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On weakly 3-permutable subgroups of finite groups II

Received: 6 February 2015 / Accepted: 28 May 2015 / Published online: 13 June 2015

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Abstract Let G be a finite group. We say that $\mathfrak{F}$ is a complete set of Sylow subgroups of G if for each prime p dividing the order of G , $\mathfrak{F}$ contains exactly one Sylow p -subgroup of G , G_p say. A subgroup of G is said to be $\mathfrak{F}$ -permutable in G if it permutes with every member of $\mathfrak{F}$. A subgroup H of G is said to be weakly $\mathfrak{F}$ -permutable in G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_3$, where H_3 is the subgroup of H generated by all those subgroups of H which are $\mathfrak{F}$ -permutable in G . In this paper, we prove that G is supersolvable if the maximal subgroups of $G_p \cap F^*(G)$ are weakly $\mathfrak{F}$ -permutable in G , for every $G_p \in \mathfrak{F}$, where $F^*(G)$ is the generalized Fitting subgroup of G . Also, we prove that if $\mathfrak{F}$ is a saturated formation containing the class of all supersolvable groups, then $G \in \mathfrak{F}$ if and only if there is a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are weakly $\mathfrak{F}$ -permutable in G , for every $G_p \in \mathfrak{F}$.

Mathematics Subject Classification 20D10 · 20D15 · 20D20 · 20F16

المخلص

لتكن G زمرة منتهية. نقول إن $\mathfrak{F}$ مجموعة تامة من زمر سيلو الجزئية من G إذا احتوت $\mathfrak{F}$ لكل قاسم أولي p من قواسم مرتبة G على زمرة سيلو جزئية واحدة بالضبط من G ، ولتكن G_p تسمى زمرة جزئية من G زمرة $\mathfrak{F}$ -قابلة للتبديل في G إذا كانت إبدالاً مع كل عنصر من عناصر $\mathfrak{F}$. تسمى الزمرة الجزئية H من G زمرة $\mathfrak{F}$ -قابلة للتبديل بشكل ضعيف في G إذا وجدت زمرة جزئية K ناظمية جزئياً من G بحيث يكون $G = HK$ و $H \cap K \leq H_3$ ، حيث H_3 الزمرة الجزئية من H المولدة بواسطة كل الزمر الجزئية من H التي تكون $\mathfrak{F}$ -قابلة للإبدال في G . في هذه الورقة، نثبت أن G فوق حلولة إذا كانت الزمر الجزئية الأعظمية من $G_p \cap F^*(G)$ من الزمر $\mathfrak{F}$ -القابلة للإبدال بشكل ضعيف في G لكل $G_p \in \mathfrak{F}$ ، حيث $F^*(G)$ زمرة فتنغ الجزئية المعممة من G . نثبت أيضاً أنه إذا كان $\mathfrak{F}$ تكويناً مشبعاً يحتوي صف الزمر فوق الحلولة، فإن $G \in \mathfrak{F}$ إذا وجدت فقط إذا وجدت زمرة جزئية ناظمية H من G بحيث تكون $G/H \in \mathfrak{F}$ وكانت الزمر الجزئية الأعظمية من $G_p \cap F^*(H)$ هي $\mathfrak{F}$ -قابلة للإبدال بشكل ضعيف في G لكل $G_p \in \mathfrak{F}$.

1 Introduction and statement of results

All groups considered in the sequel will be finite. Most of the notation is standard and can be found in Ballester-Bolinches et al. [2] and Doerk and Hawkes [3]. In addition, $\pi(G)$ denotes the set of distinct primes dividing $|G|$ and G_p is a Sylow p -subgroup of the group G for some prime $p \in \pi(G)$.

