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The impact of the flatness condition on the prescribed Webster scalar curvature

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Abstract In this paper, we give existence and multiplicity results for the problem of prescribing the Webster scalar curvature on the three CR sphere of $\mathbb{C}^2$ under mixed conditions: non-degeneracy and flatness.

Mathematics Subject Classification 32V20 · 32V05 · 58E05 · 57R70 · 35J65

المخلص

في هذه الورقة، نعطي نتائج تتعلق بالوجود والتعددية لمسألة وصف تقوس وبستر السلمي على كرة CR الثلاثية في $\mathbb{C}^2$ بافتراض شروط مختلطة: عدم الانحلال والانبساط.

1 Introduction

Let $\mathbb{S}^3$ be the unit sphere of $\mathbb{C}^2$ endowed with its standard contact form θ and K be a given $\mathcal{C}^2$ positive function on $\mathbb{S}^3$. We will give conditions on the set of critical points of K to ensure to this function to be the Webster scalar curvature for a contact form $\tilde{\theta}$ conformal to θ . This problem is equivalent to solve the following differential equation

$$(P_K) \begin{cases} L_\theta u = Ku^3 & \text{on } \mathbb{S}^3, \\ u > 0. \end{cases}$$

where L_θ is the conformal laplacian, $L_\theta = 4\Delta_\theta + R_\theta$, Δ_θ is the sublaplacian operator on $(\mathbb{S}^3, \theta)$ and R_θ is the Webster scalar curvature of $(\mathbb{S}^3, \theta)$.

The problem (P_K) has a variational structure, however the associated Euler functional does not satisfy the Palais–Smale condition. Therefore, the compactness methods cannot be applied in the present case instead we will use a method due to Bahri [2].

Such problems of existence and multiplicity results were investigated by several authors one can see [5–7, 9–11, 14, 16, 17].

To state our result, we set the following notations.

Let K be a $\mathcal{C}^2$ positive function on $\mathbb{S}^3$ such that the set of its critical points which we denote by $\mathcal{K}$ is finite. We say that K satisfies condition (H_1) if for all $y \in \mathcal{K}$ we have either:

$$y \text{ is a nondegenerate critical point with } \Delta K(y) \neq 0 \quad (1.1)$$

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or there exist $\beta = \beta(y) \in]2, 4[$ and $b_1 = b_1(y)$, $b_2 = b_2(y)$, $b_0 = b_0(y) \in \mathbb{R}^*$ such that

$$\sum_{k=1}^2 b_k + \tau b_0 \neq 0, \quad \sum_{k=1}^2 b_k + \tau' b_0 \neq 0$$

and in some pseudohermitian normal coordinates centered at y , K has the following expansion:

$$K(x) = K(y) + \sum_{k=1}^2 b_k |x_k|^\beta + b_0 |t|^{\frac{\beta}{2}} + R(z, t), \quad \text{with} \quad \sum_{s=0}^{[\beta]} |\nabla^s R(z, t)| |(z, t)|_{\mathbb{H}^1}^{-\beta-s} = o(1), \quad (1.2)$$

where

$$\tau = \frac{\int_{\mathbb{H}^1} |t|^{\frac{\beta}{2}} \frac{1-|z|^2-|t|^2}{|1+|z|^2-it|^6} \theta_0 \wedge d\theta_0}{\int_{\mathbb{H}^1} |x_1|^\beta \frac{1-|z|^2-|t|^2}{|1+|z|^2-it|^6} \theta_0 \wedge d\theta_0}, \quad \tau' = \frac{\int_{\mathbb{H}^1} \frac{|t|^{\frac{\beta}{2}}}{|1+|z|^2-it|^4} \theta_0 \wedge d\theta_0}{\int_{\mathbb{H}^1} \frac{|x_1|^\beta}{|1+|z|^2-it|^4} \theta_0 \wedge d\theta_0},$$

∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β .

We denote by

$$I_1 = \{y \in \mathcal{K}; \quad \Delta K(y) < 0\},$$

$$I_2 = \left\{ y \in \mathcal{K}; \quad y \text{ satisfies (1.2) and } \sum_{k=1}^2 b_k + \tau' b_0 < 0 \right\}.$$

For each p -tuple of critical points of K , $(y_{i_1}, \dots, y_{i_p}) \in I_1^p$, we associate the matrix $M(y_{i_1}, \dots, y_{i_p}) = (m_{jq})$ defined by

$$m_{jj} = -\frac{\Delta K(y_{i_j})}{3(K(y_{i_j}))^2}, \quad m_{jq} = -2 \frac{G(y_{i_j}, y_{i_q})}{(K(y_{i_j})K(y_{i_q}))^{\frac{1}{2}}} \quad \text{for } j \neq q \quad (1.3)$$

where G is the Green's function of the conformal Laplacian $L_\theta = 4\Delta_\theta + R_\theta$.

We say that K satisfies **(H₂)** if for each $(y_{i_1}, \dots, y_{i_p}) \in I_1^p$ the corresponding matrix $M(y_{i_1}, \dots, y_{i_p})$ is nondegenerate.

For each $(y_{i_1}, \dots, y_{i_p}) \in I_1^p$ we denote by $\varrho(y_{i_1}, \dots, y_{i_p})$ the least eigenvalue of $M(y_{i_1}, \dots, y_{i_p})$.

