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Hermite–Hadamard type inequalities for differentiable h_φ -preinvex functions

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Abstract In this paper, we introduce a new class of convex functions which is called h_φ -preinvex functions. We prove several Hermite–Hadamard inequalities for h_φ -preinvex functions. Some special cases are also discussed. Results proved in this paper continue to hold for these special cases. Our results may stimulate further investigation regarding variant forms of Hermite–Hadamard type inequalities.

Mathematics Subject Classification 26D15 · 26A51

المخلص

نقدم في هذه الورقة صفا جديدا من الدوال المحدبة التي تسمى دوال محدبة- h_φ . نثبت عدة متباينات هيرميت – هادامارد للدوال المحدبة- h_φ . يتم نقاش بعض الحالات الخاصة. تنطبق النتائج التي تم إثباتها في هذه الورقة على تلك الحالات الخاصة. من الممكن اعتبار أن نتائجنا تحفز إجراء مزيد من الفحص لأشكال بديلة لمتباينات من نوع هيرميت – هادامارد.

1 Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite–Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Hermite–Hadamard inequality can be considered as necessary and sufficient condition for a function to be convex. It provides estimates of the mean value of continuous convex function. For further useful details on Hermite–Hadamard inequalities readers are referred to [1,3–5,7–14,16–23].

In the past few years, several extensions and generalizations have been proposed for classical convexity (see [2,3,5,6,13–22,24,25]). A significant generalization of classical convex functions is that of φ -convex functions introduced by Noor [13]. Noor [13] has investigated the basic properties of φ -convex functions

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and showed that φ -convex functions are nonconvex functions. Noor [12] extended Hermite–Hadamard type inequalities for φ -convex functions.

Motivated by the ongoing research on generalizations and extensions of classical convexity, Varosanec [24] introduced the class of h -convex functions. She studied the basic properties of h -convex functions and noticed that along with classical convexity this class also generalizes several other classes of convex functions such as, s -convex functions [2], Q functions [6] and P functions [5] respectively. Noor et al. [16] introduced another generalization of classical convexity which is called as h_φ -convex functions. Several Hermite–Hadamard type inequalities are proved for h_φ -convex functions.

Weir and Mond [25] investigated the class of preinvex functions. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. For recent investigation on preinvexity, (see [1, 8–11, 20]).

Motivated and inspired by the recent research in this field, we introduce and consider a new class of convex functions, which is called h_φ -preinvex functions. This class includes several known and new classes of convex functions such as φ -convex functions [13], preinvex functions [25], h -convex functions [24] as special cases. We derive several new Hermite–Hadamard type integral inequalities for h_φ -preinvex functions and their variant forms. Results obtained in this paper continue to hold for these special cases. Our results represent significant refinement and improvement of the previous results. The interested readers are encouraged to find novel and innovative applications of h_φ -preinvex functions.

2 Preliminaries

In this section, we define and recall some basic concepts and results. Let $\mathbb{R}^n$ be the finite dimensional Euclidian space. Also let $0 \leq \varphi \leq \frac{\pi}{2}$ be a continuous function.

Definition 2.1 ([15]) The set $K_{\varphi\eta}$ in $\mathbb{R}^n$ is said to be φ -invex at u with respect to $\varphi(\cdot)$, if there exists a bifunction $\eta(\cdot, \cdot) : K_{\varphi\eta} \times K_{\varphi\eta} \rightarrow \mathbb{R}$, such that

$$u + te^{i\varphi}\eta(v, u) \in K_{\varphi\eta}, \quad \forall u, v \in K_{\varphi\eta}, t \in [0, 1]. \quad (1)$$

The φ -invex set $K_{\varphi\eta}$ is also called $\varphi\eta$ -connected set. Note that the convex set with $\varphi = 0$ and $\eta(v, u) = v - u$ is a φ -invex set, but the converse is not true (see [15]).