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Recall that a subgroup H of a group G is said to be S -permutable in G if it permutes with every Sylow subgroup of G . This concept was introduced by Kegel [7], who called these subgroups S -quasinormal, and has been studied extensively by many authors. Asaad and Heliel [1] generalized S -permutability property by requiring permutability only with the members of a complete set of Sylow subgroups. We say that $\mathfrak{S}$ is a complete set of Sylow subgroups of G if for each prime p dividing $|G|$, $\mathfrak{S}$ contains exactly one Sylow p -subgroup of G . A subgroup H of G is said to be $\mathfrak{S}$ -permutable in G if H permutes with every member of $\mathfrak{S}$. Following Wang [11], we say that a subgroup H of a group G is c -normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = \text{Core}_G(H)$ is the largest normal subgroup of G contained in H . One can easily find groups with $\mathfrak{S}$ -permutable subgroups that are not c -normal and conversely there are also groups with c -normal subgroups that are not $\mathfrak{S}$ -permutable subgroups; see Examples 1, 2 and 3 in Heliel et al. [4]. In fact, there is no inclusion relationship between c -normality and $\mathfrak{S}$ -permutability. Consequently the authors in [4] unified and generalized both of $\mathfrak{S}$ -permutability and c -normality concepts as follows: a subgroup H of a group G is said to be weakly $\mathfrak{S}$ -permutable in G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{\mathfrak{S}}$, where $H_{\mathfrak{S}}$ is the subgroup of H generated by all those subgroups of H which are $\mathfrak{S}$ -permutable in G . Using this new subgroup embedding property, the authors in [4] studied the structure of a group G when all maximal subgroups of certain or every member of a complete set of Sylow subgroups of some normal subgroup of G are weakly $\mathfrak{S}$ -permutable in G . There they achieved results unified and generalized several recent results in the literature. The present paper may be viewed as a continuation of [4]; to be more precise, the following results have been proved in [4]:

Theorem 1.1 *Let $\mathfrak{S}$ be a complete set of Sylow subgroups of a group G and let p be the smallest prime dividing $|G|$. If the maximal subgroups of $G_p \in \mathfrak{S}$ are weakly $\mathfrak{S}$ -permutable in G , then G is p -nilpotent.*

Theorem 1.2 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{A}$ and let $\mathfrak{S}$ be a complete set of Sylow subgroups of a group G . Then the following are equivalent:*

- (a) $G \in \mathfrak{F}$.
- (b) *There is a solvable normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of the Fitting subgroup $F(H)$ are weakly $\mathfrak{S}$ -permutable in G .*

The main object here is to take the above-mentioned results further by proving:

Theorem 1.3 *Let $\mathfrak{S}$ be a complete set of Sylow subgroups of a group G . If the maximal subgroups of $G_p \cap F^*(G)$ are weakly $\mathfrak{S}$ -permutable in G , for all $G_p \in \mathfrak{S}$, then G is supersolvable.*

Theorem 1.4 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{A}$ and let $\mathfrak{S}$ be a complete set of Sylow subgroups of a group G . Then the following are equivalent:*

- (a) $G \in \mathfrak{F}$.
- (b) *There is a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are weakly $\mathfrak{S}$ -permutable in G , for all $G_p \in \mathfrak{S}$.*

Theorems 1.3 and 1.4 improve and extend some well-known results in the literature; see Corollaries 3.1, 3.2, 3.3, 3.4 and 3.5.

Recall that for any group G , the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every chief factor of G . $F^*(G)$ is an important characteristic subgroup of G and it is a natural generalization of the Fitting subgroup $F(G)$. By [5, X 13], $F^*(G) \neq 1$ if $G \neq 1$. The basic properties of $F^*(G)$ can be found in [5, X 13]. The reader is referred to [3] for basic properties and results about saturated formations.

2 Known results

Lemma 2.1 *Let $\mathfrak{S}$ be a complete set of Sylow subgroups of a group G . If the maximal subgroups of G_p are weakly $\mathfrak{S}$ -permutable in G , for all $G_p \in \mathfrak{S}$, then G is supersolvable.*

Proof See [4, Corollary 3.4]. □

Lemma 2.2 *Let G be a group. Then:*

- (a) $F^*(G) = F(G)E(G)$ and $[F(G), E(G)] = 1$, where $E(G)$ is the layer subgroup of G .