To give our multiplicity result, we introduce the following definition and notations:

- We recall that a solution u of (P_K) is said to be nondegenerate if the linearized operator $\mathcal{L}(\varphi) := L_\theta(\varphi) - 3u^2\varphi$ does not have zero as an eigenvalue.
- Let

$$l^+ := \max \{p \in \mathbb{N} \text{ s.t. } \exists (y_{i_1}, \dots, y_{i_p}) \in I_1^p, \varrho(y_{i_1}, \dots, y_{i_p}) > 0\}$$

and

$$L_0 := \max \left\{ \left\{ 4p-1 - \sum_{j=1}^p \text{Morse}(K, y_{i_j}); (y_{i_1}, \dots, y_{i_p}) \in I_1^p, \varrho(y_{i_1}, \dots, y_{i_p}) > 0 \right\} \cup \{3-m(y); y \in I_2\} \right\}$$

where for $y \in I_1$, $\text{Morse}(K, y)$ denotes the Morse index of K at y and for $y \in I_2$, $m(y) = \#\{b_k(y); b_k(y) < 0\}$.

Now we are ready to state our multiplicity result.

Theorem 1.1 *Let K be a C^2 positive function on $\mathbb{S}^3$ satisfying **(H₁)** and **(H₂)**. Assume that all solutions to the problem (P_K) having their Morse indices less or equal than $L_0 + 1$ are nondegenerate and denote their number by N . Then*

$$N \geq \left| 1 - \sum_{y \in I_2} (-1)^{3-m(y)} - \sum_{p=1}^{l^+} \sum_{\varrho(y_{i_1}, \dots, y_{i_p}) > 0} (-1)^{4p-1-\sum_{j=1}^p \text{Morse}(K, y_{i_j})} \right|.$$

The rest of this paper is organized as follows. In Sect. 2, we introduce the Heisenberg group and the Cayley transform which give an equivalence between $\mathbb{S}^3$ minus a point and the Heisenberg group $\mathbb{H}^1$ (for details see [8, 12]). We give in Sect. 3 the Euler–Lagrange functional J associated to the problem (P_K) and some connected facts. The characterization of the critical points at infinity of J is the main tool of Sect. 4. Finally, Sect. 5 is devoted to the proof of Theorem 1.1.

2 The Heisenberg group and Cayley transform

The Heisenberg group $\mathbb{H}^1$ is the Lie group whose underlying manifold is $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ with coordinates $(z, t) = (x_1, x_2, t)$ and whose group law is given by

$$(z, t)(z', t') = (z + z', t + t' + 2\operatorname{Im}(z\bar{z}')) = (x_1 + x'_1, x_2 + x'_2, t + t' + x'_1x_2 - x_1x'_2).$$

The $\mathbb{H}^1$ -dilations are the $\mathbb{H}^1$ -transformations given by:

$$d_\lambda : \mathbb{H}^1 \longrightarrow \mathbb{H}^1, \quad (z, t) \longmapsto (\lambda z, \lambda^2 t), \quad \lambda > 0.$$

The Jacobian determinant of d_λ is λ^4 , so that the homogeneous dimension of the group is equal to 4. The homogeneous norm of the space is given by

$$|(z, t)|_{\mathbb{H}^1} = (|z|^4 + t^2)^{\frac{1}{4}}$$

and the natural distance is accordingly defined by

$$d_{\mathbb{H}^1}((z, t), (z', t')) = |(z', t')^{-1}(z, t)|_{\mathbb{H}^1}.$$

The space $T_{1,0} = \operatorname{span} \left\{ \frac{\partial}{\partial z} + i\bar{z} \frac{\partial}{\partial t} \right\}$ gives the CR structure of $\mathbb{H}^1$.

Let $H = \operatorname{Re}(T_{1,0} \oplus T_{0,1})$ where $T_{0,1} = \overline{T_{1,0}}$. The bundle H is spanned by $X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial t}$ and $X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial t}$.

The tangent bundle of $\mathbb{H}^1$ has the following natural decomposition $T\mathbb{H}^1 = H \oplus \mathbb{R}T$, where $T = \frac{\partial}{\partial t}$. The complex structure on $\mathbb{H}^1$ is given by

$$J : H \longrightarrow H, \quad V + \bar{V} \longmapsto J(V + \bar{V}) = i(V - \bar{V}) \quad \forall V \in T_{1,0}$$

and (H, J) gives a real CR structure on $\mathbb{H}^1$.

The real 1-form

$$\theta_0 = dt + izd\bar{z} - i\bar{z}dz = dt + 2(x_2dx_1 - x_1dx_2)$$

annihilates $T_{1,0}$, we take it to be the contact form for the CR structure.

The Cayley transform defined on $\mathbb{S}^3$ minus $(0, -1)$ is the following transformation:

$$F : \mathbb{S}^3 \setminus (0, -1) \longrightarrow \mathbb{H}^1$$

$$\zeta = (\zeta_1, \zeta_2) \longmapsto (z, t) = (x_1, x_2, t) = \left(\frac{\zeta_1}{1 + \zeta_2}, \frac{2\operatorname{Im}\zeta_2}{|1 + \zeta_2|^2} \right).$$

with inverse:

$$F^{-1} : \mathbb{H}^1 \longrightarrow \mathbb{S}^3 \setminus (0, -1)$$

$$(z, t) \longmapsto \left(\frac{2z}{1 + |z|^2 - it}, i \frac{1 - |z|^2 + it}{1 + |z|^2 - it} \right).$$

Recall that F is a CR diffeomorphism between $(\mathbb{S}^3 \setminus (0, -1), \theta)$ and $(\mathbb{H}^1, \theta_0)$, where $\theta = j^*[i(\bar{\partial} - \partial)|\zeta|^2]$ and $j : \mathbb{S}^3 \longrightarrow \mathbb{C}^2$ is the inclusion map (for more details one can see [8]).