Remark 2.2 Definition 2.1 says that there is a path starting from a point u which is contained in $K_{\varphi\eta}$. We do not require that the point v should be one of the end point of the path. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in K_{\varphi\eta}$, then $e^{i\varphi}\eta(v, u) = v - u$, if and only if, $e^{i\varphi} = 1$ and $\eta(v, u) = v - u$. Then the φ -invex $K_{\varphi\eta}$ becomes the convex set K . It is clear that every convex set is a φ -invex set, but the converse is not necessarily true.

Definition 2.3 ([13]) Let K_φ be a set in $\mathbb{R}^n$. Then the set K_φ is said to be φ -convex with respect to φ , if and only if

$$u + te^{i\varphi}(v - u) \in K_\varphi, \quad \forall u, v \in K_\varphi, t \in [0, 1].$$

For $\varphi = 0$, the set K_φ reduces to the classical convex set K . That is,

$$u + t(v - u) \in K, \quad \forall u, v \in K, t \in [0, 1].$$

Definition 2.4 ([25]) A set K_η is said to be invex set with respect to bifunction $\eta(\cdot, \cdot)$, if

$$u + t\eta(v, u) \in K_\eta, \quad \forall u, v \in K_\eta, t \in [0, 1].$$

The invex set K_η is also called η -connected set.

Definition 2.5 Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function. A function f on the set $K_{\varphi\eta}$ is said to be h_φ -preinvex function with respect to φ and bifunction η , if

$$f(u + te^{i\varphi}\eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v), \quad \forall u, v \in K_{\varphi\eta}, t \in [0, 1]. \quad (2)$$

Remark 2.6 One can deduce several known concepts from Definition 2.5 as:

- (1) For $h(t) = t$ Definition 2.5 reduces to the definition for φ -preinvex functions (see [15]).
- (2) For $\varphi = 0$ Definition 2.5 reduces to the definition for h -preinvex functions (see [20]).
- (3) For $\eta(v, u) = v - u$ Definition 2.5 reduces to the definition for h_φ -convex functions (see [16]).
- (4) For $\varphi = 0$ and $\eta(v, u) = v - u$ Definition 2.5 reduces to the definition for h -convex functions (see [24]).

Now we discuss some special cases of Definition 2.5.

I. For $h(t) = t^s$ where $s \in (0, 1]$ in (2) we have the definition for s_φ -preinvex functions.

Definition 2.7 A function f on the set $K_{\varphi\eta}$ is said to be s_φ -preinvex function with respect to φ and η , if

$$f(u + te^{i\varphi}\eta(v, u)) \leq (1-t)^s f(u) + t^s f(v), \quad \forall u, v \in K_{\varphi\eta}, t \in [0, 1], s \in (0, 1]. \quad (3)$$

II. For $h(t) = 1$ in (2) we have the definition for P_φ -preinvex functions.

Definition 2.8 A function f on the set $K_{\varphi\eta}$ is said to be P_φ -preinvex function with respect to φ and η , if

$$f(u + te^{i\varphi}\eta(v, u)) \leq f(u) + f(v), \quad \forall u, v \in K_{\varphi\eta}, t \in [0, 1]. \quad (4)$$

From the above discussion, it is obvious that the class of h_φ -preinvex functions is quite general and unifying one. This is one of the main motivation of this paper.

3 Main results

In this section, we will discuss our main results.

Using the technique of [1], we prove the following Lemma which play a key role in our results.

Lemma 3.1 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \rightarrow \mathbb{R}$ where $\eta(b, a) > 0$. If $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$, then

$$\begin{aligned} & -\frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} + \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \\ & = \frac{e^{i\varphi}\eta(b, a)}{2} \int_0^1 (1-2t) f'(a + e^{i\varphi}\eta(b, a))dt. \end{aligned}$$