- (b) $E(G)$ is a perfect quasinilpotent characteristic subgroup of G .
- (c) If N is a normal subgroup of G , then $F^*(N) \leq F^*(G)$.
- (d) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.
- (e) Suppose that P is a normal subgroup of G contained in $O_p(G)$ for some prime p , then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.
- (f) Suppose that K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.

Proof For (a), (b) and (c), see [5, p. 128, Definition 13.14 and p. 127, Corollary 13.11(c)]. For the first part of (d), see [5, p. 127, Corollary 13.11(a)], and the rest comes from the fact that $F^*(G) = F(G)E(G)$ by (a) and in case $F^*(G)$ is solvable, we have that $E(G)$ is a perfect solvable group by (b) which implies that $E(G) = 1$ and hence $F^*(G) = F(G)$.

For (e) and (f), see [10, Lemma 2.3(6) and (7)]. $\square$

Lemma 2.3 Let H and N be subgroups of a group G such that N is normal in G and let $\mathfrak{S}$ be a complete set of Sylow subgroups of G . Then:

- (a) If $H \leq N$ and H is weakly $\mathfrak{S}$ -permutable in G , then H is weakly $\mathfrak{S} \cap N$ -permutable in N .
- (b) If $N \leq H$, then H is weakly $\mathfrak{S}$ -permutable in G if and only if H/N is weakly $\mathfrak{S}N/N$ -permutable in G/N .
- (c) If $(|H|, |N|) = 1$ and H is weakly $\mathfrak{S}$ -permutable in G , then HN/N is weakly $\mathfrak{S}N/N$ -permutable in G/N .
- (d) If H is a p -subgroup of G for some prime p such that H is weakly $\mathfrak{S}$ -permutable in G but it is not $\mathfrak{S}$ -permutable in G , then there exists a normal subgroup M of G with $|G : M| = p$ and $G = HM$.

Proof See [4, Lemma 2.3]. $\square$

Lemma 2.4 Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{A}$ and let $\mathfrak{S}$ be a complete set of Sylow subgroups of a group G . Then the following are equivalent:

- (a) $G \in \mathfrak{F}$.
- (b) There is a solvable normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F(H)$ are weakly $\mathfrak{S}$ -permutable in G .

Proof See [4, Theorem 1.6]. $\square$

Lemma 2.5 Let N be a normal subgroup of a group G and let $\mathfrak{S}$ be a complete set of Sylow subgroups of G . Suppose that P is a p -subgroup of G for some prime p . Then:

- (a) If M/N is a maximal subgroup of PN/N , then $M = (M \cap P)N$, where $M \cap P$ is a maximal subgroup of P .
- (b) If $(|P|, |N|) = 1$ and the maximal subgroups of P are weakly $\mathfrak{S}$ -permutable in G , then the maximal subgroups of PN/N are weakly $\mathfrak{S}N/N$ -permutable in G/N .

Proof See [4, Lemma 2.4]. $\square$

Lemma 2.6 Let N be a normal nilpotent subgroup of a group G and let $\mathfrak{S}$ be a complete set of Sylow subgroups of G . Suppose that P is a normal p -subgroup of G , for some prime p , with $P \leq N$. If the maximal subgroups of the Sylow subgroups of N are weakly $\mathfrak{S}$ -permutable in G , then the maximal subgroups of the Sylow subgroups of N/P are weakly $\mathfrak{S}P/P$ -permutable in G/P . In other words, if the maximal subgroups of the members of $\mathfrak{S} \cap N$ are weakly $\mathfrak{S}$ -permutable in G , then the maximal subgroups of the members of $(\mathfrak{S}P/P) \cap (N/P)$ are weakly $\mathfrak{S}P/P$ -permutable in G/P .

