 1$	$(y_3, x_2), (y_2, x_1)$	0
$2b - 2a$	(y_3, y_1)	(y_3, x_1^2)
a	$(x_1, 1)$	$-2(y_1, x_1) - (y_2, x_2)$
$2b - 2a - 1$	(y_3, x_1^2)	0
$a - 1$	$(y_2, x_2), (y_1, x_1)$	0
$b - a$	$(y_3, y_2), (y_2, y_1), (x_2, x_1)$	$(y_3, x_1x_2), (y_2, x_1^2), -(y_2, x_1^2) - 2(y_3, x_1x_2)$
$b - a - 1$	$(y_3, x_1x_2), (y_2, x_1^2)$	0
$2a - b - 1$	(y_1, x_2)	0

Case 2. Suppose X_2 satisfies $\frac{3}{2}a < b < 2a$. Then, $\text{Der } M(X_2)$ is given by

Degree	Generators	$\text{Im } \partial$
$2b - 1$	$(y_3, 1)$	0
$a + b - 1$	$(y_2, 1)$	0
$2b - a - 1$	(y_3, x_1)	0
$2a - 1$	$(y_1, 1)$	0
b	$(x_2, 1)$	$-(y_2, x_1) - 2(y_3, x_2)$
$b - 1$	$(y_3, x_2), (y_2, x_1)$	0
$2b - 2a$	(y_3, y_1)	(y_3, x_1^2)
a	$(x_1, 1)$	$-2(y_1, x_1) - (y_2, x_2)$
$2b - 2a - 1$	(y_3, x_1^2)	0
$a - 1$	$(y_2, x_2), (y_1, x_1)$	0
$b - a$	$(y_3, y_2), (y_2, y_1), (x_2, x_1)$	$(y_3, x_1x_2), (y_2, x_1^2), -(y_2, x_1^2) - 2(y_3, x_1x_2)$
$b - a - 1$	$(y_3, x_1x_2), (y_2, x_1^2)$	0
$2a - b - 1$	(y_1, x_2)	0
$2b - 3a$	(y_3, x_1y_1)	(y_3, x_1^3)
$2b - 3a - 1$	(y_3, x_1^3)	0

Thus, $\text{Der } M(X_2) \cong \text{Der } M(X_1) \oplus \mathbb{Q}(y_3, x_1y_1) \oplus \mathbb{Q}(y_3, x_1^3)$ as non-graded Lie algebras in which

$$\partial((y_3, x_1y_1)) = (y_3, x_1^3) \quad \text{and} \quad \partial((y_3, x_1^3)) = 0.$$

Thus, $H_*(\text{Der } M(X_1)) \cong H_*(\text{Der } M(X_2))$ as non-graded Lie algebras. $\square$

From Lemma 4.4, we have the following.

Proposition 4.5 Let $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = a, |x_2| = b, |y_1| = 2a-1, |y_2| = a+b-1, |y_3| = 2b-1, dx_1 = dx_2 = 0, dy_1 = x_1^2, dy_2 = x_1x_2, dy_3 = x_2^2$, where a and b are both even and $a < b < 2a'D$. Then

$$\begin{aligned} M(\text{Baut}_1 X) &\cong (\Lambda(U_{y_3,1}, U_{y_2,1}, U_{y_1,1}, U_{y_3,x_1}, U_a, U_b, U_\sigma, U_{y_1,x_2}), d) \quad \text{where} \\ |U_{y_3,1}| &= 2b, \quad |U_{y_2,1}| = a+b, \quad |U_{y_1,1}| = 2a, \quad |U_{y_3,x_1}| = 2b-a \\ |U_a| &= a, \quad |U_b| = b, \quad |U_\sigma| = b-a+1, \quad |U_{y_1,x_2}| = 2a-b \\ d(U_{y_3,1}) &= U_{y_2,1}U_\sigma \\ d(U_{y_2,1}) &= U_{y_1,1}U_\sigma \\ d(U_{y_1,1}) &= 0 \\ d(U_{y_3,x_1}) &= U_bU_\sigma \\ d(U_b) &= U_aU_\sigma \\ d(U_a) &= U_{y_1,x_2}U_\sigma \\ d(U_\sigma) &= 0 \\ d(U_{y_1,x_2}) &= 0, \end{aligned}$$

where $U_{y_i,f}$ corresponds to the derivation (y_i, f) and U_σ is an element corresponding to the non-exact cycle $2(y_3, y_2) + (y_2, y_1) + (x_2, x_1)$.