Proof Let

$$\begin{aligned} & \int_0^1 (1-2t) f'(a + te^{i\varphi}\eta(b, a))dt \\ & = \left| \frac{(1-2t)f(a + te^{i\varphi}\eta(b, a))}{e^{i\varphi}\eta(b, a)} \right|_0^1 + 2 \int_0^1 \frac{f(a + te^{i\varphi}\eta(b, a))}{e^{i\varphi}\eta(b, a)} dt, \\ & = -\frac{f(a + e^{i\varphi}\eta(b, a))}{e^{i\varphi}\eta(b, a)} - \frac{f(a)}{e^{i\varphi}\eta(b, a)} + \frac{2}{e^{i\varphi}\eta(b, a)} \int_0^1 f(a + te^{i\varphi}\eta(b, a))dt, \\ & = \frac{2}{e^{i\varphi}\eta(b, a)} \left[-\frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} + \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right]. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.2 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \rightarrow \mathbb{R}$. Suppose $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. Let $|f'|$ be h_φ -preinvex function. Then, for $\eta(b, a) > 0$,

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ & \leq \frac{e^{i\varphi}\eta(b, a)}{2} \{ |f'(a)| + |f'(b)| \} \int_0^1 |1 - 2t| h(t) dt. \end{aligned} \quad (5)$$

Proof Using Lemma 3.1, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ & = \left| \frac{e^{i\varphi}\eta(b, a)}{2} \int_0^1 (1 - 2t) f'(a + e^{i\varphi}\eta(b, a)t) dt \right|, \\ & \leq \frac{e^{i\varphi}\eta(b, a)}{2} \left\{ \int_0^1 |1 - 2t| |f'(a + te^{i\varphi}\eta(b, a))| dt \right\}, \end{aligned}$$

now using h_φ -preinvexity of $|f'|$, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ & \leq \frac{e^{i\varphi}\eta(b, a)}{2} \left\{ \int_0^1 |1 - 2t| \{ h(1-t) |f'(a)| + h(t) |f'(b)| \} dt \right\}, \\ & = \frac{e^{i\varphi}\eta(b, a)}{2} \{ |f'(a)| + |f'(b)| \} \int_0^1 |1 - 2t| h(t) dt. \end{aligned}$$

This completes the proof. $\square$

Now we have some special cases.

I. If $h(t) = t$, then we have the result for φ -preinvexity.

Corollary 3.3 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|$ is φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{8} \{ |f'(a)| + |f'(b)| \}.$$

II. If $h(t) = t^s$, then we have the result for s_φ -preinvexity.

Corollary 3.4 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|$ is s_φ -preinvex on I , then, for $\eta(b, a) > 0$ and $s \in (0, 1]$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2} \{ |f'(a)| + |f'(b)| \} \frac{1 + s2^s}{2^s(s+1)(s+2)}.$$

III. If $h(t) = 1$, then we have the result for P_φ -preinvexity.

Corollary 3.5 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|$ is P_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{4} \{ |f'(a)| + |f'(b)| \}.$$

Theorem 3.6 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. Let $|f'|^p$ be h_φ -preinvex on I , $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta(b, a) > 0$. Then

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left\{ \left(|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right) \int_0^1 h(t)dt \right\}^{\frac{p-1}{p}}. \quad (6)$$

Proof Using Lemma 3.1, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ &= \left| \frac{e^{i\varphi}\eta(b, a)}{2} \int_0^1 (1-2t) f'(a + e^{i\varphi}\eta(b, a)) dt \right| \\ &\leq \frac{e^{i\varphi}\eta(b, a)}{2} \left\{ \int_0^1 |1-2t| |f'(a + te^{i\varphi}\eta(b, a))| dt \right\}. \end{aligned}$$

Now using well-known Holders inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ &\leq \frac{e^{i\varphi}\eta(b, a)}{2} \left(\int_0^1 |1-2t|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(a + te^{i\varphi}\eta(b, a))|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Using h_φ -preinvexity of $|f'|^p$, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x) dx \right| \\ & \leq \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left(\int_0^1 [h(1-t)|f'(a)|^q + h(t)|f'(b)|^q] dt \right)^{\frac{1}{q}} \\ & = \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left\{ (|f'(a)|^q + |f'(b)|^q) \int_0^1 h(t) dt \right\}^{\frac{1}{q}} \\ & = \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left\{ \left(|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right) \int_0^1 h(t) dt \right\}^{\frac{p-1}{p}}. \end{aligned}$$

This completes the proof. $\square$

Now we have some special cases.

