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On nonlinear inclusions in non-smooth domains

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Abstract This paper reviews the old and new landmark extensions of the famous intermediate value theorem (IVT) of Bolzano and Poincaré to a set-valued operator $\Phi : E \supset X \rightrightarrows E$ defined on a possibly non-convex, non-smooth, or even non-Lipschitzian domain X in a normed space E . Such theorems are most general solvability results for nonlinear inclusions: $\exists x_0 \in X$ with $0 \in \Phi(x_0)$. Naturally, the operator Φ must have continuity properties (essentially upper semi- or hemi-continuity) and its values (assumed to be non-empty closed sets) may be convex or have topological properties that extend convexity. Moreover, as the one-dimensional IVT simplest formulation tells freshmen calculus students, to have a zero, the mapping must also satisfy “direction conditions” on the boundary ∂X which, when $X = [a, b] \subset E = \mathbb{R}$, $\Phi(x) = f(x)$ is an ordinary single-valued continuous mapping, consist of the traditional “sign condition” $f(a)f(b) \leq 0$. When X is a convex subset of a normed space, this sign condition is expressed in terms of a tangency boundary condition $\Phi(x) \cap T_X(x) \neq \emptyset$, where $T_X(x)$ is the tangent cone of convex analysis to X at $x \in \partial X$. Naturally, in the absence of convexity or smoothness of the domain X , the tangency condition requires the consideration of suitable local approximation concepts of non-smooth analysis, which will be discussed in the paper in relationship to the solvability of general dynamical systems.

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المخلص

تستعرض هذه الورقة البحثية المعالم القديمة والحديثة لتمديدات نظرية القيمة المتوسطة الشهيرة لبولزانو وبوانكاريه إلى مؤثر ذي قيم مجموعية $\Phi: E \supset X \rightrightarrows E$ ربما يكون معرفاً على مجال غير محدب، أو غير أملس، أو حتى غير ليبيشيتزي X في فضاء معياري E . مثل هذه المبرهنات هي النتائج الأعم لحلول الاحتواءات غير الخطية: $\exists x_0 \in X$ بحيث أن $0 \in \Phi(x_0)$. بالطبع، يجب أن يكون للمؤثر Φ بعض خصائص الاستمرار (أساساً استمرار شبه علوي أو نصف استمرار) وربما تكون قيمه (التي من المفترض أن تكون مجموعات مغلقة وغير خالية) محدبة أو لها خصائص توبولوجية تمدد التحدّب. بالإضافة إلى ذلك، وحيث أن أسهل صيغة لنظرية القيمة المتوسطة في البعد الواحد تبين لطلاب الصفوف الأولى الجامعية أن للدالة قيمة صفرية، فإن الراسم يجب أن يحقق "شروط الاتجاه" على الحدود ∂X ، التي عندما تكون $X = [a, b] \subset \mathbb{R}$ و $\Phi(x) = f(x)$ دالة عادية في متغير واحد ومتصلة، تتكون من "شرط الإشارة" التقليدي $f(a)f(b) \leq 0$. عندما تكون X مجموعة جزئية محدبة من فضاء معياري E ، فإنه يعبر عن شرط الإشارة بدلالة شرط تماس حدي $\Phi(x) \cap T_X(x) \neq \emptyset$ حيث $T_X(x)$ مخروط التماس للتحليل المحدب لـ X عند $x \in \partial X$. بالطبع، عند غياب التحدّب والملوسة للمجال X ، فإن شرط التماس يستوجب اعتبار بعض المفاهيم من التحليل غير الأملس لتقريبات محلية مناسبة، والتي سوف نناقشها في الورقة من ناحية العلاقة مع قابلية الحل لأنظمة ديناميكية عامة.

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1 Historical background and preliminaries

In 1817, the Bohemian philosopher, theologian and mathematician Bernhard Bolzano published a series of investigations consecrating him as one of the forefathers of modern analysis [19]. This work contained among other things the first modern criterion of convergence and the celebrated Bolzano-Weierstrass theorem,¹ which he used as a lemma to provide the first “*purely analytic proof of the theorem that between any two values, which give results of opposite sign, there lies at least one real root of the equation.*” This is known as the *intermediate value theorem* (IVT) in dimension one: the first existence theorem in a freshmen calculus course. In today’s notation, it reads: *if $f \in \mathbf{c}[a, b], \mathbb{R}$ and $f(a)f(b) \leq 0$, then $f(x_0) = 0$ for some $x_0 \in [a, b]$.*

As part of his program of rigorizing analysis, Augustin Louis Cauchy provided 4 years later (1821) another proof of the IVT in his *Cours d’analyse*.² Cauchy was the first to contribute significantly (in 1831 and 1837) to the study of the planar case ($f_1(x_1, x_2) = 0$, $f_2(x_1, x_2) = 0$) with scalar functions f_i “*defined inside a domain on whose boundary they do not vanish simultaneously.*” He defined the Cauchy’s index, precursor of the notion of degree of a mapping. Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882) worked to make Cauchy’s work more rigorous and precise, laying the bases for the analytic definition of the topological degree of a continuous mapping.

Building on the contributions of Cauchy, Sturm-Liouville and later Leopold Kronecker generalized in 1869 the Cauchy’s index to C^1 vector fields on $\mathbb{R}^n$ into what became known as the *Kronecker index* $i(f, a)$, which counts the generalized multiplicity of an isolated root a of a general equation $f(x) = 0$, $f \in \mathbf{c}^1(\mathbb{R}^n, \mathbb{R})$. ($i(f, a)$ is precisely the Brouwer degree $\deg(f, B(a, \epsilon), 0)$ for $\epsilon > 0$ small enough.) The reader is referred to Dinca–Mahwin [34] for a detailed modern exposition on the Kronecker index and the Brouwer degree.

