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Note on valuative dimension in power series rings

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Abstract We give necessary and sufficient conditions for the power series ring $R[[x_1, \dots, x_n]]$ to be a Jaffard domain, where R is an almost pseudo-valuation domain.

Mathematics Subject Classification 13F25 · 13C15 · 13F05 · 13A15

المخلص

نعطي شروطاً ضرورية وكافية على مجال شبه تقييمي تقريباً R بحيث تكون حلقة متسلسلات القوى $R[[x_1, \dots, x_n]]$ مجال جَفَّارْد.

All rings considered below are (commutative with identity) integral domains. The dimension of a ring R , denoted by $\dim R$, means its Krull dimension. In [10], Jaffard defines the valuative dimension, denoted by $\dim_v R$, of an integral domain R to be the maximal rank of the valuation overrings of R . Following [1], an integral domain R is said to be a Jaffard ring if $\dim_v R = \dim R < \infty$. Most standard examples of Jaffard domains are finite-dimensional Noetherian domains and finite-dimensional Prüfer domains [10].

In this paper, we are interested in when the power series ring $R[[x]]$ is a Jaffard domain. It was shown by Arnold in [2] that if a ring R fails to have the SFT (strong finite type)-property, then $R[[x]]$ has infinite dimension, and so has infinite valuative dimension. Recall that an ideal I of a ring R is an SFT-ideal if there exist a positive integer k and a finitely generated ideal F of R such that $F \subseteq I$ and $x^k \in F$ for each $x \in I$; moreover, if each ideal of R is an SFT-ideal, then we say that R is an SFT-ring.

Let R be an integral domain. When is $R[[x]]$ a Jaffard domain? The answer is known in two cases. First, if R is a Noetherian ring with finite dimension, then $R[[x]]$ is also Noetherian with finite dimension [7, Lemma 2.6], and so a Jaffard domain. The second case was shown by Kang and Park. In [11], they computed the dimension of mixed extension $R[[x_1]] \dots [[x_n]]$ where R is a finite dimensional SFT-Prüfer domain. In [15], Park generalized the result to the case, where R is a finite dimensional SFT-globalized pseudo-valuation domain (for short, GPVD) as shown in the following:

Theorem 1 [15, Theorem 2.15] *Let R be an m -dimensional SFT-GPVD with associated Prüfer domain T and let $I = (R : T)$. For each maximal ideal M of R , let N_M be the maximal ideal of T such that $N_M \cap R = M$, set $k_M := R/M$ and $K_M := T/N_M$, and let $k_{0,M}$ denote the maximal separable extension of k_M in K_M . Assume that $[x_i] = [[x_i]]$ for some i . Then, $\dim R[[x_1]] \dots [[x_n]] = nm + 1$ if for each maximal ideal M of R with $ht M = m$ and $M \supseteq I$, K_M has finite exponent over $k_{0,M}$, and $[k_{0,M} : k_M] < \infty$ and $\dim R[[x_1]] \dots [[x_n]] = nm + n$ otherwise.*

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Recall that an integral domain R is a GPVD [6] if there exists a Prüfer overring T of R such that (i) $R \subseteq T$ is a unbranched extension, and (ii) there exists a nonzero radical ideal I common to T and R such that the rings T/I and R/I are zero-dimensional. The domain T is uniquely determined [6, page 156] and is called the associated Prüfer domain of R . Most standard examples of GPVD are valuation domains, PVDs and Prüfer domains [6]. It is well known that $\dim_v R = \lim_{n \rightarrow \infty} (\dim R[n] - n)$ [5] where $R[n]$ means the ring of polynomials in n indeterminates over R . It follows immediately from Park's result that:

Corollary 2 *Let R be a GPVD (which is not a field) with associated Prüfer domain T and let $I = (R : T)$. For each maximal ideal M of R , let N_M be the maximal ideal of T such that $N_M \cap R = M$; set $k_M := R/M$ and $K_M := T/N_M$, and let $k_{0,M}$ denote the maximal separable extension of k_M in K_M . For $m \geq 1$, the following assertions are equivalent:*

- (1) $R[[x_1, \dots, x_m]]$ is a Jaffard domain.
- (2) $\dim_v R[[x_1, \dots, x_m]] < \infty$.
- (3) R is a one-dimensional SFT-ring, and for each maximal ideal M of R with $M \supseteq I$, K_M has finite exponent over $k_{0,M}$ and $[k_{0,M} : k_M] < \infty$.