Proof We compute $C^*(\text{Der}M(X_i))$ for $\text{Der}M(X_i)$ ($i = 1, 2$) of Lemma 4.4 according to Theorem 2.1. Let $V_{v,f}$ be the element corresponding to a derivation (v, f) by C^* .

First, consider Case 1 ($a < b \leq \frac{3}{2}a$).

d_1 is given as follows:

$$\begin{aligned} d_1(V_{y_3,1}) &= d_1(V_{y_2,1}) = d_1(V_{y_3,x_1}) = d_1(V_{y_1,1}) = d_1(V_{x_2,1}) = 0 \\ d_1(V_{y_3,x_2}) &= 2V_{x_2,1}, \quad d_1(V_{y_2,x_1}) = V_{x_2,1}, \quad d_1(V_{y_3,y_1}) = d_1(V_{x_1,1}) = 0 \\ d_1(V_{y_3,x_1^2}) &= -V_{y_3,y_1}, \quad d_1(V_{y_2,x_2}) = V_{x_1,1}, \quad d_1(V_{y_1,x_1}) = 2V_{x_1,1} \\ d_1(V_{y_3,y_2}) &= d_1(V_{y_2,y_1}) = d_1(V_{x_2,x_1}) = 0, \quad d_1(V_{y_3,x_1x_2}) = -V_{y_3,y_2} + 2V_{x_2,x_1} \\ d_1(V_{y_2,x_1^2}) &= -V_{y_2,y_1} + V_{x_2,x_1}, \quad d_1(V_{y_1,x_2}) = 0. \end{aligned}$$

d_2 is given as follows:

$$\begin{aligned} d_2(V_{y_3,1}) &= -V_{x_1,1}V_{y_3,x_1} - V_{x_2,1}V_{y_3,x_2} + V_{y_1,1}V_{y_3,y_1} + V_{y_2,1}V_{y_3,y_2} \\ d_2(V_{y_2,1}) &= -V_{x_1,1}V_{y_2,x_1} - V_{x_2,1}V_{y_2,x_2} + V_{y_1,1}V_{y_2,y_1} \\ d_2(V_{y_3,x_1}) &= -V_{x_2,x_1}V_{y_3,x_2} + V_{y_1,x_1}V_{y_3,y_1} - 2V_{x_1,1}V_{y_3,x_1^2} + V_{y_2,x_1}V_{y_3,y_2} - V_{x_2,1}V_{y_3,x_1x_2} \\ d_2(V_{y_1,1}) &= -V_{x_1,1}V_{y_1,x_1} - V_{x_2,1}V_{y_1,x_2}, \quad d_2(V_{x_2,1}) = -V_{x_1,1}V_{x_2,x_1} \\ d_2(V_{y_3,x_2}) &= V_{y_1,x_2}V_{y_3,y_1} + V_{y_2,x_2}V_{y_3,y_2} - V_{x_1,1}V_{y_3,x_1x_2} \\ d_2(V_{y_2,x_1}) &= -V_{x_2,x_1}V_{y_2,x_2} + V_{y_1,x_1}V_{y_2,y_1} - 2V_{x_1,1}V_{y_2,x_1^2} \\ d_2(V_{y_3,y_1}) &= -V_{y_2,y_1}V_{y_3,y_2}, \quad d_2(V_{x_1,1}) = 0 \\ d_2(V_{y_3,x_1^2}) &= -V_{x_2,x_1}V_{y_3,x_1x_2} + V_{y_2,x_1^2}V_{y_3,y_2} \\ d_2(V_{y_2,x_2}) &= V_{y_1,x_2}V_{y_2,y_1}, \quad d_2(V_{y_1,x_1}) = -V_{x_2,x_1}V_{y_1,x_2} \\ d_2(V_{y_3,y_2}) &= d_2(V_{y_2,y_1}) = d_2(V_{x_2,x_1}) = d_2(V_{y_3,x_1x_2}) = d_2(V_{y_2,x_1^2}) = d_2(V_{y_1,x_2}) = 0. \end{aligned}$$