This paper aims at reviewing, in some detail, the most important extensions of the Bolzano IVT to spaces of arbitrary dimensions and point to set mappings defined on compact domains that may or may not be convex. The remainder of this section is devoted to a survey of landmark generalizations of the IVT to n -dimensional Euclidean spaces and to general normed spaces, followed by continuity concepts for set-valued maps. Tangency conditions for convex and non-smooth domains are discussed in Sect. 2. Section 3 outlines a general method, based on the Browder–Ky Fan fixed point theorem, for the solvability of the inclusion $0 \in \Phi(x_0)$ where Φ is an upper hemicontinuous point to set map with closed convex values defined on a compact convex subset of a locally convex topological linear space. Section 4 discusses the case of non-smooth domains with or without Lipschitzian behavior and maps with convex or non-convex values.

1.1 Extensions of the IVT to arbitrary dimensions

In a qualitative study of nonlinear ordinary differential equations, using the Kronecker index, Jules Henri Poincaré extended in 1883 the IVT to $\mathbb{R}^n$ [58]. “*Mr. Kronecker, Poincaré writes, has presented to the Berlin Academy, in 1869, a memoir on functions of several variables, including an important theorem from which the following result follows easily: let $\xi_1, \xi_2, \dots, \xi_n$ be n continuous functions in the n variables $x_1, x_2, \dots, x_n$, the variable x_i restricted to range among the limits $-a_i$ and $+a_i$. Let us suppose that for $x_i = a_i$ the function ξ_i is always positive, and for $x_i = -a_i$ the function ξ_i is always negative. Then, I say that there is a system of values for the x at which all the ξ_i vanish. This result can be applied to the three-body problem to prove it has infinitely many special solutions . . .*”

The IVT in n -dimension was later used in 1910 by Jacques Hadamard to give the first proof of the Brouwer fixed point theorem for arbitrary n .³

¹ It was only 50 years later that Karl Weierstrass proved the theorem again and stressed its importance in the foundations of analysis.

² The idea was, as one can imagine, older than Bolzano and Cauchy’s. Earlier mathematicians considered it intuitively obvious, thus requiring no proof. For instance, Dutch mathematician Simon Stevin (1548–1620) proved the IVT for polynomials, anticipating Cauchy’s algorithmic proof. There are views that Cauchy was more than inspired by Bolzano’s earlier work (without acknowledgment). Others see in the similarities (e.g., in the definition of continuity, the convergence criterion, and the proof of the IVT) nothing more than common earlier influences, particularly those of Lagrange. We refer to Grabiner [39] for an insightful discussion.

³ The Italian mathematician, Carlo Miranda [53] proved in 1940 that the Brouwer fixed point theorem is equivalent to Poincaré’s intermediate value theorem, and thus the appellation, used by some, of the *Poincaré–Miranda Theorem*.

The sign conditions $\xi_i(a_i, x^i) > 0$, $\xi_i(-a_i, x^i) < 0$ are boundary conditions on $X = \prod[-a_i, a_i]$ that can be expressed geometrically for the mapping $\xi = (-\xi_i)$ as: $\xi(x) \in S_X(x)$, $\forall x \in \partial X$, where $S_X(x)$ is the cone $\bigcup_{t>0} \frac{1}{t}(X - x)$, $x \in \partial X$. Note that $S_X(x) = \mathbb{R}^n$ whenever $x \in \text{int}(X)$.

Note also that the Brouwer fixed point theorem follows at once from Poincaré's IVT: if $f \in \mathbf{c}(X, X)$, then $\xi(x) = f(x) - x \in (X - x) \subset S_X(x)$; hence a zero $\xi(x_0) = 0$, $x_0 \in X$, is a fixed point $f(x_0) = x_0$ for f .

Landmark solvability results in infinite dimension, with X a subset of a Banach space E , $\xi : X \rightarrow E$ a compact field, i.e., $\xi(x) = f(x) - x$, $x \in X$, where $f \in \mathbf{c}(X, E)$, $f(X) \subset K$ compact $\subset E$ are due to:

- **Rothe [60]:** if X is a closed ball centered at 0_E and ξ is a compact field with $(\xi + Id)(\partial X) \subset X$, then ξ has a zero in X . Clearly, if X is a closed ball centred at 0 in $\mathbb{R}^n$ equipped with the maximum norm and if ξ satisfies Rothe's boundary condition, then the IVT sign condition on ∂X holds. Rothe's theorem remains valid if X is a convex subset of a topological vector space E with $0 \in U$ open $\subset X$, ξ a compact field, and $p(f(x)) \leq p(x)$ for all $x \in \partial X$, where $p : E \rightarrow [0, +\infty)$ is a (not necessarily continuous) positively homogeneous functional satisfying $p^{-1}(0) = \{0\}$ (e.g., p is a seminorm.)
- **Altman [1]:** if X is a closed ball centered at 0_E and ξ is a compact field with $\|\xi(x) + x\|^2 \leq \|\xi(x)\|^2 + \|x\|^2$ for all $x \in \partial X$, then ξ has a zero in X . In the case where E is a Hilbert space, this theorem was first established by Krasnoselsky [50]. Clearly, if X is a closed ball centred at 0 in $\mathbb{R}^n$ equipped with the Euclidean norm, then $(\xi$ satisfies Altman's condition on $\partial X) \iff \langle \xi(x), x \rangle \leq 0$, $\forall x \in \partial X \iff \xi(x) \in S_X(x)$, $\forall x \in \partial X$, i.e., the IVT sign condition on ∂X holds.
- **Yamamoto [66]:** let $X = cl(G)$, G being an open subset in E and ξ a compact field verifying: [there exists $a \in G$ such that if $\xi(x) = \lambda(a - x)$ for some $x \in \partial G$, then $\lambda \geq 0$] and ξ has a zero in X .