I. If $h(t) = t$, then we have the result for φ -preinvexity.

Corollary 3.7 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. Let $|f'|^p$ be φ -preinvex on I , $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta(b, a) > 0$. Then

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x) dx \right| \\ & \leq \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}. \end{aligned}$$

II. If $h(t) = t^s$, then we have the result for s_φ -preinvexity.

Corollary 3.8 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. Let $|f'|^p$ be s_φ -preinvex on I , $\frac{1}{p} + \frac{1}{q} = 1$, $\eta(b, a) > 0$ and $s \in (0, 1]$. Then

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x) dx \right| \\ & \leq \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{s+1} \right)^{\frac{p-1}{p}}. \end{aligned}$$

III. If $h(t) = 1$, then we have the result for P_φ -preinvexity.

Corollary 3.9 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. Let $|f'|^p$ be P_φ -preinvex on I , $\frac{1}{p} + \frac{1}{q} = 1$,

$\eta(b, a) > 0$. Then

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left(|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.$$

Theorem 3.10 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|^q$, $q > 1$ is h_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \left(|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right) \int_0^1 |1-2t|h(t)dt \right\}^{\frac{p-1}{p}}.$$

Proof Using Lemma 3.1, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ &= \left| \frac{e^{i\varphi}\eta(b, a)}{2} \int_0^1 (1-2t)f'(a + e^{i\varphi}\eta(b, a))dt \right| \\ &\leq \frac{e^{i\varphi}\eta(b, a)}{2} \left\{ \int_0^1 |1-2t||f'(a + te^{i\varphi}\eta(b, a))| \right\}. \end{aligned}$$

Using power-mean inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ &\leq \frac{e^{i\varphi}\eta(b, a)}{2} \left\{ \left(\int_0^1 |1-2t| \right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2t||f'(a + te^{i\varphi}\eta(b, a))|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Using h_φ -preinvexity of $|f'|^q$, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ &\leq \frac{e^{i\varphi}\eta(b, a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \left(|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right) \int_0^1 |1-2t|h(t)dt \right\}^{\frac{p-1}{p}}. \end{aligned}$$

This completes the proof. $\square$

Now we have some special cases.

I. If $h(t) = t$, then we have the result for φ -preinvexity.

Corollary 3.11 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|^q$, $q > 1$ is φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{4} \right)^{\frac{p-1}{p}}.$$

II. If $h(t) = t^s$, then we have the result for s_φ -preinvexity.

Corollary 3.12 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|^q$, $q > 1$ is s_φ -preinvex on I , then, for $\eta(b, a) > 0$ and $s \in (0, 1]$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \frac{1+s2^s}{2^s(s+1)(s+2)} \right)^{\frac{p-1}{p}}.$$

III. If $h(t) = 1$, then we have the result for P_φ -preinvexity.

Corollary 3.13 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Suppose that $f : I \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f'|^q$, $q > 1$ is P_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{e^{i\varphi}\eta(b, a)}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}.$$

Using the technique of [1], we prove the following result which helps us in proving our next results.

Lemma 3.14 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$ where $\eta(b, a) > 0$. If $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$, then

$$\begin{aligned} & \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \\ &= \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \int_0^1 t(1-t)f''(a + e^{i\varphi}\eta(b, a))dt. \end{aligned}$$

Proof Let

$$\begin{aligned}
 & \int_0^1 t(1-t)f''(a+e^{i\varphi}\eta(b,a))dt \\
 &= \left| \frac{f'(a+te^{i\varphi}\eta(b,a))}{e^{i\varphi}\eta(b,a)} t(1-t) \right|_0^1 - \frac{1}{e^{i\varphi}\eta(b,a)} \int_0^1 (1-2t)f'(a+te^{i\varphi}\eta(b,a))dt \\
 &= -\frac{1}{e^{i\varphi}\eta(b,a)} \int_0^1 (1-2t)f'(a+te^{i\varphi}\eta(b,a))dt \\
 &= -\frac{1}{e^{i\varphi}\eta(b,a)} \left\{ \frac{2}{e^{i\varphi}\eta(b,a)} \left[-\frac{f(a)+f(a+e^{i\varphi}\eta(b,a))}{2} + \frac{1}{e^{i\varphi}\eta(b,a)} \int_a^{a+e^{i\varphi}\eta(b,a)} f(x)dx \right] \right\} \\
 &= \frac{2}{[e^{i\varphi}\eta(b,a)]^2} \left[\frac{f(a)+f(a+e^{i\varphi}\eta(b,a))}{2} - \frac{1}{e^{i\varphi}\eta(b,a)} \int_a^{a+e^{i\varphi}\eta(b,a)} f(x)dx \right].
 \end{aligned}$$