Proof Let Q be a Sylow q -subgroup of N . Since Q is characteristic in N and $N \trianglelefteq G$, we have $Q \trianglelefteq G$. Assume that $q \neq p$. As $(|Q|, |P|) = 1$ and the maximal subgroups of Q are weakly $\mathfrak{S}$ -permutable in G , we have, by Lemma 2.5(b), that the maximal subgroups of QP/P are weakly $\mathfrak{S}P/P$ -permutable in G/P . Hence, we may assume that $q = p$ and so $P \leq Q$. Let M/P be a maximal subgroup of Q/P . By Lemma 2.5(a), M is a maximal subgroup of Q . The hypothesis and Lemma 2.3(b) imply that M/P is weakly $\mathfrak{S}P/P$ -permutable in G/P . Therefore, the maximal subgroups of Q/P are weakly $\mathfrak{S}P/P$ -permutable in G/P . Thus, the maximal subgroups of the Sylow subgroups of N/P are weakly $\mathfrak{S}P/P$ -permutable in G/P . Clearly, $\mathfrak{S} \cap N$ is the set of all Sylow subgroups of N as N is nilpotent. Since $\mathfrak{S}P/P$ is a complete set of Sylow subgroups of G/P and N/P is a normal nilpotent subgroup of G/P , it follows that $(\mathfrak{S}P/P) \cap (N/P)$ is the set of all Sylow subgroups of N/P . Thus, the maximal subgroups of $(\mathfrak{S}P/P) \cap (N/P)$ are weakly $\mathfrak{S}P/P$ -permutable in G/P . $\square$

Lemma 2.7 Let H be a subgroup of a group G and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G . Then $H_{\mathfrak{Z}}$ is $\mathfrak{Z}$ -permutable in G and $H_G \leq H_{\mathfrak{Z}}$.

Proof See [4, Lemma 2.2(a)]. $\square$

Lemma 2.8 Let H and N be subgroups of a group G such that N is normal in G , and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G . If H is $\mathfrak{Z}$ -permutable in G , then $H \cap N$ is $\mathfrak{Z}$ -permutable in G .

Proof See [4, Lemma 2.1(e)]. $\square$

Lemma 2.9 Let G be a group. Assume that N is a normal subgroup of G ($N \neq 1$) and $N \cap \Phi(G) = 1$. Then the Fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroups of G which are contained in $F(N)$.

Proof See [9, Lemma 2.6]. $\square$

3 Proofs

Proof of Theorem 1.3. Assume that the result is false and let G be a counterexample of minimal order. Then the following statements about G are true:

(1) $F^*(G) \neq G$.

Suppose that $F^*(G) = G$. Since the maximal subgroups of G_p are weakly $\mathfrak{Z}$ -permutable in G , for all $G_p \in \mathfrak{Z}$, it follows, by Lemma 2.1, that G is supersolvable, a contradiction. Thus, $F^*(G) \neq G$.

(2) Every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

By (1), $F^*(G) \neq G$ and so there exists a proper normal subgroup N of G containing $F^*(G)$. Since $F^*(G) \trianglelefteq N \trianglelefteq G$, it follows, by Lemma 2.2(c), that $F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$. By Lemma 2.2(d), $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$ and hence $F^*(G) = F^*(N)$. Lemma 2.3(a) implies that the maximal subgroups of $G_p \cap F^*(N)$ are weakly $\mathfrak{Z} \cap N$ -permutable in N , for all $G_p \cap N \in \mathfrak{Z} \cap N$. Therefore, N is supersolvable by the minimal choice of G . Thus, every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

(3) $F^*(G) = F(G)$.

Since $F^*(G) \neq G$ by (1), it follows that $F^*(G)$ is supersolvable by (2). Thus, $F^*(G) = F(G)$ by Lemma 2.2(d).

(4) G has no normal subgroup of prime order and, therefore, $Z(G) = 1$.