In this paper, we are interested in the case where R is an almost pseudo-valuation domain (for short, APVD). Recall that an integral domain R is called an almost pseudo-valuation domain if R is quasi-local with maximal ideal M such that $(M : M)$ is a valuation overring of R with M primary to the maximal ideal of $(M : M)$ [3]. A domain R is called $*$ -domain if it has (at least) a height one prime ideal and for each height one prime ideal P of R , R satisfies the ascending chain condition on ideals of the form aP (where $a \in R$) [12]. We proved that for an APVD R with nonzero finite dimension, $R[[x_1, \dots, x_n]]$ has finite dimension if and only if R is a residually $*$ -domain (i.e., R/P is a $*$ -domain for each nonmaximal prime ideal P of R) [12, Theorem 2.9], if and only if the integral closure of R is an SFT-ring [13, Theorem 2.3]. Note that every PVD is an APVD, APVD's class contains strictly the PVD's class [3, Example 3.9] and an APVD is a GPVD if and only if it is a PVD [13, beginning of Section 3]. The following result gives us a necessary and sufficient condition on an APVD R so that $R[[x]]$ is a Jaffard domain.

Theorem 3 *Let R be an APVD with maximal ideal M , residue field K and nonzero dimension, L be the residue field of $(M : M)$ and K_0 be the maximal separable extension of K in L . For $m \geq 1$, the following assertions are equivalent:*

- (1) $R[[x_1, \dots, x_m]]$ is a Jaffard domain.
- (2) $\dim_v R[[x_1, \dots, x_m]] < \infty$.
- (3) R is one-dimensional with SFT-integral closure, $[K_0 : K] < \infty$, and L has finite exponent over K_0 .
- (4) R satisfies the ascending chain condition on principal ideals, the integral closure of R is an SFT-ring, $[K_0 : K] < \infty$, and L has finite exponent over K_0 .

Proof Let N be the maximal ideal of the valuation domain $V := (M : M)$.

(1) $\Rightarrow$ (2) trivial. (2) $\Rightarrow$ (3) Since $\dim R[[x_1, \dots, x_m]] < \infty$, R has an SFT-integral closure and $N^s \subseteq M$ for some positive integer s by Khalifa and Benhissi [13, Theorem 2.3]. Thus, the prime ideal $N[[x_1, \dots, x_m]]$ is minimal over $M[[x_1, \dots, x_m]]$ in $V[[x_1, \dots, x_m]]$. Then $\dim R[[x_1, \dots, x_m]][n] \geq \text{ht}_{V[[x_1, \dots, x_m]][1]} N[[x_1, \dots, x_m]][1] + \inf\{d, n-1\} + \dim K[[x_1, \dots, x_m]][n] = (m+1)(\dim R - 1) + 1 + \inf\{d, n-1\} + m + n$ by Cahen [5, Lemme 3], Khalifa and Benhissi [13, Lemma 2.1], and Park [15, Lemma 2.7], where d is the transcendence degree of $L[[x_1, \dots, x_m]]$ over $K[[x_1, \dots, x_m]]$. If $d = \infty$, then $\dim R[[x_1, \dots, x_m]][n] \geq (m+1)(\dim R - 1) + 1 + n - 1 + m + n \geq 2n$ which contradicts that $R[[x_1, \dots, x_m]]$ has finite valuative dimension. Thus, $d < \infty$ and so $[K_0 : K] < \infty$ and L has finite exponent over K_0 by Park [14, Lemma 2.1]. To show that R is 1-dimensional, we apply again [5, Lemme 3], [13, Lemma 2.1], and [15, Lemma 2.7]. Then we have $\dim R[[x_1, \dots, x_m]][n] - n \geq \text{ht}_{V[[x_1, \dots, x_m]][n]} N[[x_1, \dots, x_m]][n] + \dim K[[x_1, \dots, x_m]][n] - n = (n+m)(\dim R - 1) + 1 + m + n - n = n(\dim R - 1) + m \dim R + 1$. Hence, $\dim R = 1$. (3) $\Rightarrow$ (4) see [13, Lemma 2.2 and Theorem 2.3]. (4) $\Rightarrow$ (1) Recall that a domain D is called Archimedean in case $\bigcap_{n=1}^{\infty} a^n D = 0$ for each nonunit a of D . It is well known that rings satisfying the ascending chain condition on principal ideals are Archimedean and a divided Archimedean domain must have Krull dimension ≤ 1 , and so $\dim R = 1$ by Badawi and Houston [3, Proposition 3.2]. Thus, $N^s \subseteq M$ for some positive integer s by Khalifa and Benhissi [13, Lemma 2.2]. By Park [14, Lemma 2.1], $L[[x_1, \dots, x_m]][n]$ is integral over $K[[x_1, \dots, x_m]][n]$. Thus, $V[[x_1, \dots, x_m]][n]$ is integral over $R[[x_1, \dots, x_m]][n]$. Hence, $\dim R[[x_1, \dots, x_m]][n] - n = (n+m) \dim R + 1 - n = m + 1$ for every positive integer n by Kang and Park [11, Theorem 14]. Then $\dim_v R[[x_1, \dots, x_m]] = m + 1 (< \infty) = \dim R[[x_1, \dots, x_m]]$. $\square$