This completes the proof. $\square$

Theorem 3.15 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a+e^{i\varphi}\eta(b,a)]$. If $|f''|$ is h_φ -preinvex on I , then, for $\eta(b,a) > 0$,

$$\begin{aligned}
 & \left| \frac{f(a)+f(a+e^{i\varphi}\eta(b,a))}{2} - \frac{1}{e^{i\varphi}\eta(b,a)} \int_a^{a+e^{i\varphi}\eta(b,a)} f(x)dx \right| \\
 & \leq \frac{[e^{i\varphi}\eta(b,a)]^2}{2} \left\{ (|f''(a)|+|f''(b)|) \int_0^1 t(1-t)h(t)dt \right\}. \quad (7)
 \end{aligned}$$

Proof Using Lemma 3.14, we have

$$\begin{aligned}
 & \left| \frac{f(a)+f(a+e^{i\varphi}\eta(b,a))}{2} - \frac{1}{e^{i\varphi}\eta(b,a)} \int_a^{a+e^{i\varphi}\eta(b,a)} f(x)dx \right| \\
 &= \left| \frac{[e^{i\varphi}\eta(b,a)]^2}{2} \int_0^1 t(1-t)f''(a+e^{i\varphi}\eta(b,a))dt \right| \\
 &\leq \frac{[e^{i\varphi}\eta(b,a)]^2}{2} \left[\int_0^1 t(1-t)|f''(a+e^{i\varphi}\eta(b,a))|dt \right] \\
 &\leq \frac{[e^{i\varphi}\eta(b,a)]^2}{2} \left[\int_0^1 t(1-t)\{h(1-t)|f''(a)|+h(t)|f''(b)|\}dt \right] \\
 &= \frac{[e^{i\varphi}\eta(b,a)]^2}{2} \left[\{|f''(a)|+|f''(b)|\} \int_0^1 t(1-t)h(t)dt \right].
 \end{aligned}$$

This completes the proof. $\square$

Now, we discuss some special cases.

I. If $h(t) = t$, then we have the result for φ -preinvexity.

Corollary 3.16 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is φ -preinvex on I then, for, $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{24} (|f''(a)| + |f''(b)|). \quad (8)$$

II. If $h(t) = t^s$, then we have the result for φ -preinvexity.

Corollary 3.17 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is s_φ -preinvex on I , then, for $\eta(b, a) > 0$ and $s \in (0, 1]$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left\{ \frac{|f''(a)| + |f''(b)|}{(s+2)(s+3)} \right\}. \quad (9)$$

III. If $h(t) = 1$, then we have the result for P_φ -preinvexity.

Corollary 3.18 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is P_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{12} \{|f''(a)| + |f''(b)|\}. \quad (10)$$

Theorem 3.19 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is h_φ -preinvex on I then for $\eta(b, a) > 0$,

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ & \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\{|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}\} \int_0^1 h(t)dt \right)^{\frac{p-1}{p}}. \quad (11) \end{aligned}$$

Proof Using Lemma 3.14, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x) dx \right| \\
 &= \left| \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \int_0^1 t(1-t) f''(a + e^{i\varphi}\eta(b, a)) dt \right| \\
 &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left[\int_0^1 t(1-t) |f''(a + e^{i\varphi}\eta(b, a))| dt \right] \\
 &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left(\int_0^1 (t-t^2)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \\
 &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left(\frac{2^{-1-2p} \sqrt{\pi} \Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\int_0^1 \{h(1-t)|f''(a)|^q + h(t)|f''(b)|^q\} dt \right)^{\frac{1}{q}} \\
 &= \frac{[e^{i\varphi}\eta(b, a)]^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}} \int_0^1 h(t) dt \right)^{\frac{p-1}{p}}.
 \end{aligned}$$

□

We have some special cases.