Let $V_{w_1} = 2V_{y_2,x_1} - V_{y_3,x_2}$ and $V_{w_2} = 2V_{y_2,x_2} - V_{y_1,x_1}$. Then

$$\begin{aligned} C^*(\text{Der}M(X_1)) &= (\Lambda(V_{y_3,1}, V_{y_2,1}, V_{y_3,x_1}, V_{y_1,1}, V_{x_2,1}, V_{w_1}, V_{y_2,x_1}, V_{y_3,y_1}, V_{x_1,1}, \\ &\quad V_{y_3,x_1^2}, V_{y_2,x_2}, V_{w_2}, V_{y_3,y_2}, V_{y_2,y_1}, V_{x_2,x_1}, V_{y_3,x_1x_2}, V_{y_2,x_1^2}, V_{y_1,x_2}), D) \end{aligned}$$

where $|V_{y_3,1}| = 2b$, $|V_{y_2,1}| = a + b$, $|V_{y_3,x_1}| = 2b - a$, $|V_{y_1,1}| = 2a$, $|V_{x_2,1}| = b + 1$, $|V_{w_1}| = |V_{y_2,x_1}| = b$, $|V_{y_3,y_1}| = 2b - 2a + 1$, $|V_{x_1,1}| = a + 1$, $|V_{y_3,x_1^2}| = 2b - 2a$, $|V_{y_2,x_2}| = |V_{w_2}| = a$, $|V_{y_3,y_2}| = |V_{y_2,y_1}| = |V_{x_2,x_1}| = b - a + 1$, $|V_{y_3,x_1x_2}| = |V_{y_2,x_1^2}| = b - a$, $|V_{y_1,x_2}| = 2a - b$ and

$$\begin{aligned} D(V_{y_3,1}) &= -V_{x_1,1}V_{y_3,x_1} + V_{x_2,1}V_{w_1} - 2V_{x_2,1}V_{y_2,x_1} + V_{y_1,1}V_{y_3,y_1} + V_{y_2,1}V_{y_3,y_2} \\ D(V_{y_2,1}) &= -V_{x_1,1}V_{y_2,x_1} - V_{x_2,1}V_{y_2,x_2} + V_{y_1,1}V_{y_2,y_1} \\ D(V_{y_3,x_1}) &= V_{x_2,x_1}V_{w_1} - 2V_{x_2,x_1}V_{y_2,x_1} - V_{w_2}V_{y_3,y_1} + 2V_{y_2,x_2}V_{y_3,y_1} \\ &\quad - 2V_{x_1,1}V_{y_3,x_1^2} + V_{y_2,x_1}V_{y_3,y_2} - V_{x_2,1}V_{y_3,x_1x_2} \\ D(V_{y_1,1}) &= V_{x_1,1}V_{w_2} - 2V_{x_1,1}V_{y_2,x_2} - V_{x_2,1}V_{y_1,x_2}, \quad D(V_{x_2,1}) = -V_{x_1,1}V_{x_2,x_1} \\ D(V_{w_1}) &= -2V_{x_2,x_1}V_{y_2,x_2} - 4V_{x_1,1}V_{y_2,x_1^2} + 4V_{y_2,x_2}V_{y_2,y_1} - 2V_{w_2}V_{y_2,y_1} \\ &\quad - V_{y_1,x_2}V_{y_3,y_1} - V_{y_2,x_2}V_{y_3,y_2} + V_{x_1,1}V_{y_3,x_1x_2} \\ D(V_{y_2,x_1}) &= V_{x_2,1} - V_{x_2,x_1}V_{y_2,x_2} - V_{w_2}V_{y_2,y_1} + 2V_{y_2,x_2}V_{y_2,y_1} - 2V_{x_1,1}V_{y_2,x_1^2} \\ D(V_{y_3,y_1}) &= -V_{y_2,y_1}V_{y_3,y_2}, \quad D(V_{x_1,1}) = 0 \\ D(V_{y_3,x_1^2}) &= -V_{y_3,y_1} - V_{x_2,x_1}V_{y_3,x_1x_2} + V_{y_2,x_1^2}V_{y_3,y_2} \\ D(V_{y_2,x_2}) &= V_{x_1,1} + V_{y_1,x_2}V_{y_2,y_1}, \quad D(V_{w_2}) = 2V_{y_1,x_2}V_{y_2,y_1} + V_{x_2,x_1}V_{y_1,x_2} \\ D(V_{y_3,y_2}) &= D(V_{y_2,y_1}) = D(V_{x_2,x_1}) = 0 \\ D(V_{y_3,x_1x_2}) &= -V_{y_3,y_2} + 2V_{x_2,x_1}, \quad D(V_{y_2,x_1^2}) = -V_{y_2,y_1} + V_{x_2,x_1}, \quad D(V_{y_1,x_2}) = 0. \end{aligned}$$