Remark 4 Park [15] showed that if R is an m -dimensional SFT-PVD with maximal ideal M and $[x_i] = [[x_i]]$ for some i , then $\dim R[x_1] \dots [x_n] = nm + 1$ if L has finite exponent over K_0 and $[K_0 : K] < \infty$ (where $K = R/M$, L is the residue field of $(M : M)$ and K_0 is the maximal separable extension of K in L), and $\dim R[x_1] \dots [x_n] = nm + n$ otherwise.

We show that Park's results [15, Lemma 2.8 and Theorem 2.9] remain true also if we replace a finite-dimensional SFT-PVD by a nonzero finite-dimensional APVD R with SFT-integral closure. The proof is the same as Park's proof (in the proof, we need also [13, Theorem 2.3]: $N^s \subseteq M$ for some positive integer $s > 0$ where M is the maximal ideal of R and N is the maximal ideal of the ring $(M : M)$), and so we do not rewrite it. Then, we have: if R is an APVD with SFT-integral closure, maximal ideal M , nonzero finite dimension m , residue field K , L is the residue field of $(M : M)$, K_0 is the maximal separable extension of K in L and $[x_i] = [[x_i]]$ for some i , then $\dim R[x_1] \dots [x_n] = nm + 1$ if L has finite exponent over K_0 and $[K_0 : K] < \infty$, and $\dim R[x_1] \dots [x_n] = nm + n$ otherwise. This gives us another proof for Theorem 3 using Park's technique.

The following example shows that there exists a non-Noetherian domain R such that $R[[x]]$ is a Jaffard domain.

Example 5 If R is a one-dimensional Prüfer domain, then $R[[x_1, \dots, x_n]]$ is a Jaffard domain if and only if R is Noetherian (because any one-dimensional SFT-Prüfer domain is a Dedekind domain). This is not the case for PVDs. More generally, there exists a non-Noetherian PVD R such that $R[[x]]$ is a Jaffard domain. Let F be a field with nonzero characteristic p , $\{x_1, \dots, x_n, \dots\}$ a countably infinite set of indeterminates over F , $L = F(x_1, \dots, x_n, \dots)$ and $K = L^p$ (K is the set of elements λ^p where $\lambda \in L$). Let y be an indeterminate over L , $V := L[y]_{(y)} = L + yV$ and $R := K + yV$. It is well known that V is a rank-one discrete valuation domain, and R is a one-dimensional SFT-PVD with maximal ideal yV and associated valuation overring V . Since L/K is purely inseparable with exponent 1, $R[[x_1, \dots, x_n]]$ is a Jaffard domain by Corollary 2. But R is not Noetherian since L/K has infinite degree [5, Proposition 3.6].