3 Variational structure and preliminaries

In this section, we recall the associated Euler–Lagrange functional of (P_K)

$$J(u) = \frac{\int_{\mathbb{S}^3} L_\theta u u \theta \wedge d\theta}{\left(\int_{\mathbb{S}^3} K u^4 \theta \wedge d\theta\right)^{\frac{1}{2}}}, \quad (3.1)$$

where u is on the unit sphere of $\mathcal{S}_1^2(\mathbb{S}^3)$ equipped with the norm

$$\|u\| = \left(\int_{\mathbb{S}^3} L_\theta u u \theta \wedge d\theta \right)^{\frac{1}{2}},$$

$\mathcal{S}_1^2(\mathbb{S}^3)$ is a Folland–Stein space (see [13]).

To investigate a solution for Problem (P_K) is equivalent to search a critical point of J subjected to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma; u \geq 0\}, \quad \Sigma = \{u \in \mathcal{S}_1^2(\mathbb{S}^3); \|u\| = 1\}.$$

For $a \in \mathbb{S}^3$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda}{|1 + \lambda^2(|z - z_a|^2 - i(t - (a)_0 - 2\text{Im}z\bar{z}_a))|},$$

where

$$(z, t) = (x_1, x_2, t), \quad (z_a, (a)_0) = ((a)_1, (a)_2, (a)_0) \in \mathbb{H}^1$$

are the coordinates of x and a , respectively, in some pseudohermitian normal coordinates and the constant c_0 is such that the following equation is satisfied

$$L_\theta \delta_{(a,\lambda)} = \delta_{(a,\lambda)}^3 \quad \text{on } \mathbb{S}^3.$$

We define now the set of potential critical points at infinity associated to the functional J . For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let

$$V(p, \varepsilon) = \left\{ u \in \Sigma; \exists a_1, \dots, a_p \in \mathbb{S}^3, \lambda_1, \dots, \lambda_p > 0, \alpha_1, \dots, \alpha_p > 0 \text{ s.t. } \left\| u - \sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} \right\| < \varepsilon, \right. \\ \left. \left| \frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 \right| < \varepsilon, \varepsilon_{ij} < \varepsilon, \lambda_i^{-1} < \varepsilon \right\},$$

here $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d(a_i, a_j) \right)^{-1}$ denotes the interaction between the two masses a_i and a_j , $d(a_i, a_j) = d_{\mathbb{H}^1}(((a_i)_1, (a_i)_2, (a_i)_0), ((a_j)_1, (a_j)_2, (a_j)_0))$, where $((a_i)_1, (a_i)_2, (a_i)_0)$ and $((a_j)_1, (a_j)_2, (a_j)_0)$ are, respectively, the components of a_i and a_j in some pseudohermitian normal coordinates.

For w a solution of (P_K) , let also

$$V(p, \varepsilon, w) = \{u \in \Sigma; \exists \alpha_0 > 0, \text{ s.t. } u - \alpha_0 w \in V(p, \varepsilon) \text{ and } |\alpha_0^2 J(u)^2 - 1| < \varepsilon\}. \quad (3.2)$$

The failure of the Palais–Smale condition for the functional J is characterized as follows.

Proposition 3.1 *Let $(u_k) \in \Sigma^+$ be a sequence such that $\partial J(u_k)$ tends to zero and $J(u_k)$ is bounded. Then, there exist an integer $p \in \mathbb{N}^*$, a sequence (ε_k) , ε_k tends to zero, and an extracted subsequence of (u_k) 's, again denoted (u_k) , such that $u_k \in V(p, \varepsilon_k, w)$ where w is zero or a solution of (P_K) .*

If a function u belongs to $V(p, \varepsilon)$, we consider the following minimization problem for $u \in V(p, \varepsilon)$ with ε small

$$\min \left\{ \left\| u - \sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} \right\|, \quad \alpha_i > 0, \quad \lambda_i > 0, \quad a_i \in \mathbb{S}^3 \right\}. \quad (3.3)$$

Then we have the following results.

Proposition 3.2 *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p, \varepsilon)$, the minimization problem (3.3) has a unique solution (up to permutation). In particular, we can write $u \in V(p, \varepsilon)$ as follows*

$$u = \sum_{i=1}^p \bar{\alpha}_i \delta_{\bar{a}_i, \bar{\lambda}_i} + v,$$

where $(\bar{\alpha}_1, \dots, \bar{\alpha}_p, \bar{\lambda}_1, \dots, \bar{\lambda}_p, \bar{a}_1, \dots, \bar{a}_p)$ is the solution of (3.3) and $v \in \mathcal{S}_1^2(\mathbb{S}^3)$ satisfying the following condition

$$(V_0) \quad \langle v, \psi \rangle_{L_\theta} = 0 \quad \text{for all } \psi \in \left\{ \delta_{a_i, \lambda_i}, \frac{\partial \delta_{a_i, \lambda_i}}{\partial \lambda_i}, (A_i)_1 \delta_{a_i, \lambda_i}, (A_i)_2 \delta_{a_i, \lambda_i}, (A_i)_0 \delta_{a_i, \lambda_i} \right\}.$$