Second, consider Case 2 ($\frac{3}{2}a < b < 2a$).

d_1 is given as follows:

$$\begin{aligned} d_1(V_{y_3,1}) &= d_1(V_{y_2,1}) = d_1(V_{y_3,x_1}) = d_1(V_{y_1,1}) = d_1(V_{x_2,1}) = 0, \quad d_1(V_{y_3,x_2}) = 2V_{x_2,1} \\ d_1(V_{y_2,x_1}) &= V_{x_2,1}, \quad d_1(V_{y_3,y_1}) = d_1(V_{x_1,1}) = 0, \quad d_1(V_{y_3,x_1^2}) = -V_{y_3,y_1} \\ d_1(V_{y_2,x_2}) &= V_{x_1,1}, \quad d_1(V_{y_1,x_1}) = 2V_{x_1,1}, \quad d_1(V_{y_3,y_2}) = 0, \quad d_1(V_{y_2,y_1}) = 0, \quad d_1(V_{x_2,x_1}) = 0 \\ d_1(V_{y_3,x_1x_2}) &= -V_{y_3,y_2} + 2V_{x_2,x_1}, \quad d_1(V_{y_2,x_1^2}) = -V_{y_2,y_1} + V_{x_2,x_1}, \quad d_1(V_{y_1,x_2}) = 0 \\ d_1(V_{y_3,x_1^3}) &= -V_{y_3,x_1y_1}, \quad d_1(V_{y_3,x_1y_1}) = 0. \end{aligned}$$

d_2 is given as follows:

$$\begin{aligned} d_2(V_{y_3,1}) &= -V_{x_1,1}V_{y_3,x_1} - V_{x_2,1}V_{y_3,x_2} + V_{y_1,1}V_{y_3,y_1} + V_{y_2,1}V_{y_3,y_2} \\ d_2(V_{y_2,1}) &= -V_{x_1,1}V_{y_2,x_1} - V_{x_2,1}V_{y_2,x_2} + V_{y_1,1}V_{y_2,y_1} \\ d_2(V_{y_3,x_1}) &= -V_{x_2,x_1}V_{y_3,x_2} + V_{y_1,x_1}V_{y_3,y_1} - 2V_{x_1,1}V_{y_3,x_1^2} \\ &\quad + V_{y_2,x_1}V_{y_3,y_2} - V_{x_2,1}V_{y_3,x_1x_2} + V_{y_1,1}V_{y_3,x_1y_1} \\ d_2(V_{y_1,1}) &= -V_{x_1,1}V_{y_1,x_1} - V_{x_2,1}V_{y_1,x_2}, \quad d_2(V_{x_2,1}) = -V_{x_1,1}V_{x_2,x_1} \\ d_2(V_{y_3,x_2}) &= V_{y_1,x_2}V_{y_3,y_1} + V_{y_2,x_2}V_{y_3,y_2} - V_{x_1,1}V_{y_3,x_1x_2} \\ d_2(V_{y_2,x_1}) &= -V_{x_2,x_1}V_{y_2,x_2} + V_{y_1,x_1}V_{y_2,y_1} - 2V_{x_1,1}V_{y_2,x_1^2} \\ d_2(V_{y_3,y_1}) &= -V_{y_2,y_1}V_{y_3,y_2} - V_{x_1,1}V_{y_3,x_1y_1}, \quad d_2(V_{x_1,1}) = 0 \\ d_2(V_{y_3,x_1^2}) &= -V_{x_2,x_1}V_{y_3,x_1x_2} + V_{y_2,x_1^2}V_{y_3,y_2} - 3V_{x_1,1}V_{y_3,x_1^3} + V_{y_1,x_1}V_{y_3,x_1y_1} \\ d_2(V_{y_2,x_2}) &= V_{y_1,x_2}V_{y_2,y_1}, \quad d_2(V_{y_1,x_1}) = -V_{x_2,x_1}V_{y_1,x_2} \\ d_2(V_{y_3,y_2}) &= d_2(V_{y_2,y_1}) = d_2(V_{x_2,x_1}) = 0, \quad d_2(V_{y_3,x_1x_2}) = V_{y_1,x_2}V_{y_3,x_1y_1} \\ d_2(V_{y_2,x_1^2}) &= d_2(V_{y_1,x_2}) = d_2(V_{y_3,x_1^3}) = d_2(V_{y_3,x_1y_1}) = 0. \end{aligned}$$

Let $V_{w_1} = 2V_{y_2,x_1} - V_{y_3,x_2}$ and $V_{w_2} = 2V_{y_2,x_2} - V_{y_1,x_1}$. Then

$$C^*(\text{Der}M(X_2)) = (\Lambda(V_{y_3,1}, V_{y_2,1}, V_{y_3,x_1}, V_{y_1,1}, V_{x_2,1}, V_{w_1}, V_{y_2,x_1}, V_{y_3,y_1}, V_{x_1,1}, V_{y_3,x_1^2}, V_{y_2,x_2}, V_{w_2}, V_{y_3,y_2}, V_{y_2,y_1}, V_{x_2,x_1}, V_{y_3,x_1x_2}, V_{y_2,x_1^2}, V_{y_1,x_2}, V_{y_3,x_1y_1}, V_{y_3,x_1^3}), D)$$

where $|V_{y_3,1}| = 2b$, $|V_{y_2,1}| = a + b$, $|V_{y_3,x_1}| = 2b - a$, $|V_{y_1,1}| = 2a$, $|V_{x_2,1}| = b + 1$, $|V_{w_1}| = |V_{y_2,x_1}| = b$, $|V_{y_3,y_1}| = 2b - 2a + 1$, $|V_{x_1,1}| = a + 1$, $|V_{y_3,x_1^2}| = 2b - 2a$, $|V_{y_2,x_2}| = |V_{w_2}| = a$, $|V_{y_3,y_2}| = |V_{y_2,y_1}| = |V_{x_2,x_1}| = b - a + 1$, $|V_{y_3,x_1x_2}| = |V_{y_2,x_1^2}| = b - a$, $|V_{y_1,x_2}| = 2a - b$, $|V_{y_3,x_1y_1}| = 2b - 3a + 1$, $|V_{y_3,x_1^3}| = 2b - 3a$ and