Let K be a normal subgroup of G of prime order. Assume that $C_G(K)$ is a proper subgroup of G . By [3, p. 36, Theorem 10.6(b)] and (3), $F^*(G) = F(G) \leq C_G(K)$. Therefore, $C_G(K)$ is supersolvable by (2) and, since $G/C_G(K)$ is isomorphic to a subgroup of the cyclic group $\text{Aut}(K)$, it follows that G is solvable. Thus, G is supersolvable by Lemma 2.4, a contradiction. Consequently we may assume that $C_G(K) = G$ and so $K \leq Z(G)$. Lemma 2.2(f) and (3) imply that $F^*(G/K) = F^*(G)/K = F(G)/K$. By Lemma 2.6, the maximal subgroups of the Sylow subgroups of $F^*(G/K) = F(G)/K$ are weakly $\mathfrak{Z}K/K$ -permutable in G/K . Therefore, G/K satisfies the hypothesis of the theorem and hence G/K is supersolvable by the minimal choice of G . This implies that G is supersolvable as K has a prime order, a contradiction. Thus, G has no normal subgroup of prime order and it follows easily that $Z(G) = 1$.

Let P be a Sylow p -subgroup of $F(G)$, then the following holds:

(5) $\Phi(P) = 1$ and, therefore, P is abelian.

Suppose that $\Phi(P) \neq 1$. Clearly, as $\Phi(P)$ is characteristic in P and $P \trianglelefteq G$, $\Phi(P) \trianglelefteq G$. By Lemma 2.2(e) and (3), $F^*(G/\Phi(P)) = F^*(G)/\Phi(P) = F(G)/\Phi(P)$. Lemma 2.6 implies that the maximal subgroups of the Sylow subgroups of $F^*(G/\Phi(P)) = F(G)/\Phi(P)$ are weakly $\mathfrak{Z}\Phi(P)/\Phi(P)$ -permutable in $G/\Phi(P)$. Therefore, $G/\Phi(P)$ is supersolvable by the minimal choice of G . Since $\Phi(P) \leq \Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable. But since the class of supersolvable groups is a saturated formation, then G is supersolvable, a contradiction. Thus, $\Phi(P) = 1$ and it follows that P is abelian.

(6) $G = PO^p(G)$.

Suppose that $G \neq PO^p(G)$. Since $F^*(G) = F(G) \leq PO^p(G)$ by (3), it follows that $PO^p(G)$ is supersolvable by (2) and hence $O^p(G)$ is supersolvable. Therefore, G is solvable as $G/O^p(G)$ is a p -group. By Lemma 2.4, G is supersolvable, a contradiction. Thus, $G = PO^p(G)$.

(7) $U \cap O^p(G)$ is a normal subgroup of G for any maximal subgroup U of P .

Let x be an element of G and let $V = U^{x^{-1}}$. It is clear that V is a maximal subgroup of P as P is normal in G and $|V| = |U|$. By hypothesis, V is weakly $\mathfrak{Z}$ -permutable in G . Then there exists a subnormal subgroup K of G such that $G = VK$ and $V \cap K \leq V_{\mathfrak{Z}}$. Since P is abelian by (5) and $G = PK$, it follows that $P \cap K$ is a normal subgroup of G . Clearly, $O^p(G) \leq K$ as K is subnormal in G and $|G : K|$ is a power of p . By (6), $K = PO^p(G) \cap K = (P \cap K)O^p(G)$ and hence K is a normal subgroup of G . Therefore, $V_{\mathfrak{Z}}K$ is a subgroup of G . Obviously, $V \cap V_{\mathfrak{Z}}K = V_{\mathfrak{Z}}(V \cap K) = V_{\mathfrak{Z}}$, and hence $V \cap V_{\mathfrak{Z}}K$ is $\mathfrak{Z}$ -permutable in G by Lemma 2.7. Therefore, $V \cap O^p(G) = (V \cap V_{\mathfrak{Z}}K) \cap O^p(G)$ is $\mathfrak{Z}$ -permutable in G by Lemma 2.8. Thus, $(V \cap O^p(G))G_q = (U^{x^{-1}} \cap O^p(G))G_q$ is a subgroup of G , for all $x \in G$ and for all $G_q \in \mathfrak{Z}$. This implies that $(U \cap O^p(G))G_q^x = ((U^{x^{-1}})^x \cap O^p(G)^x)G_q^x = (U^{x^{-1}} \cap O^p(G))^x G_q^x = ((U^{x^{-1}} \cap O^p(G))G_q)^x$ is a subgroup of G , for all $x \in G$ and for all $G_q \in \mathfrak{Z}$. As the Sylow subgroups of G are conjugate, we have that $U \cap O^p(G)$ is an S -permutable in G . By [2, p. 17, Lemma 1.2.16], $O^p(G) \leq N_G(U \cap O^p(G))$. Also, $U \cap O^p(G)$ is a normal subgroup of P as P is abelian by (5). Consequently, $G = PO^p(G) \leq N_G(U \cap O^p(G))$ by (6) and hence $U \cap O^p(G)$ is normal in G . Thus, $U \cap O^p(G)$ is a normal subgroup of G for any maximal subgroup U of P .