Here $\langle, \rangle_{L_\theta}$ denotes the scalar product defined on $\mathcal{S}_1^2(\mathbb{S}^3)$ by

$$\langle u, v \rangle_{L_\theta} = \int_{\mathbb{S}^3} L_\theta u v \theta \wedge d\theta \quad (3.4)$$

and

$$\begin{aligned} (A_i)_1 &= \frac{\partial}{\partial (a_i)_1} - 2(a_i)_2 \frac{\partial}{\partial (a_i)_0}, \\ (A_i)_2 &= \frac{\partial}{\partial (a_i)_2} + 2(a_i)_1 \frac{\partial}{\partial (a_i)_0}, \\ (A_i)_0 &= \frac{\partial}{\partial (a_i)_0}, \end{aligned}$$

where $(a_i)_1, (a_i)_2, (a_i)_0$ are the coordinates of a_i in some pseudohermitian normal coordinates.

Proposition 3.3 *There exists a $\mathcal{C}^1$ map which, to each $(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$ such that $\sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} \in V(p, \varepsilon)$, with ε small enough, associates $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i)$ satisfying*

$$J \left(\sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} + \bar{v} \right) = \min_{v \text{ satisfies } (V_0)} J \left(\sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} + v \right).$$

Moreover, there exists $c > 0$ such that the following holds

$$\|\bar{v}\| \leq c \left(\sum_{i=1}^p \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) + \sum_{i \neq j} \varepsilon_{ij} (\text{Log}(\varepsilon_{ij})^{-1})^{\frac{1}{2}} \right).$$

Proposition 3.4 *There is $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $u \in V(p, \varepsilon, w)$, then the problem*

$$\min_{\alpha_i > 0, \lambda_i > 0, a_i \in \mathbb{S}^3, h \in T_w(W_u(w))} \left\| u - \sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} - \alpha_0(w + h) \right\|$$

has a unique solution $(\bar{\alpha}, \bar{\lambda}, \bar{a}, \bar{h})$. Thus, we write u as follows:

$$u = \sum_{i=1}^p \bar{\alpha}_i \delta_{\bar{a}_i, \bar{\lambda}_i} + \bar{\alpha}_0(w + \bar{h}) + v,$$

where v belongs to $\mathcal{S}_1^2(\mathbb{S}^3) \cap T_w(W_s(w))$ and satisfies (V_0) , $T_w(W_u(w))$ and $T_w(W_s(w))$ are the tangent spaces at w to the unstable and stable manifolds of w .

4 Critical points at infinity of the variational problem

In the sequel, ∂J designates the gradient of J with respect to the scalar product $\langle \cdot, \cdot \rangle_{L_\theta}$, that is

$$\forall u, v \in S_1^2(\mathbb{S}^3), \text{ we have } \langle \partial J(u), v \rangle_{L_\theta} = J'(u)v.$$

Following Bahri (see [2]), we set the following definitions and notations.

Definition 4.1 A critical point at infinity of J on Σ^+ is a limit of a flow line $u(s)$ of the equation

$$\begin{cases} \frac{\partial u}{\partial s} = -\partial J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s), w)$ for $s \geq s_0$.

Here w is either zero or a solution of (P_K) and $\varepsilon(s)$ is some function tending to zero when $s \rightarrow \infty$. Using Proposition 3.4, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^p \alpha_i(s) \delta_{a_i(s), \lambda_i(s)} + \alpha_0(s)(w + h(s)) + v(s).$$

Denoting $a_i := \lim_{s \rightarrow \infty} a_i(s)$ and $\alpha_i = \lim_{s \rightarrow \infty} \alpha_i(s)$, we let

$$(a_1, \dots, a_p, w)_\infty \quad \text{or} \quad \sum_{i=1}^p \alpha_i(s) \delta_{a_i, \infty} + \alpha_0 w$$

such a critical point at infinity. If $w \neq 0$ it is called of w -type.

Definition 4.2 A critical point at infinity z_∞ is said to be dominated by another critical point at infinity z'_∞ if

$$W_u(z_\infty) \cap W_s(z'_\infty) \neq \emptyset.$$

If we assume that the intersection is transverse, then we obtain

$$\text{index}(z'_\infty) \geq \text{index}(z_\infty) + 1.$$

4.1 Ruling out the existence of critical points at infinity in $V(p, \varepsilon, w)$ for $w \neq 0$

Given a solution w of (P_K) , following the lines of the method given in [16] we prove a non existence result for critical points or critical points at infinity of J in $V(p, \varepsilon, w)$ for $p \in \mathbb{N}^*$ and $\varepsilon > 0$.

Proposition 4.3 Let K be a C^2 positive function on $\mathbb{S}^3$ and let w be a nondegenerate critical point of J in Σ^+ . Then, for each $p \in \mathbb{N}^*$, there is no critical points or critical points at infinity in the set $V(p, \varepsilon, w)$, that means we can construct a pseudogradient of J so that the Palais–Smale condition is satisfied along the decreasing flow lines.