$$\begin{aligned} D(V_{y_3,1}) &= -V_{x_1,1}V_{y_3,x_1} + V_{x_2,1}V_{w_1} - 2V_{x_2,1}V_{y_2,x_1} + V_{y_1,1}V_{y_3,y_1} + V_{y_2,1}V_{y_3,y_2} \\ D(V_{y_2,1}) &= -V_{x_1,1}V_{y_2,x_1} - V_{x_2,1}V_{y_2,x_2} + V_{y_1,1}V_{y_2,y_1} \\ D(V_{y_3,x_1}) &= +V_{x_2,x_1}V_{w_1} - 2V_{x_2,x_1}V_{y_2,x_1} - V_{w_2}V_{y_3,y_1} + 2V_{y_2,x_2}V_{y_3,y_1} - 2V_{x_1,1}V_{y_3,x_1^2} \\ &\quad + V_{y_2,x_1}V_{y_3,y_2} - V_{x_2,1}V_{y_3,x_1x_2} + V_{y_1,1}V_{y_3,x_1y_1} \\ D(V_{y_1,1}) &= +V_{x_1,1}V_{w_2} - 2V_{x_1,1}V_{y_2,x_2} - V_{x_2,1}V_{y_1,x_2}, \quad D(V_{x_2,1}) = -V_{x_1,1}V_{x_2,x_1} \\ D(V_{w_1}) &= -2V_{x_2,x_1}V_{y_2,x_2} - 4V_{x_1,1}V_{y_2,x_1^2} + 4V_{y_2,x_2}V_{y_2,y_1} - 2V_{w_2}V_{y_2,y_1} - V_{y_1,x_2}V_{y_3,y_1} \\ &\quad - V_{y_2,x_2}V_{y_3,y_2} + V_{x_1,1}V_{y_3,x_1x_2} \\ D(V_{y_2,x_1}) &= +V_{x_2,1} - V_{x_2,x_1}V_{y_2,x_2} - V_{w_2}V_{y_2,y_1} + 2V_{y_2,x_2}V_{y_2,y_1} - 2V_{x_1,1}V_{y_2,x_1^2} \\ D(V_{y_3,y_1}) &= -V_{y_2,y_1}V_{y_3,y_2} - V_{x_1,1}V_{y_3,x_1y_1}, \quad D(V_{x_1,1}) = 0 \\ D(V_{y_3,x_1^2}) &= -V_{y_3,y_1} - V_{x_2,x_1}V_{y_3,x_1x_2} + V_{y_2,x_1^2}V_{y_3,y_2} - 3V_{x_1,1}V_{y_3,x_1^3} - V_{w_2}V_{y_3,x_1y_1} \\ &\quad + 2V_{y_2,x_2}V_{y_3,x_1y_1}, \quad D(V_{y_2,x_2}) = +V_{x_1,1} + V_{y_1,x_2}V_{y_2,y_1} \\ D(V_{w_2}) &= +2V_{y_1,x_2}V_{y_2,y_1} + V_{x_2,x_1}V_{y_1,x_2}, \quad D(V_{y_3,y_2}) = D(V_{y_2,y_1}) = D(V_{x_2,x_1}) = 0 \\ D(V_{y_3,x_1x_2}) &= -V_{y_3,y_2} + 2V_{x_2,x_1} + V_{y_1,x_2}V_{y_3,x_1y_1} \\ D(V_{y_2,x_1^2}) &= -V_{y_2,y_1} + V_{x_2,x_1} \\ D(V_{y_1,x_2}) &= 0, \quad D(V_{y_3,x_1^3}) = -V_{y_3,x_1y_1}, \quad D(V_{y_3,x_1y_1}) = 0. \end{aligned}$$

In both cases, we obtain the minimal model of this proposition. $\square$

For a connected closed sub-Lie group H of a compact connected Lie group G with inclusion $j : H \rightarrow G$, there is the induced map $Bj : BH \rightarrow BG$ between the classifying spaces. It induces a map $\psi_j : M(BG) = (\Lambda V_{BG}, 0) = (\mathbb{Q}[z_1, \dots, z_k], 0) \rightarrow (\Lambda V_{BH}, 0) = M(BH)$ between the models. Here, $|z_i|$ are even and $\text{rank} G = k$. Let $V_G^n = V_{BG}^{n+1}$ by corresponding y_i to z_i with $|y_i| = |z_i| - 1$.