Perhaps, the first most striking generalizations of the Poincaré's IVT to locally convex topological vector spaces are the ones due to Halpern and Halpern-Bergman.

- **Halpern [43]:** X is a non-empty convex compact subset of a strictly convex normed linear E and $f \in \mathbf{c}(X, E)$ satisfies the inwardness condition $\forall x \in \partial X$, $\exists 0 < \lambda < 1$ with $\lambda x + (1 - \lambda)f(x) \in X$. Then f has a fixed point $f(x_0) = x_0$, $x_0 \in X$, i.e., the field $\xi = f - Id$ has a zero in X . The inwardness condition of Halpern also amounts to $\xi(x) \in S_X(x) = \bigcup_{t>0} \frac{1}{t}(X - x)$, $\forall x \in X$. Note that here, X need not have a non-empty interior, but also the tangency condition $\xi(x) \in S_X(x)$ seems to be less general than the Yamamoto boundary condition.
- **Halpern and Bergman [46]:** if X is a non-empty convex compact subset of a topological vector space E having separating dual and $\xi \in \mathbf{c}(X, E)$ satisfies the weak inwardness condition $\xi(x) \in T_X(x) = cl(S_X(x))$, $\forall x \in \partial X$, then ξ has a zero in X .
- The strict convexity assumption was removed first by Halpern and Bergman [46] before the publication of Felix Browder's paper [24].

Before turning our attention to the set-valued case $0 \in \Phi(x_0)$, Φ being a set-valued map, let us fix a few definitions, conventions and notations and define crucial concepts for what follows.

1.2 Continuity concepts and classes of set-valued maps

Throughout this paper, it is assumed that topological (vector) spaces (t.v.s. for short) are Hausdorff (and real).

Set-valued maps are assumed to have non-empty values and are denoted by capital greek letters Φ , Ψ , etc., using double arrows $\rightrightarrows$. A zero, $0 \in \Phi(x_0)$, for a set-valued map is also called an *equilibrium* for Φ . Single-valued maps are denoted by small letters f , ξ , s , etc. The space of continuous single-valued maps from a topological space X into a topological space Y is denoted by $\mathbf{c}(X, Y)$.

Definition 1.1 A set-valued map $\Phi : X \rightrightarrows Y$ between two topological spaces is:

- lower semicontinuous (l.s.c.) at $x_0 \in X$, if for any open set V of Y such that $\Phi(x_0) \cap V \neq \emptyset$, there exists an open neighborhood U of x_0 such that $\Phi(x) \cap V \neq \emptyset$ for all $x \in U$ (this amounts to $\Phi(x_0) \subset \liminf_{x \rightarrow x_0} \Phi(x)$ ⁴);

⁴ Recall that, given a map (with non-empty values) $\Phi : X \rightrightarrows Y$ of topological spaces and $x_0 \in X$, the (Kuratowski-Painlevé) inferior limit $\liminf_{x \rightarrow x_0} \Phi(x)$ is the set $\{y \in Y : \text{for any net } x_i \rightarrow x_0, \text{ there exists a net } y_i \rightarrow y \text{ with } y_i \in \Phi(x_i), \text{ for all } i\}$. The (Kuratowski-Painlevé) superior limit $\limsup_{x \rightarrow x_0} \Phi(x) := \{y \in Y : \exists x_i \rightarrow x_0 \text{ and } \exists y_i \rightarrow y \text{ with } y_i \in \Phi(x_i), \text{ for all } i\}$. If Y is metrizable, $\liminf_{x \rightarrow x_0} \Phi(x) = \{y : \limsup_{x \rightarrow x_0} d(y; \Phi(x)) = 0\}$ and $\limsup_{x \rightarrow x_0} \Phi(x) = \{y : \liminf_{x \rightarrow x_0} d(y; \Phi(x)) = 0\}$. Both limiting sets are closed (possibly empty) and $\liminf_{x \rightarrow x_0} \Phi(x) \subset cl(\Phi(x)) \subset \limsup_{x \rightarrow x_0} \Phi(x)$. When equality occurs, the common set is denoted $\lim_{x \rightarrow x_0} \Phi(x)$. For a countable family, $\liminf_{n \rightarrow \infty} \Phi(x_n) = \bigcap_{\epsilon > 0} \bigcup_{N > 0} \bigcap_{n \geq N} B(\Phi(x_n), \epsilon)$ and $\limsup_{n \rightarrow \infty} \Phi(x_n) = \bigcap_{\epsilon > 0} \bigcap_{N > 0} \bigcup_{n \geq N} B(\Phi(x_n), \epsilon)$ (see [6]).

- (ii) *upper semicontinuous (u.s.c.)* at x_0 if for any open set $V \supset \Phi(x_0)$, there exists an open neighborhood U of x_0 such that $\Phi(U) \subset V$.
- (iii) *l.s.c. (u.s.c.)* on X if it is l.s.c. (u.s.c.) at every point of X .
- (iv) an *usco* is a u.s.c. map with (non-empty) compact values.
- (v) compact, if $\Phi(X) \subseteq K$ compact $\subseteq Y$.