(8) $P \cap O^p(G) \neq 1$.

Suppose that $P \cap O^p(G) = 1$. Since $[P, O^p(G)] \leq P \cap O^p(G) = 1$, it follows that $O^p(G) \leq C_G(P)$. Also, $P \leq C_G(P)$ as P is abelian by (5). Therefore, $G = PO^p(G) \leq C_G(P)$ by (6) and so $P \leq Z(G)$ which is a contradiction with (4). Thus, $P \cap O^p(G) \neq 1$.

(9) $P \cap \Phi(G) \neq 1$.

Suppose that $P \cap \Phi(G) = 1$. By Lemma 2.9, $P = L_1 \times L_2 \times \cdots \times L_t$, where each L_i is a minimal normal subgroup of G , for $i = 1, 2, \dots, t$. Since $P \cap O^p(G) \neq 1$ by (8) and $\Phi(P) = 1$ by (5), then there exists a maximal subgroup U of P such that $P = (P \cap O^p(G))U$. This implies that $|(P \cap O^p(G)) : (U \cap O^p(G))| = |P : U| = p$. By (7), $U \cap O^p(G)$ is a normal subgroup of G . Therefore, $(P \cap O^p(G))/(U \cap O^p(G))$ is a chief factor of G with order p . Consider P as an operator group with operator domain $\Omega = \text{Inn}(G)$. Then the series $1 \leq L_1 \leq L_1 L_2 \leq \cdots \leq L_1 L_2 \cdots L_t = P$ is an Ω -composition series of the Ω -group P . Since $(P \cap O^p(G))/(U \cap O^p(G))$ is an Ω -composition factor of P , it follows, by the Jordan–Hölder Theorem, that $(P \cap O^p(G))/(U \cap O^p(G)) \cong L_i$, for some i ($1 \leq i \leq t$). Therefore, $|L_i| = p$, for some i ($1 \leq i \leq t$), a contradiction with (4). Thus, $P \cap \Phi(G) \neq 1$.

Let L be a minimal subgroup of G contained in $P \cap \Phi(G)$, then the following holds:

(10) $F(G/L) = F(G)/L$.

Let $K/L = F(G/L)$. Then K is a normal nilpotent subgroup of G by [6, p. 270, Satz 3.5] and so $K \leq F(G)$. Thus, $F(G/L) = F(G)/L$.

(11) L is the unique minimal normal subgroup of G contained in P .