4.2 Morse lemma at infinity in $V(p, \varepsilon)$ for $p \geq 2$

Proposition 4.4 Let K be a C^2 positive function on $\mathbb{S}^3$ satisfying (H_1) and (H_2) and let $\beta := \max\{\beta(y), y \in K\}$. For any $p \geq 2$, there exists a pseudogradient W so that the following holds:

There is a positive constant c independent of $u = \sum_{i=1}^p \alpha_i \delta_{a_i, \lambda_i} \in V(p, \varepsilon)$ such that

- (i) $\langle -\partial J(u), W \rangle_{L_\theta} \geq c \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{c}{\lambda_i^\beta} + c \sum_{i \neq j} \varepsilon_{ij}$
- (ii) $\left\langle -\partial J(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W) \right\rangle_{L_\theta} \geq c \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{c}{\lambda_i^\beta} + c \sum_{i \neq j} \varepsilon_{ij}$
- (iii) $|W|$ is bounded. Furthermore, the only cases where the maximum of the λ_i 's is not bounded is when the concentration points $(a_1, \dots, a_p)$ satisfy: each point a_j is close to a critical point y_{i_j} of K satisfying (1.1) with $i_j \neq i_k$ for $j \neq k$ and $\varrho(y_{i_1}, \dots, y_{i_p}) > 0$, where $\varrho(y_{i_1}, \dots, y_{i_p})$ denotes the least eigenvalue of $M(y_{i_1}, \dots, y_{i_p})$.

As immediate consequence of this proposition, we have:

Corollary 4.5 *for each $p \geq 2$, in $V(p, \varepsilon)$, the only critical points at infinity are $(y_{i_1}, \dots, y_{i_p})_\infty$ such that the matrix $M(y_{i_1}, \dots, y_{i_p})$, defined by (1.3), is positive definite, where the y_{i_j} 's are critical points of K satisfying (1.1) with $i_j \neq i_k$ for $j \neq k$.*

Such a critical point at Infinity has a Morse index equal to

$$4p - 1 - \sum_{j=1}^p \text{Morse}(K, y_{i_j}).$$

Proof of Proposition 4.4. Without loss of generality, we can assume that $\lambda_1 \leq \dots \leq \lambda_p$. Let M be a large positive constant and define

$$I = \{1\} \cup \{i \leq p : \lambda_k \leq M\lambda_{k-1} \forall k \leq i\} \quad (4.1)$$

$$I' = \{i \in I : a_i \text{ is close to a critical point of } K \text{ satisfying (1.2)}\} \quad (4.2)$$

Note that the set I contains the indices i such that λ_i and λ_1 are of the same order. We divide the set $V(p, \varepsilon)$ into four subsets:

$$\begin{aligned} F_1 &= \left\{ \sum \alpha_i \delta_{(a_i, \lambda_i)} : \exists i \in I \text{ such that } \lambda_i |\nabla K(a_i)| \geq C \right\}, \\ F_2 &= \left\{ \sum \alpha_i \delta_{(a_i, \lambda_i)} : \forall i \in I \quad \lambda_i |\nabla K(a_i)| \leq 2C \text{ and } I' = \emptyset \right\} \\ F_3 &= \left\{ \sum \alpha_i \delta_{(a_i, \lambda_i)} : I = I' = \{1\} \text{ and } \lambda_1 |\nabla K(a_1)| \leq 2C \right\}, \\ F_4 &= \left\{ \sum \alpha_i \delta_{(a_i, \lambda_i)} : \forall i \in I \quad \lambda_i |\nabla K(a_i)| \leq 2C, \quad \#I \geq 2 \text{ and } I' \neq \emptyset \right\}, \end{aligned}$$

where C is a large positive constant. In each subset we will define a pseudogradient and the vector-field W required in the proposition will be defined as a convex combination of all the pseudogradients. Let us denote by

$$X_i := \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|} \psi_1(\lambda_i |\nabla K(a_i)|) \quad \text{and} \quad Z_i := \lambda_i \frac{\partial \delta_i}{\partial \lambda_i},$$

where ψ_1 is a cut off function defined by $\psi_1(t) = 0$ if $t \leq \rho$ and $\psi_1(t) = 1$ if $t \geq 2\rho$, where ρ is a large positive constant ($2\rho \leq C$).

First, by an easy computation we obtain

$$-2\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq c \varepsilon_{ij} \quad \text{for } \lambda_j \leq \lambda_i, \quad (4.3)$$

$$\lambda_j^{-2} = o(\varepsilon_{ij}) \quad \text{if } \lambda_i = o(\lambda_j), \quad (4.4)$$

$$\lambda_j^{-2} = O(\varepsilon_{ij}) \quad \text{if } \lambda_i = O(\lambda_j). \quad (4.5)$$

Hence using Proposition A.1. of [16], for each $q \leq p$ since C is large, it is easy to obtain

$$\langle -\partial J(u), X_i \rangle_{L_\theta} \geq c \psi_1(\lambda_i |\nabla K(a_i)|) \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + O\left(\sum \varepsilon_{ij}\right) \right), \quad (4.6)$$

$$\left\langle -\partial J(u), -\sum_{i=q}^p 2^i Z_i \right\rangle_{L_\theta} \geq c \sum_{k \geq q, r \neq k} \varepsilon_{kr} + O\left(\sum_{j=q}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i \neq j} \varepsilon_{ij}\right). \quad (4.7)$$

- In F_1 , we define $W_1 = \sum_{i=1}^p X_i - C_1 \sum_{i=1}^p 2^i Z_i$, where C_1 is a large positive constant ($C_1 \leq \frac{C}{M^p}$). Using (4.6), (4.7) and the fact that there exists an index $i \in I$ such that $\lambda_i |\nabla K(a_i)| \geq C$, we derive the estimate of claim (i) in this case.