In case X is a topological space and E a t.v.s. with topological dual E' , the following regularity properties for a set-valued map $\Phi : X \rightrightarrows E$ are more general than upper semicontinuity:

Definition 1.2 (i) Φ is *upper demicontinuous (u.d.c.)* at $x \in X$ if for any open half-space H in E containing $\Phi(x)$, there exists an open neighborhood N_x of x in X such that $\Phi(u) \subset H$, $\forall u \in N_x$. This concept has been presented by Fan [35].

(ii) Φ is *upper hemicontinuous on X (u.h.c.)* if for each $p \in E'$, the support functional $x \mapsto \sigma_{\Phi(x)}(p) = \sup_{y \in \Phi(x)} \langle p, y \rangle$ is upper semicontinuous as an extended real-valued function on X , i.e., $\forall \lambda \in \mathbb{R} \cup \{\infty\}$, the set $\{x \in X : \sigma_{\Phi(x)}(p) < \lambda\}$ is open in X . This concept has been presented, as far as we can tell, by Cornet [29].

The relationships are:

Proposition 1.3

$$\begin{array}{ccc}
 \text{u.s.c.} & \implies & \text{u.d.c.} \\
 4 \uparrow \downarrow 5 & & \downarrow \uparrow 1 \\
 \text{closed graph} & \begin{array}{c} \xleftarrow{2} \\ \xrightarrow{3} \end{array} & \text{u.h.c.}
 \end{array}$$

where: (1) Φ has compact values;⁵ (2) Φ has closed convex values; (3) Φ is bounded and has convex graph and weakly compact values; (4) Φ is locally compact (this also implies Φ is usco); (5) Φ has closed values + codomain is regular. In particular, if $\Phi(x)$ is convex and weakly compact for all $x \in X$, then $\text{u.s.c.} \Leftrightarrow \text{u.d.c.} \Leftrightarrow \text{u.h.c.}$.

Proof We only prove (3), in the simple case of metrizable X , by showing that, for any fixed $p \in E'$, the extended real function $\sigma_{\Phi(x)}(p)$ is u.s.c. at $\bar{x}$ for any given $\bar{x}$, i.e., for any sequence $x_v \rightarrow \bar{x}$, $\limsup_v \sup_{y \in \Phi(x_v)} \langle p, y \rangle \leq \sup_{y \in \Phi(\bar{x})} \langle p, y \rangle$. Since $\langle p, y \rangle$ is weakly continuous in y , for each v , there exists $y_v \in \Phi(x_v)$ with $\langle p, y_v \rangle = \sup_{y \in \Phi(x_v)} \langle p, y \rangle$. Since Φ is bounded, the sequence $\{y_v\}$ is also bounded, hence it has a weakly convergent subsequence, again denoted by $\{y_v\}$, $y_v \rightarrow \bar{y}$. Thus, $\limsup_v \langle p, y_v \rangle = \langle p, \bar{y} \rangle$. The sequence $\{(x_v, y_v)\}_v \subset \text{graph}(\Phi)$ which is closed and convex, hence is weakly closed. Therefore, $(\bar{x}, \bar{y}) \in \text{graph}(\Phi)$, and $\limsup_v \sup_{y \in \Phi(x_v)} \langle p, y \rangle = \limsup_v \langle p, y_v \rangle = \langle p, \bar{y} \rangle \leq \sup_{y \in \Phi(\bar{x})} \langle p, y \rangle$. $\square$

The interesting fact about u.h.c. maps is that a linear combination of u.h.c. maps is also u.h.c. (one can easily verify that (i) the support functional of a sum of sets $\sigma_{A_1+A_2}$ is the sum of the support functionals $\sigma_{A_1} + \sigma_{A_2}$ and that the sum of u.s.c. extended real-valued functions is also a u.s.c. function; and (ii) that for a given set A and a real λ , $\sigma_{\lambda A} = \lambda \sigma_A$ if $\lambda \geq 0$ and $\sigma_{\lambda A}(p) = |\lambda| \sigma_A(-p)$, $\forall p \in E'$, if $\lambda < 0$, so that $x \mapsto \sigma_{\lambda \Phi(x)}(p)$ is indeed a u.s.c. function for any $\lambda \in \mathbb{R}$).

We shall consider in the sequel the classes of convex valued maps:

Definition 1.4 $\mathbf{K}(X, E) := \{\Phi : X \rightrightarrows E : \Phi \text{ is u.s.c. and has closed convex values}\}.$

$\mathbf{D}(X, E) := \{\Phi : X \rightrightarrows E : \Phi \text{ is u.d.c. and has closed convex values}\}.$

$\mathbf{H}(X, E) := \{\Phi : X \rightrightarrows E : \Phi \text{ is u.h.c. and has closed convex values}\}.$

Clearly, $\mathbf{K}(X, E) \subset \mathbf{D}(X, E) \subset \mathbf{H}(X, E)$.

⁵ Yang and Yuan [67] proved less: a u.h.c. map with non-empty convex values is u.s.c. if and only if it has no interior asymptotic plane.

2 Tangency and dynamics on non-convex domains

This section discusses boundary conditions extending the IVT's sign condition to convex and non-convex subsets of normed spaces. These boundary conditions are expressed in terms of tangent or normal cones and have an inherent dynamical nature as they appear as necessary and sufficient conditions for the existence of viable solutions to dynamical systems. Although one can conceivably work in the context of topological linear spaces with separating duals, for the sake of simplicity we assume in this section that the underlying space E is a real normed space with topological dual E' .