By Lemma 2.2(a) and (10), we have that $F^*(G/L) = F(G/L)E(G/L) = F(G)/L \cdot S/L$ and $[F(G)/L, S/L] = 1$, where $S/L = E(G/L)$ is the layer subgroup of G/L . Since $[F(G)/L, S/L] = 1$, then $[F(G), S] \leq L$. Let K be a minimal normal subgroup of G contained in P distinct from L . Then $[K, S] \leq K \cap L = 1$ and hence $S \leq C_G(K)$. By (4), $C_G(K)$ is a proper subgroup of G . Since $F^*(G) = F(G) \leq C_G(K)$ by (3) and [3, p. 36, Theorem 10.6(b)], we have that $C_G(K)$ is supersolvable by (2) and so S is solvable. Since S/L is a solvable perfect group by Lemma 2.2(b), then $S/L = 1$ and it follows that $F^*(G/L) = F(G)/L$. By Lemma 2.6, the maximal subgroups of the Sylow subgroups of $F^*(G/L) = F(G)/L$ are weakly $\mathfrak{Z}L/L$ -permutable in G/L . Thus, G/L satisfies the hypothesis of the theorem and hence G/L is supersolvable by the minimal choice of G . Since $L \leq \Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable. This implies that G is supersolvable as the class of supersolvable groups is a saturated formation, a contradiction. Thus, L is the unique minimal normal subgroup of G contained in P .

(12) The final contradiction.

By (5), there exists a maximal subgroup U of P such that $P = LU$. If $L \leq U \cap O^p(G)$, then $L \leq U$ and hence $P = U$, a contradiction. Therefore, L is not contained in $U \cap O^p(G)$. Since $U \cap O^p(G)$ is a normal subgroup of G by (7) and $L \not\leq U \cap O^p(G)$, it follows, by (11), that $U \cap O^p(G) = 1$. Clearly, $P = (P \cap O^p(G))U$ as $L \leq P \cap O^p(G)$ by (8) and (11). Thus, $|P \cap O^p(G)| = |P : U| = p$, a contradiction with (4). This completes the proof of the theorem. $\square$

Proof of Theorem 1.4. (a) $\Rightarrow$ (b). If $G \in \mathfrak{F}$, then (b) is true with $H = 1$.

(b) $\Rightarrow$ (a). By Lemma 2.3(a), the maximal subgroups of $G_p \cap F^*(H)$ are weakly $\mathfrak{Z} \cap H$ -permutable in H , for all $G_p \cap H \in \mathfrak{Z} \cap H$. Theorem 1.3 implies that H is supersolvable and hence $F^*(H) = F(H)$. Therefore, H is a solvable normal subgroup of G with $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F(H)$ are weakly $\mathfrak{Z}$ -permutable in G . By Lemma 2.4, $G \in \mathfrak{F}$. This completes the proof of the theorem. $\square$

The following well-known results in the literature are immediate consequences of Theorems 1.3 and 1.4.

Corollary 3.1 [8, Theorem 3.1] *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G and let H be a normal subgroup of G such that $G/H \in \mathfrak{U}$. If the maximal subgroups of $G_p \cap F^*(H)$ are $\mathfrak{Z}$ -permutable in G , for all $G_p \in \mathfrak{Z}$, then $G \in \mathfrak{U}$.*

Corollary 3.2 [9, Theorem 3.1] *Let G be a group with a normal subgroup H such that $G/H \in \mathfrak{U}$. If the maximal subgroups of the Sylow subgroups of $F^*(H)$ are S -permutable in G , then $G \in \mathfrak{U}$.*

Corollary 3.3 [8, Main Theorem] *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G . Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are $\mathfrak{Z}$ -permutable in G , for all $G_p \in \mathfrak{Z}$.*

Corollary 3.4 [9, Theorem 3.4] *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let G be a group. If G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F^*(H)$ are S -permutable in G , then $G \in \mathfrak{F}$.*

Corollary 3.5 [10, Theorem 3.1] *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let G be a group. If G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F^*(H)$ are c -normal in G , then $G \in \mathfrak{F}$.*

Acknowledgments The authors thank the referees for the careful reading of the paper and the helpful comments and suggestions, which have improved the original manuscript up to its present form. The authors also thank M. Asaad for his helpful suggestions and comments. Finally, the authors are grateful to financial support from the Deanship of Scientific Research at Northern Borders University, Saudi Arabia.

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