- In F_2 , we remark that we can write u as

$$u = u_1 + u_2 \quad \text{where} \quad u_1 = \sum_{i \in I} \alpha_i \delta_{(a_i, \lambda_i)} \quad \text{and} \quad u_2 = \sum_{i \notin I} \alpha_i \delta_{(a_i, \lambda_i)}.$$

Observe that $u_1 \in V(\#I, \varepsilon)$. Since $I' = \emptyset$, we can apply the vector field defined in [11] in this set, we will denote it by $Y(u_1)$. Hence we define $W_2 = Y(u_1) - M \sum_{i \notin I} 2^i Z_i + \sum_{i \notin I} X_i$, where M is defined in 4.1. Using [11] and (4.6), (4.7), we obtain

$$\langle -\partial J(u), W_2 \rangle_{L_\theta} \geq c \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + c \sum_{i=1}^p \frac{1}{\lambda_i^2} + c \sum_{k \neq r} \varepsilon_{kr}.$$

- In F_3 , we have $I = I' = \{1\}$. Let y_1 be the critical point which is close to a_1 , for simplicity, we can assume that $F(y_1) = 0$, so the Cayley transform gives a pseudohermitian normal coordinates centered at y_1 . In this case we define

$$W_3^1 = \sum_{k=1}^2 \frac{1}{\lambda_1} (A_1)_k \delta_1 \frac{\Xi_k}{1 + |\Xi_k|} + \frac{1}{\lambda_1} (A_1)_0 \delta_1 \frac{\Xi_0}{1 + |\Xi_0|}, \quad (4.8)$$

$$W_3^2 = \left(-\sum_{k=1}^2 b_k - \tau b_0 \right) \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \psi_2(\lambda_1 |a_1|), \quad (4.9)$$

where for $k \in \{1, 2\}$ and $k' = \begin{cases} 1 & \text{if } k = 2 \\ 2 & \text{if } k = 1 \end{cases}$, we have

$$\begin{aligned} \Xi_k &= b_k \int_{\mathbb{H}^1} \frac{|x_k + \lambda_{i_1}(a_1)_k|^\beta}{|1 + |z|^2 - it|^\beta} x_k (1 + |z|^2) \theta_0 \wedge d\theta_0 \\ &\quad + b_0 \int_{\mathbb{H}^1} \frac{|t + \lambda_{i_1}^2(a_1)_0 + 2\lambda_1(x_2(a_1)_1 - x_1(a_1)_2)|^{\frac{\beta}{2}}}{|1 + |z|^2 - it|^\beta} \left(x_k (1 + |z|^2) + (-1)^{k'} x_{k'} t \right) \theta_0 \wedge d\theta_0, \\ \Xi_0 &= b_0 \int_{\mathbb{H}^1} \frac{|t + \lambda_1^2(a_1)_0 + 2\lambda_1(x_2(a_1)_1 - x_1(a_1)_2)|^{\frac{\beta}{2}}}{|1 + |z|^2 - it|^\beta} t \theta_0 \wedge d\theta_0, \end{aligned}$$

and ψ_2 is a cut off function defined by $\psi_2(t) = 1$ if $t \leq \mu$ and $\psi_2(t) = 0$ if $t \geq 2\mu$, where μ is a small positive constant.

From [16], we have

$$\langle -\partial J(u), W_3^1 + W_3^2 \rangle_{L_\theta} \geq \frac{c}{\lambda_1^\beta} + c \frac{|\nabla K(a_1)|}{\lambda_1} + O\left(\sum_{k \neq r} \varepsilon_{kr}\right). \quad (4.10)$$

In this region, the pseudogradient will be defined as $W_3 = W_3^1 + W_3^2 - C_2 \sum_{i=2}^p 2^i Z_i + \sum_{i=1}^p X_i$, where C_2 is a large positive constant. Using (4.6), (4.7), (4.10) and the fact that $\lambda_i^{-2} = o(\varepsilon_{ij})$ for each $i \notin I$ and $j \in I$, we derive

$$\langle -\partial J(u), W_3 \rangle_{L_\theta} \geq c \sum_{k \neq r} \varepsilon_{kr} + c \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + c \sum_{i=2}^p \frac{1}{\lambda_i^2} + \frac{c}{\lambda_1^\beta}.$$

Thus the estimate of claim (i) follows in this case.

- In F_4 , we decrease all the variables λ_i with different speeds. First, observe that $\lambda_i^{-2} \leq c_M \varepsilon_{ij}$ for each $i, j \in I$ and for each $i \in I'$ we have $\Delta K(a_i) = o(1)$. In this region, the pseudogradient will be defined by

$$W_4 := - \sum_{i \in I'} Z_i - m_1 \sum_{i \in I \setminus I'} Z_i - \sum_{i \notin I} 2^i Z_i + m_2 \sum_{i=1}^p X_i,$$

where m_1 and m_2 are small positive constants ($m_1 c_M$ and m_2/m_1 are small).

Using Proposition A.1. of [16], formulas (4.6) and (4.7) and the fact that $\lambda_i^{-2} = o(\varepsilon_{ij})$ for each $i \notin I$ and $j \in I$, we derive

$$\langle -\partial J(u), W_4 \rangle_{L_\theta} \geq c \sum_{k \neq r} \varepsilon_{kr} + c \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + c \sum_{i=1}^p \frac{1}{\lambda_i^2}.$$

which implies the estimate of claim (i) in this case.