2.1 Tangent and normal cones

We start with a brief discussion of some local approximation concepts of tangent cones to a subset X of E near a point $x \in cl(X)$.

As defined earlier, given $x \in cl(X)$, let $S_X(x) = \mathbb{R}_+(X - x) = \bigcup_{t>0} \frac{1}{t}(X - x)$ be the cone pointed at 0 with base $X - x$. When X is convex, $cl(S_X(x))$ is the *tangent cone of convex analysis* to X at x , and its negative polar cone is precisely the *normal cone of convex analysis*:

$$\begin{aligned} N_X(x) &= S_X(x)^- = \{p \in E' : \langle p, y \rangle \leq 0, \forall y \in S_X(x)\} = cl(S_X(x))^- \\ &= \left\{ p \in E' : p(x) = \sup_{u \in X} \langle p, u \rangle \right\}. \end{aligned}$$

The *Bouligand–Severi contingent cone* $T_X(x)$ is the upper limit in the sense of Painlevé–Kuratowski (i.e., the set of all cluster points) when $t \downarrow 0$, of the family $\{\frac{1}{t}(X - x)\}_{t>0}$, ([23, 62]). It is characterized as:

$$\begin{aligned} T_X(x) &= \{v \in E : \liminf_{t \downarrow 0^+} d_X(x + tv) = 0\} \\ &= \{v \in E : \exists t_n \rightarrow 0^+, \exists v_n \rightarrow v \text{ s.t. } x + t_n v_n \in X, \forall n\}, \end{aligned}$$

where $d_X(x) = \inf\{\|x - u\| : u \in X\}$, $x \in E$, is the distance from the point x to the set X .

The *Clarke (circatangent) cone* (see [61]) $T_X^C(x)$ is the lower limit (i.e., the set of all limit points), when $t \downarrow 0$ and $x' \rightarrow_X x$, of the family $\{\frac{1}{t}(X - x')\}_{t>0, x' \in X}$ and is characterized as:

$$\begin{aligned} T_X^C(x) &= \{v \in E : \limsup_{t \downarrow 0^+, x' \rightarrow_X x} d_X(x' + tv) = 0\} \\ &= \{v \in E : \forall t_n \rightarrow 0^+, \forall x_n \rightarrow_X x, \exists v_n \rightarrow v \text{ s.t. } x_n + t_n v_n \in X, \forall n\}. \end{aligned}$$

Note the following facts:

- If $x \in \text{int}(X)$, then $T_X^C(x) = T_X(x) = S_X(x) = cl(S_X(x)) = E$, the whole space.
- The mapping $d_X(\cdot)$ being globally Lipschitz continuous on E ($|d_X(x) - d_X(x')| \leq \|x - x'\|$, $\forall x, x' \in E$), its Clarke's directional derivative $d_X^0(x)(v) := \limsup_{t \downarrow 0^+, x' \rightarrow_X x} \frac{1}{t}[d_X(x' + tv) - d_X(x')]$ exists and Clarke's cone is written for a given $x \in cl(X)$, as $T_X^C(x) = \{v \in E : d_X^0(x)(v) = 0\}$. Note that the mapping $v \mapsto d_X^0(x)(v)$ is finite, positively homogeneous, subadditive and Lipschitz continuous on E : $|d_X^0(x)(v)| \leq \|v\|$. Moreover, $d_X^0(x)(v)$ is u.s.c. in (x, v) .
- The generalized gradient $\partial^0 d_X(x)$ is the convex weak*-compact set of linear forms defined as $\partial^0 d_X(x) := \{p \in E' : \langle p, v \rangle \leq d_X^0(x)(v), \forall v \in E\}$. It follows that $d_X^0(x)(v) = \sigma_{\partial^0 d_X(x)}(v)$ the support functional of $\partial^0 d_X(x)$ and that $T_X^C(x)$ is the negative polar cone $\partial^0 d_X(x)^-$.
- The *Clarke's normal cone* to X at x is the negative polar cone to $T_X^C(x)$:

$$N_X^C(x) = T_X^C(x)^- = \{p \in E' : \langle p, v \rangle \leq 0, \forall v \in T_X^C(x)\}.$$

Note that $\partial^0 d_X(x) \subset N_X^C(x) = (\partial^0 d_X(x)^-)^-$. The Bipolar theorem (see e.g., [6]) implies that $N_X^C(x) = cl^*(\mathbb{R}_+ \partial^0 d_X(x))$ the weak*-closure of the cone with basis $\partial^0 d_X(x)$.

- $T_X^C(x)$ is a closed convex cone contained in $T_X(x)$, which is itself a closed cone contained in $cl(S_X(x))$.

- The negative polar cone to the Bouligand–Severi contingent cone is the *prenormal cone* (of regular normals) to X at x given by:

$$\begin{aligned}\widehat{N}_X(x) &= T_X(x)^- = \{p \in E' : \langle p, v \rangle \leq 0, \forall v \in T_X(x)\} \\ &= \left\{ p \in E' : \limsup_{u \rightarrow x} \frac{\langle p, u - x \rangle}{\|u - x\|} \leq 0 \right\},\end{aligned}$$

and the (*basic*) *normal cone* to X at x is the cone of limiting proximal normals $N_X(x) = \limsup_{x' \rightarrow x} \widehat{N}_X(x')$. Note that the basic normal cone is not convex and that its closed convex hull is the Clarke's normal cone $N_X^C(x) = cl(conv(N_X(x)))$.