Proposition 4.6 [4, Proposition 15.16] *The (non-minimal) model of G/H is given as $(\Lambda V_{BH} \otimes \Lambda V_G, d)$, where $dx_i = 0$ and $dy_i = \psi_j(z_i)$ for $i = 1, \dots, k$.*

Recall the cohomology induced maps $\psi_{j_1} : H^*(BSp(6); \mathbb{Q}) \rightarrow H^*(BSU(6); \mathbb{Q})$ of the inclusion $j_1 : SU(6) \rightarrow Sp(6)$ is given as

$$\psi_{j_1} : \mathbb{Q}[q_1, q_2, q_3, q_4, q_5, q_6] \rightarrow \mathbb{Q}[c_2, c_3, c_4, c_5, c_6]$$

where $|q_i| = 4i$, $|c_i| = 2i$, $\psi_{j_1}(q_1) = c_2$, $\psi_{j_1}(q_2) = c_2^2 + c_4$, $\psi_{j_1}(q_3) = 2c_2c_4 + c_3^2 + c_6$, $\psi_{j_1}(q_4) = 2c_2c_6 + 2c_3c_5 + c_4^2$, $\psi_{j_1}(q_5) = 2c_4c_6 + c_5^2$, $\psi_{j_1}(q_6) = c_6^2$, and $\psi_{j_2} : H^*(BSO(12); \mathbb{Q}) \rightarrow H^*(BSU(6); \mathbb{Q})$ for $j_2 : SU(6) \rightarrow SO(12)$ is given as

$$\psi_{j_2} : \mathbb{Q}[p_1, p_2, p_3, p_4, p_5, \chi] \rightarrow \mathbb{Q}[c_2, c_3, c_4, c_5, c_6]$$

where $|p_i| = 4i$, $|\chi| = 12$, $|c_i| = 2i$, $\psi_{j_2}(p_1) = c_2$, $\psi_{j_2}(p_2) = c_2^2 + c_4$, $\psi_{j_2}(p_3) = 2c_2c_4 + c_3^2 + c_6$, $\psi_{j_2}(p_4) = 2c_2c_6 + 2c_3c_5 + c_4^2$, $\psi_{j_2}(p_5) = 2c_4c_6 + c_5^2$, $\psi_{j_2}(\chi) = c_6$ [13, Theorem 5.8, Lemma 5.15]. Here, q_i is the i th symplectic Pontrjagin class, c_i is the i th Chern class, and p_i is the i th Pontrjagin class. Refer [8, 9] for the geometrical realizations of some elements of $H^*(\text{Baut}_1 G/H; \mathbb{Q})$.

- Proof of Theorem 1.6* (1) Let $X = SU(2) \times SU(2) \times SU(2)/T^2$. Then, $M(X) \cong (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = |x_2| = 2$, $|y_1| = |y_2| = |y_3| = 3$, $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_1x_2$, and $dy_3 = x_2^2$ from [15]. Then, it satisfies neither (I) or (II) of Theorem 3.2 as [14, Example 1.15 (1)].
- (2) Let $X = SU(n+3)/SU(n) \times SU(3)$. Then, we have done from Theorem 4.3 using Theorem 3.2. Indeed, when $n > 3$, it does not satisfy (I) since $|x_i| + |y_j| \geq |y_3|$ for $i = 1, 2$ and $j = 1, 2$ and it does not satisfy (II), since dy_1 contains x_2 . When $n = 3$, it satisfies (II) for $k = 1$ (see [14, Example 1.14 (2)]).
- (3) Let $X = Sp(6)/SU(6)$. Then, $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = 6$, $|x_2| = 10$, $|y_1| = 15$, $|y_2| = 19$, $|y_3| = 23$, $dx_1 = dx_2 = 0$, $dy_1 = x_1x_2$, $dy_2 = x_2^2$, and $dy_3 = x_1^4$ from Proposition 4.6. Then, it holds from Theorem 3.2, since it does not satisfy (I), since $|x_2y_1 - x_1y_2| = 25 \geq |y_3|$ and does not satisfy (II), since $dy_1 \neq x^l$ for any $l > 0$.
- (4) Let $X = SO(12)/SU(6)$ or $X = Q(5)/SU(5)$. Then, $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = 6$, $|x_2| = 10$, $|y_1| = 11$, $|y_2| = 15$, $|y_3| = 19$, $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_1x_2$, and $dy_3 = x_2^2$ from Proposition 4.6 or [6, p. 489], respectively. Then, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is not a polynomial algebra, since it satisfies (II) of Theorem 3.2. Furthermore, $M(\text{Baut}_1 X) \cong M(\text{Baut}_1 SU(6)/SU(3) \times SU(3))$ as differential (non-graded) algebras from Proposition 4.5. Here, X is a space of Case 2 from $(a, b) = (6, 10)$ and $SU(6)/SU(3) \times SU(3)$ is a space of Case 1 from $(a, b) = (4, 6)$ in the proof of Proposition 4.5. $\square$