Finally, the pseudogradient W will be defined by a convex combination of W_1, W_2, W_3, W_4 . This vector field satisfies claim (i).

Regarding claim (ii), as in [11], it follows from claim (i) and the estimates of $\|\bar{v}\|^2$ and $\|\partial J(u + \bar{v})\| \|\bar{v}\|$. Concerning claim (iii), it is easy to get that $|W|$ is bounded. Furthermore, from the definition of the W_i 's, we remark that the maximum of the λ_i 's is a decreasing function on the sets F_1, F_3 and F_4 . However, in F_2 , if $I \neq \{1, \dots, p\}$, the maximum of the λ_i 's is a decreasing function. But if $I = \{1, \dots, p\}$, we have the same case as [11]. Thus claim (iii) follows. The proof of the proposition is thereby complete. $\square$

4.3 Morse Lemma at infinity in $V(1, \varepsilon)$

In this section, we will characterize the critical points at infinity in $V(1, \varepsilon)$.

Proposition 4.6 *Let K be a C^2 positive function on $\mathbb{S}^3$ satisfying (H_1) and (H_2) and let $\beta = \max\{\beta(y), y \in \mathcal{K}\}$. Then, there exists a pseudogradient W so that the following holds: there is a positive constant $c > 0$ independent of $u = \alpha \delta_{(a, \lambda)} \in V(1, \varepsilon)$, such that*

- (i) $\langle -\partial J(u), W \rangle_{L_\theta} \geq c \frac{|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^\beta}$
- (ii) $\left\langle -\partial J(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(W) \right\rangle_{L_\theta} \geq c \frac{|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^\beta}$
- (iii) $|W|$ is bounded. Furthermore, λ is an increasing function along the flow lines generated by W only if a is close to a critical point $y \in I_1 \cup I_2$.

Proof The construction of W depends on the variables a and λ . We will divide the set $V(1, \varepsilon)$ into three subsets:

$$\begin{aligned} F_1 &= \{\alpha \delta_{(a, \lambda)} : \lambda |\nabla K(a)| \geq C'\}, \\ F_2 &= \{\alpha \delta_{(a, \lambda)} : \lambda |\nabla K(a)| \leq 2C' \text{ and } a \text{ is close to a critical point } y \text{ satisfying (1.1)}\}, \\ F_3 &= \{\alpha \delta_{(a, \lambda)} : \lambda |\nabla K(a)| \leq 2C' \text{ and } a \text{ is close to a critical point } y \text{ satisfying (1.2)}\}, \end{aligned}$$

where C' is a large positive constant.

- In F_1 , we define

$$W_1 = \frac{1}{\lambda} \frac{\partial \delta}{\partial a} \frac{\nabla K(a)}{|\nabla K(a)|}$$

and using Proposition A.1 from [16], we get

$$\langle -\partial J(u), W_1 \rangle_{L_\theta} \geq c \frac{|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^2}. \quad (4.11)$$

- In F_2 , we define

$$W_2 = -\Delta K(y)\lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda}$$

and using Proposition A.1 from [16], we derive that

$$\langle -\partial J(u), W_2 \rangle_{L_\theta} \geq c \frac{|\Delta K(a)|}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right) \geq c \frac{|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^2}. \quad (4.12)$$

- In F_3 , we set $W_3 = W_3^1 + W_3^2$ where W_3^1 and W_3^2 are defined by (4.8) and (4.9), respectively. We have

$$\langle -\partial J(u), W_3 \rangle_{L_\theta} \geq c \frac{|\nabla K(a)|}{\lambda} + \frac{c}{\lambda^\beta}. \quad (4.13)$$

The required pseudogradient W will be defined by convex combination of W_1 , W_2 , W_3 . Using (4.11), (4.12) and (4.13), claim (i) follows. Regarding claim (ii), it follows from claim (i) and the estimate of $\|\bar{v}\|^2$. Concerning claim (iii), it follows from the definition of W . Hence the proof of the proposition is completed. $\square$

Once the pseudogradient is constructed, following [1, 11], one can find a change of variables which gives the normal form of the functional J on the subset $F_y = \{\alpha \delta_{(a,\lambda)} + v : a \text{ is close to } y, y \in \mathcal{K}\}$. More precisely, we have

Proposition 4.7 *For $y \in \mathcal{K}$ in $F_y = \{\alpha \delta_{(a,\lambda)} + v : a \text{ is close to } y\}$, there exists a change of variables:*

$$v - \bar{v} \longrightarrow V \quad \text{and} \quad (a, \lambda) \longrightarrow (\tilde{a}, \tilde{\lambda})$$

so that

$$J(\alpha \delta_{(a,\lambda)} + v) = \left(\frac{S}{K(\tilde{a})} \right)^{\frac{1}{2}} \begin{cases} 1 + c(1 - \mu) \frac{-\Delta_\theta K(y)}{\tilde{\lambda}^2} + \|V\|^2, & \text{if } y \text{ satisfies (1.1)} \\ 1 + c(1 - \mu) \frac{-\sum_{k=1}^2 b_k - \tau' b_0}{\tilde{\lambda}^\beta} + \|V\|^2, & \text{if } y \text{ satisfies (1.2)} \end{cases}$$

where μ is a small positive constant.

The proof of Proposition 4.7 is exactly the same as in [1, 4, 11].