- The set X is said to be *regular* at x if the contingent cone $T_X(x)$ and the normal cone $N_X(x)$ are mutually polar (thus, they are both convex and closed, i.e., $T_X(x) = T_X^C(x)$ and $N_X(x) = N_X^C(x)$).
- If X is *locally convex* at x , then it is regular: $T_X^C(x) = T_X(x) = cl(S_X(x))$ and $N_X(x) = N_X^C(x)$.
- It always holds that $\liminf_{x' \rightarrow x} T_X(x') \subseteq T_X^C(x)$; equality holds whenever $\dim(E) < \infty$.
- X is said to be *sleek* at x if $x \mapsto T_X(x)$ is l.s.c. at x ; in which case, $\liminf_{x' \rightarrow x} T_X(x') \subseteq T_X^C(x) \subseteq T_X(x) \subseteq \liminf_{x' \rightarrow x} T_X(x')$, hence $T_X^C(x) = T_X(x)$ (hence, X is regular) and $T_X(x)$ is also a convex and closed cone, and $x \mapsto T_X^C(x)$ is also l.s.c..
- Examples of non-sleek sets are:
 - $X = \{(x, y) \in \mathbb{R}^2 : \|x\| = \|y\|\}$ with $C_X(0, 0) = \{(0, 0)\} \subset X = T_X(0, 0) = cl(S_X(0, 0))$.
 - $X = X_1 \cup X_2$, the union of the annuli $X_1 = \{(x, y) \in \mathbb{R}^2 : 1 \leq (x+2)^2 + y^2 \leq 4\}$, $X_2 = \{(x, y) \in \mathbb{R}^2 : 1 \leq (x-2)^2 + y^2 \leq 4\}$ with $C_X(0, 0) = \{(0, 0)\} \times \mathbb{R} \subset \mathbb{R}^2 = T_X(0, 0) = cl(S_X(0, 0))$.

Proposition 2.1 *If X is sleek, then the convex and closed valued map $N_X^C : X \rightrightarrows E'$ is weak* - u.s.c., and hence has closed graph.*

Proof Note first that given a sequence of sets $\{T_n\}$ in a normed space, we always have $\liminf_{n \rightarrow \infty} T_n \subseteq (\sigma - \limsup_{n \rightarrow \infty} T_n^-)^-$ (equality occurs, e.g., when T_n is a closed convex cone, this is known as the Duality Theorem; see [6]). Indeed, if $x \in \liminf_{n \rightarrow \infty} T_n$, i.e., $x = \lim_n x_n$, $x_n \in T_n$, and $p \in \sigma - \limsup_{n \rightarrow \infty} T_n^-$, i.e., p is the weak*-limit $\sigma - \lim_{n_k} p_{n_k}$ of a subsequence $p_{n_k} \in T_{n_k}^-$, $\langle p_{n_k}, x_{n_k} \rangle \leq 0$, then $\langle p, x \rangle \leq 0$.

Let $x_n \rightarrow x$ in X , $T_n = T_X^C(x_n)$, $T_n^- = N_X^C(x_n)$. Since X is sleek, T_X^C is l.s.c., hence

$$T_X^C(x) \subseteq \liminf_{n \rightarrow \infty} T_n \subseteq \left(\sigma - \limsup_{n \rightarrow \infty} N_X^C(x_n) \right)^-.$$

Consequently,

$$\sigma - \limsup_{n \rightarrow \infty} N_X^C(x_n) \subseteq \left(\sigma - \limsup_{n \rightarrow \infty} N_X^C(x_n) \right)^{- -} \subseteq T_X^C(x)^- = N_X^C(x),$$

i.e., N_X^C is weak* - u.s.c.. By Proposition 1.3 (5), it has a weakly closed graph, which, being convex, is also strongly closed (Mazur's theorem). $\square$

The reader is referred to Mordukhovich [54] for a detailed and lively account of aspects of variational analysis, including tangents and normal cones to non-smooth sets.

2.2 Boundary conditions

Assume that $\Phi : cl(X) \rightrightarrows E$ and consider the following boundary conditions:

Definition 2.2 Given $x \in \partial X$:

- (R) *Rothe*: $\Phi(x) \cap (X - x) \neq \emptyset$.
- (H) *Halpern*: $\Phi(x) \cap S_X(x) \neq \emptyset$.
- (wH) *Weak Halpern*: $\Phi(x) \cap T_X^C(x) \neq \emptyset$.
- (KF) *Ky Fan*: $p \in N_X^C(x) \implies \inf_{y \in \Phi(x)} \langle p, y \rangle \leq 0$.
- (wKF1) *Weak Ky Fan 1*: $p \in \partial^0 d_X(x) \implies \inf_{y \in \Phi(x)} \langle p, y \rangle \leq 0$.

(wKF2) Weak Ky Fan 2: $\inf_{y \in \Phi(x)} d_X^0(x)(y) \leq 0$.

Observe that if $\Psi : X \rightrightarrows E$ verifies $\Psi(x) \cap X \neq \emptyset, \forall x \in \partial X$ (general case of the Brouwer and Kakutani fixed point theorems), then $\Phi = \Psi - Id$ verifies (R).

Also, note that all conditions (H, wH, KF and wKF1, 2) are meaningful only on the boundary ∂X : indeed, if $x \in \text{int}(X)$, then $T_X^C(x) = E = S_X(x)$, $N_X^C(x) = E^- = \{0_{E'}\} = \partial^0 d_X(x)$, $d_X^0(x)(y) = 0, \forall y \in E$, and the conditions are trivially satisfied.