Theorem 6 For any integral domain R , $\dim_v R[[x]] \geq 1 + \dim_v R$.

Proof Let T be an overring of R and $A = T \times_F R[[x]]_{(x)}$ be the pullback of T and $R[[x]]_{(x)}$ over F where F is the quotient field of R . By Fontana [8, Corollary 1.5-(6)], A is an overring of $R[[x]]$, and so $\dim_v R[[x]] \geq \dim A$. Since $R[[x]]_{(x)}$ is a rank-one valuation domain, $\dim A = \dim T + \dim R[[x]]_{(x)} = 1 + \dim T$ by Fontana [8, Proposition 2.1]. Hence, $\dim_v R[[x]] \geq 1 + \dim_v R$. $\square$

Remark 7 All known examples of integral domains R such that $R[[x]]$ is a Jaffard domain are D and $D[x_1] \dots [x_m]$ (it is not necessary that $[x_i] = [[x_i]]$ for some i) where $\{x_1, \dots, x_m\}$ is a finite set of indeterminates over D , and D is:

- (1) a finite-dimensional Noetherian domain, or
- (2) a one-dimensional SFT-GPVD with associated Prüfer domain T such that for each maximal ideal M of D with $M \supseteq I$, K_M has finite exponent over $k_{0,M}$ and $[k_{0,M} : k_M] < \infty$ where $I = (D : T)$, N_M is the maximal ideal of T such that $N_M \cap D = M$, $k_M := D/M$, $K_M := T/N_M$, and $k_{0,M}$ denotes the maximal separable extension of k_M in K_M , or
- (3) a one-dimensional APVD with SFT-integral closure such that $[K_0 : K] < \infty$ and L has finite exponent over K_0 , where K is the residue field of D , L the residue field of the associated valuation domain of D and K_0 the maximal separable extension of K in L .

The most natural examples of Archimedean integral domains are arbitrary accp-domains (i.e., domains satisfying the ascending chain condition on principal ideals) and also Noetherian domains, one-dimensional domains and completely integrally closed domains. It is well known that if R is an Archimedean domain, then so is the power series ring $R[[x]]$ and the polynomial ring $R[x]$. Therefore, all known examples of domains R so that $R[[x]]$ is a Jaffard domain are Archimedean. Hence, it would be interesting to know if the following statement is true or not:

$$R[[x]] \text{ is a Jaffard domain} \implies R \text{ is Archimedean?}$$

Also, all known examples of domains R so that $R[[x]]$ is a Jaffard domain are accp-domains; indeed, it is well known that if R is an accp-domain, then so is $R[[x]]$ [4, Lemma 2.1] and the polynomial ring $R[x]$. The remark is true for APVDs by Theorem 3 and it remains and suffices to show that a one-dimensional SFT-GPVD is

an accp-domain. Let R be a one-dimensional SFT-GPVD with associated Prüfer domain T . Note that T is an SFT-ring [9, Proposition 1.7] and so is Noetherian. Since an element $r \in R$ is invertible in R if and only if r is invertible in T , every increasing chain of principal ideals of R stabilizes for some n . Thus, R is an accp-domain. Hence, it would be interesting to know if the following statement is true or not:

$$R[[x]] \text{ is a Jaffard domain} \implies R \text{ is an accp-domain?}$$

Finally, all known examples of domains R so that $R[[x]]$ is a Jaffard domain are Jaffard domains. Hence, it would be interesting to know if the following statement is true or not:

$$R[[x]] \text{ is a Jaffard domain} \implies R \text{ is a Jaffard domain?}$$

It is well known that if R is an integral domain, then $\dim R[[x]] \geq 1 + \dim R$. The last inequality may be equality (take R to be Noetherian [7, Lemma 2.6]) and may be strict inequality (take $R = D[[y]]$ where D is a rank-two SFT-valuation domain, then $\dim R[[x]] = 2 + \dim R$ [11, Theorem 14]). But all known examples of domains R so that $\dim_v R[[x]] < \infty$ satisfy $\dim_v R[[x]] = 1 + \dim_v R$. Unfortunately, we are unable to find an example of domain R such that $\infty > \dim_v R[[x]] > 1 + \dim_v R$.

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