I. If $h(t) = t$, then we have the result for the class of φ -preinvexity.

Corollary 3.20 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\begin{aligned}
 & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x) dx \right| \\
 &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{16} (\sqrt{\pi})^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}. \quad (12)
 \end{aligned}$$

II. If $h(t) = t^s$, then we have the result for the class of s_φ -preinvexity.

Corollary 3.21 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is s_φ -preinvex on I , then, for $\eta(b, a) > 0$ and $s \in (0, 1]$,

$$\begin{aligned}
 & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x) dx \right| \\
 &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\frac{|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}}{s+1} \right)^{\frac{p-1}{p}}. \quad (13)
 \end{aligned}$$

III. If $h(t) = 1$, then we have the result for the class of P_φ -preinvexity.

Corollary 3.22 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is P_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{8} \left(\frac{\sqrt{\pi}}{2} \right)^{\frac{1}{p}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}. \quad (14)$$

Theorem 3.23 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is h_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{12} (6)^{\frac{p-1}{p}} \left(\{|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}\} \int_0^1 t(1-t)h(t)dt \right)^{\frac{p-1}{p}}. \quad (15)$$

Proof Using Lemma 3.14 and well-known power mean inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \\ &= \left| \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \int_0^1 t(1-t)f''(a + e^{i\varphi}\eta(b, a))dt \right|, \\ &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left[\int_0^1 t(1-t)|f''(a + e^{i\varphi}\eta(b, a))|dt \right], \\ &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left(\int_0^1 (t-t^2)dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t)|f''(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}}, \\ &\leq \frac{[e^{i\varphi}\eta(b, a)]^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} \left(\int_0^1 (t)(1-t)\{h(1-t)|f''(a)|^q + h(t)|f''(b)|^q\}dt \right)^{\frac{1}{q}}, \\ &= \frac{[e^{i\varphi}\eta(b, a)]^2}{12} (6)^{\frac{p-1}{p}} \left(\{|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}\} \int_0^1 t(1-t)h(t)dt \right)^{\frac{p-1}{p}}. \end{aligned}$$

This completes the proof. $\square$

Now we have some special cases.

I. If $h(t) = t$, then we have the class of φ -preinvexity.

Corollary 3.24 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is φ -preinvex on I , then, for every

$\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{12} \left(\frac{1}{2}\right)^{\frac{p-1}{p}} \left(|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

II. If $h(t) = t^s$, then we have the class of s_φ -preinvexity.

Corollary 3.25 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is s_φ -preinvex on I , then, for every $\eta(b, a) > 0$ and $s \in (0, 1]$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{12} (6)^{\frac{p-1}{p}} \left(\frac{|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}}{(s+2)(s+3)} \right)^{\frac{p-1}{p}}.$$

III. If $h(t) = 1$, then we have the class of P_φ -preinvexity.

Corollary 3.26 Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \rightarrow \mathbb{R}$. Let $f : I \rightarrow \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$. If $|f''|$ is P_φ -preinvex on I , then, for $\eta(b, a) > 0$,

$$\left| \frac{f(a) + f(a + e^{i\varphi}\eta(b, a))}{2} - \frac{1}{e^{i\varphi}\eta(b, a)} \int_a^{a+e^{i\varphi}\eta(b, a)} f(x)dx \right| \leq \frac{[e^{i\varphi}\eta(b, a)]^2}{12} \left(|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

4 Conclusion

In this paper, we have introduced and studied a new class of convex functions, called h_φ -preinvex functions. This class of convex functions is flexible and unifying one. Several new Hermite–Hadamard inequalities are derived. Several special cases are discussed. Results proved in this paper may stimulate further research in this field.

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