Notice that $(R) \implies (H)$. In case X is convex, its tangent cone is the cone of convex analysis $cl(S_X(x))$ which coincides with the circatangent cone of Clarke $T_X^C(x)$ and, clearly, $(H) \implies (wH)$ also holds. We have the additional relationships between the boundary conditions above:

Proposition 2.3 Given $\Phi : cl(X) \rightrightarrows E$ and $x \in \partial X$,

$$\begin{array}{ccc} (wH) & \begin{array}{c} \implies \\ \xleftarrow{1} \end{array} & (KF) \\ \downarrow \uparrow_2 & & \downarrow \uparrow_2 \\ (wKF2) & \begin{array}{c} \implies \\ \xleftarrow{3} \end{array} & (wKF1) \end{array}$$

where (1) = $\Phi(x)$ is convex and weakly compact; (2) = $\Phi(x)$ is weakly compact; (3) = $\Phi(x)$ is convex.

Proof (wH) versus (KF). If $\bar{y} \in \Phi(x) \cap T_X^C(x)$ and $p \in N_X^C(x) = T_X^C(x)^-$, then $\inf_{y \in \Phi(x)} \langle p, y \rangle \leq \langle p, \bar{y} \rangle \leq 0$. Conversely, (KF) amounts to $\sup_{p \in N_X^C(x)} \inf_{y \in \Phi(x)} \langle p, y \rangle \leq 0$; if $\Phi(x)$ is convex and weakly compact in E , then in view of the infsup inequality,⁶

$$\inf_{y \in \Phi(x)} \sup_{p \in N_X^C(x)} \langle p, y \rangle \leq \sup_{p \in N_X^C(x)} \inf_{y \in \Phi(x)} \langle p, y \rangle \leq 0.$$

Hence, for some $\bar{y} \in \Phi(x)$, $\sup_{p \in N_X^C(x)} \langle p, \bar{y} \rangle \leq 0$, which implies that

$\sup_{p \in \partial^0 d_X(x)} \langle p, \bar{y} \rangle \leq 0$ i.e., $\bar{y} \in \partial^0 d_X(x)^- = T_X^C(x)$ and (wH) thus holds.

(wH) versus (wKF2). Let $\bar{y} \in \Phi(x) \cap T_X^C(x)$. Since $T_X^C(x) = \partial^0 d_X(x)^-$, then $\langle p, \bar{y} \rangle \leq 0, \forall p \in \partial^0 d_X(x)$; i.e., $\sup_{p \in \partial^0 d_X(x)} \langle p, \bar{y} \rangle = d_X^0(x)(\bar{y}) \leq 0$. Thus $\inf_{y \in \Phi(x)} d_X^0(x)(y) \leq 0$. Conversely, if $\Phi(x)$ is weakly compact, since $y \mapsto d_X^0(x)(y)$ being continuous and convex is weakly l.s.c., there exists $\bar{y} \in \Phi(x)$ with $d^0(x)(\bar{y}) = \inf_{y \in \Phi(x)} d_X^0(x)(y) \leq 0$. Hence, $\langle p, \bar{y} \rangle \leq d^0(x)(\bar{y}) \leq 0, \forall p \in \partial^0 d_X(x)$, i.e., $\bar{y} \in \partial^0 d_X(x)^- = T_X^C(x)$.

(KF) versus (wKF1). Since $\partial^0 d_X(x) \subset N_X^C(x)$, $(KF) \implies (wKF1)$ is obvious. Conversely, define $\varphi : N_X^C(x) \longrightarrow \mathbb{R}$ as the marginal mapping $\varphi(p) := \inf_{y \in \Phi(x)} \langle p, y \rangle$. Since $N_X^C(x) = cl^*(\mathbb{R}_+ \partial^0 d_X(x))$, any given $p \in N_X^C(x)$ is the weak*-limit of a sequence $\{p_n\} \subset \mathbb{R}_+ \partial^0 d_X(x)$, for which $\forall n, \varphi(p_n) = \inf_{y \in \Phi(x)} \langle p_n, y \rangle = \langle p_n, y_n \rangle \leq 0$, for some $y_n \in \Phi(x)$. The sequence $\{y_n\}$ admits a subsequence, again denoted by $\{y_n\}$, weakly converging to some $\bar{y} \in \Phi(x)$. The set $F = \{(y, p) \in \Phi(x) \times N_X^C(x) : \langle p, y \rangle \leq 0\}$ being weakly closed and since $p_n \rightharpoonup^* p$ and $y_n \rightharpoonup \bar{y}$, it follows that $(\bar{y}, p) \in F$, i.e., $\varphi(p) \leq 0$.

(wKF1) versus (wKF2). For any $p \in \partial^0 d_X(x)$, we have $\langle p, y \rangle \leq d^0(x)(y), \forall y \in E$, in particular $\forall y \in \Phi(x)$. Hence, if (wKF2) holds, we have, for any $p \in \partial^0 d_X(x)$, $\inf_{y \in \Phi(x)} \langle p, y \rangle \leq \inf_{y \in \Phi(x)} d^0(x)(y) \leq 0$, i.e., (wKF1) also holds. Conversely, $(wKF1) \iff \sup_{p \in \partial^0 d_X(x)} \inf_{y \in \Phi(x)} \langle p, y \rangle \leq 0$. By the infsup inequality, since $\Phi(x)$ is convex and $\partial^0 d_X(x)$ is convex and weakly compact, we have $\inf_{y \in \Phi(x)} \sup_{p \in \partial^0 d_X(x)} \langle p, y \rangle \leq \sup_{p \in \partial^0 d_X(x)} \inf_{y \in \Phi(x)} \langle p, y \rangle$, thus $\inf_{y \in \Phi(x)} \sup_{p \in \partial^0 d_X(x)} \langle p, y \rangle = \inf_{y \in \Phi(x)} d^0(x)(\bar{y}) \leq 0$ and (wKF2) also holds.