From Theorem 1.6(2) and Proposition 3.1(2), we have

Corollary 4.7 When $X = SU(n+3)/SU(n) \times SU(3)$ for $n > 3$, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is given as the following polynomial algebras:

- (1) $\mathbb{Q}[u_1, u_2, \dots, u_{(m+3)^2-6}]$ ($n = 2m + 2$) and
- (2) $\mathbb{Q}[u_1, u_2, \dots, u_{(m+3)(m+4)-6}]$ ($n = 2m + 3$)

for $m \geq 1$. In particular, the numbers of generators of degree $2, 4, 6, \dots, 2l$ are

- (1) $m, m, m+1, \dots, m - [l/2] + 3$ ($4 \leq l \leq 2m + 5$) and
- (2) $m, m+1, m+1, \dots, m - [(l-1)/2] + 3$ ($4 \leq l \leq 2m + 6$),

respectively.

Example 4.8 Let $X = SU(n+3)/SU(n) \times SU(3)$ for $n = 3, 4$.

When $n = 3$, the Serre spectral sequence of any fibration with fibre X does not degenerate at E_2 term. Indeed, from Theorem 4.3, $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = 4$, $|x_2| = 6$, $|y_1| = 7$, $|y_2| = 9$, $|y_3| = 11$, $dx_1 = dx_2 = 0$, $dy_1 = x_1^2$, $dy_2 = x_1x_2$, and $dy_3 = x_2^2$. There is a rational fibration $X \xrightarrow{i} E \rightarrow S^3$ given by

$$(\Lambda z, 0) \rightarrow (\Lambda(z, x_1, x_2, y_1, y_2, y_3), D) \rightarrow (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$$

where $|z| = 3$, $Dx_1 = 0$, $Dx_2 = zx_1$, $Dy_1 = dy_1$, $Dy_2 = dy_2 + zy_1$, and $Dy_3 = dy_3 + 2zy_2$. Then, $i^* : H^*(E; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ is not an epimorphism since $[x_2] \notin H^*(E; \mathbb{Q})$.

When $n = 4$, $H^*(\text{Baut}_1 X; \mathbb{Q})$ is a polynomial algebra from Theorem 1.6. However, it does not too. Indeed, from Theorem 4.3, $M(X) = (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$ with $|x_1| = 4$, $|x_2| = 6$, $|y_1| = 9$, $|y_2| = 11$, $|y_3| = 13$, $dx_1 = dx_2 = 0$, $dy_1 = x_1x_2$, $dy_2 = x_1^3 - x_2^2$, and $dy_3 = x_1^2x_2$. There is a rational fibration $X \xrightarrow{i} E \rightarrow S^{14}$ given by

$$(\Lambda(z, v), d_Y) \rightarrow (\Lambda(z, v, x_1, x_2, y_1, y_2, y_3), D) \rightarrow (\Lambda(x_1, x_2, y_1, y_2, y_3), d)$$

where $|z| = 14$, $|v| = 27$, $d_Y v = z^2$, $d_Y z = Dx_1 = Dx_2 = 0$, $Dy_1 = dy_1$, $Dy_2 = dy_2$, and $Dy_3 = dy_3 + z$. Then, $i^* : H^*(E; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ is not an epimorphism, since the element $[x_1y_1 - y_3]$ of $H^*(X; \mathbb{Q})$ is not in $H^*(E; \mathbb{Q})$.

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