As a consequence of the above proposition, we have the following result:

Corollary 4.8 *Let K be a $\mathcal{C}^2$ positive function satisfying $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$. The only critical points at infinity in $V(1, \varepsilon)$ are y_∞ where $y \in I_1 \cup I_2$. Such a critical point has a Morse index equal to $i(y_\infty) = 3 - \text{Morse}(K, y)$, where $\text{Morse}(K, y)$ the Morse index of K at y if $y \in I_1$ and $i(y_\infty) = 3 - m(y)$ where $m(y) = \#\{b_k; b_k < 0\}$ if $y \in I_2$.*

5 Proof of the theorem

First, we will prove a topological argument which provides a lower bound on the number of critical points of J . For this purpose let $\mathcal{K}_\infty$ be the set of critical points at infinity of J and let L_0 be their maximal Morse index. We set:

$$X := \overline{\bigcup_{z_\infty \in \mathcal{K}_\infty} W_u(z_\infty)},$$

where $W_u(z_\infty)$ is the unstable manifold of the critical point at infinity z_∞ .

By a theorem of Bahri-Rabinowitz [3], we have

$$X = \bigcup_{z_\infty \in \mathcal{K}_\infty} W_u(z_\infty) \cup \bigcup_{y \text{ critical of } J \text{ dominated by a critical point } z_\infty} W_u(y), \quad (5.1)$$

It follows, that X is a stratified set of top dimension $\leq L_0$. Moreover it is contractible in Σ^+ , by taking its suspension, that is, for $a \in \Sigma^+$, we consider

$$C(X) := \{tu + (1-t)a; t \in [0, 1], u \in X\}.$$

Let U be such a contraction which is a stratified set of top dimension $L_0 + 1$. U can also be deformed using the flow of $-\partial J$. For dimension's reason the stable manifold of any critical point of Morse index $\geq L_0 + 2$ can be avoided during such a deformation (see [15]). Therefore, U is deformed onto some set

$$Z := \bigcup_{x \in X} W_u(x) \cup \bigcup_{z \text{ dominated by } X} W_u(z)$$

where x is a critical point or critical point at infinity.

We have the following result.

Proposition 5.1 *Assuming that K satisfies the condition (H_1) and (H_2) and that all the critical points of J having their Morse index less or equal to $L_0 + 1$ are nondegenerate, then it holds:*

$$\#\{\text{critical points of Morse index} \leq L_0 + 1\} \geq \left| 1 - \sum_{z_\infty \in \mathcal{K}_\infty} (-1)^{i(z_\infty)} \right|, \quad (5.2)$$

where $i(z_\infty)$ denotes the Morse index of the critical point at Infinity z_∞ .

Proof Since X is contractible in Z , we have from the exact sequence in homology that:

$$\cdots \rightarrow H_k(X) \rightarrow H_k(Z) \rightarrow H_k(Z, X) \rightarrow H_{k-1}(X) \rightarrow H_{k-1}(Z) \rightarrow \cdots$$

where $H_k(X) := H_k(X, \mathbb{Q})$ is the k th homology group with rational coefficients.

Therefore, it follows that

$$\sum_{j=0}^{L_0+1} (-1)^j (\dim H_j(Z, X) + \dim H_j(X)) = 1,$$

which implies then

$$\sum_{j=0}^{L_0+1} \dim H_j(Z, X) \geq \left| 1 - \sum_{j=0}^{L_0} (-1)^j \dim H_j(X) \right|. \quad (5.3)$$

Now observe that the pair (Z, X) is built by adding to X the unstable manifold of other critical points of index $\leq L_0 + 1$. Namely each time we add one of these unstable manifolds, starting from X and going with increasing index. At each step the new object we obtain has a total dimension of homology increased at most by one. Therefore, the total homology of (Z, X) has its dimension upper bounded by the number of critical points of index $\leq L_0 + 1$, not dominated by X . Therefore, we have the following Lemma:

Lemma 5.2 *Under the assumption that all the critical points with Morse index less or equal to $L_0 + 1$ are nondegenerate, it holds:*

$$\sum_{j=0}^{L_0+1} \dim H_j(Z, X) \leq \#\{\text{critical points of index} \leq L_0 + 1\}$$

The critical points in the above estimate are those in Z but not in X .

Now Proposition 5.1 follows from formula (5.2) and Lemma 5.2. Indeed it follows from (5.2) that:

$$\left| 1 - \sum_{j=0}^{L_0} (-1)^j \dim H_j(X) \right| \leq \sum_{j=0}^{L_0+1} \dim H_j(Z, X).$$

Now observe that:

$$\sum_{j=0}^{L_0} (-1)^j \dim H_j(X) = \chi(X), \text{ the Euler Characteristic of } X.$$

It follows from formula 5.1 that

$$\chi(X) = \sum_{z_\infty \in \mathcal{K}_\infty} (-1)^{i(z_\infty)} + \sum_{y \in X, y \text{ critical point}} (-1)^{\text{morse}(y)}$$

Therefore,

$$\left| 1 - \sum_{z_\infty \in \mathcal{K}_\infty} (-1)^{i(z_\infty)} \right| \leq \sum_{j=0}^{L_0+1} \dim H_j(Z, X) + \#\{\text{critical points in } X\}.$$

Using now the upperbound of Lemma 5.2 on the homology on the right hand side of the above formula, our claim in Proposition 5.1 follows. $\square$

Once Proposition 5.1 is proved, Theorem 1.1 follows immediately from Proposition 5.1.

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