(Note that one can prove directly $(wKF2) \implies (KF)$ if $\Phi(x)$ is weakly compact.) $\square$

⁶ Given two convex subsets X, Y in linear topological spaces, a function $f : X \times Y \longrightarrow \mathbb{R}$ which is u.s.c. and quasi-concave in x , and l.s.c. and quasi-convex in y , then $\inf_Y \sup_X f(x, y) \leq \sup_X \inf_Y f(x, y)$, provided either X or Y is compact. This is the analytic formulation of the coincidence between an $\mathbf{F}^*$ and an $\mathbf{F}$ map in the sense of [14]. Since the reverse inequality always holds, this yields the Maurice Sion version of the von Neumann minimax theorem (cf., e.g., [9, 14]).

2.3 Tangency and viability of trajectories

The dynamical nature of the tangency condition (wH) is remarkably captured by the *Nagumo viability theorem*.

Definition 2.4 A subset X of $\mathbb{R}^n$ is said to be locally viable with respect to a differential system $x'(t) = f(x(t))$, where $f \in \mathbf{c}(X, \mathbb{R}^n)$, iff

$$\forall x_0 \in X, \exists T > 0, \exists x \in C^1 \text{ such that } \begin{cases} x'(t) = f(x(t)), 0 \leq t \leq T, \\ x(0) = x_0, \\ x(t) \in X, \forall t \in [0, T]. \end{cases}$$

Theorem 2.5 (Nagumo [56]) If $\mathbb{R}^n \supseteq X$ is locally closed (i.e., $\forall x \in X, \exists r_x > 0$, with $B(x, r_x) \cap X$ is closed) and $f \in \mathbf{c}(X, \mathbb{R}^n)$ then:

$$\left[\begin{array}{l} X \text{ is locally viable} \\ \text{w.r.t. } x'(t) = f(x(t)) \end{array} \right] \iff [\forall x \in X, f(x) \in T_X(x)],$$

Here, $T_X(x)$ is the Bouligand–Severi contingent cone.

The set-valued extension of the Nagumo’s theorem is from *Bebernes–Schuur* [18].

Theorem 2.6 ([18]) Let $\mathbb{R}^n \supseteq X$ be locally compact and $\Phi \in \mathbf{K}(X, \mathbb{R}^n)$. Then

$$[X \text{ is locally viable w.r.t. } x'(t) \in \Phi(x(t)) \text{ a.e.}] \iff [\forall x \in X, \Phi(x) \cap T_X(x) \neq \emptyset].$$

Here, $x(\cdot) \in \mathbf{ac}([0, T], \mathbb{R}^n)$, the space of absolutely continuous functions.

- Remark 2.7* (i) If X is compact, then it is viable w.r.t. $x'(t) \in \Phi(x(t))$, i.e., there exists a trajectory $x(t) \in X, \forall t \in [0, \infty)$.
(ii) The *convexity of values is important*: if $X = \overline{B}(0, 1) \subset \mathbb{R}^2$, $\Phi(x) = \{(-1, 0), (1, 0)\}$ is an usco. Surely, $\Phi(x) \cap T_X(x) \neq \emptyset$ for all $x \in X$, but X is not locally viable for $x'(t) \in \{(-1, 0), (1, 0)\}$.
(iii) The lack of convexity of values of Φ can be compensated by a stronger regularity of Φ in addition to a stronger tangency condition as shown by Aubin and Cellina (see [4]): if $\mathbb{R}^n \supseteq X$ is locally closed⁷ and $\Phi : X \rightrightarrows \mathbb{R}^n$ is both u.s.c. and l.s.c. with closed values, then

$$[\forall x \in X, \Phi(x) \subset T_X(x)] \implies \left[\begin{array}{l} X \text{ is locally viable} \\ \text{w.r.t. } x'(t) \in \Phi(x(t)). \end{array} \right]$$

Observe that if Φ has closed convex values and is l.s.c., then viability follows from Michael’s selection theorem and Nagumo’s viability theorem.

Noteworthy extensions of the Nagumo viability theorem are from:

- Urescu [65]: Caratheodory versions.
- Haddad [42]: functional differential inclusions.
- Shi [64]: partial differential inclusions.
- Frankowska (see [6]), Plaskacz [57], Frankowska–Plaskasz–Rzezuchowski [38] and others: more Caratheodory versions.
- Cârjă–Vrabie (see [26] and references there), Aubin–Da Prato ([5], 1995), Buckdahn–Quincampoix–Rascanu [25] and others: stochastic control systems and Banach spaces of arbitrary dimensions.

The reader is referred to [6, 26, 33, 48] for extensive references on and proofs of viability theorems.

⁷ $\mathbb{R}^n \supseteq X$ is *locally closed* $\Leftrightarrow X = O \cap F$ with O open and F closed $\Leftrightarrow X$ is relatively open in its closure. A closed subset of $\mathbb{R}^n$ is relatively closed. An open subset of $\mathbb{R}^n$ is relatively closed. There are relatively closed subsets of $\mathbb{R}^n$, other than closed or open sets. For examples: $X := \{(x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } x^2 + y^2 < 1\}$ relatively closed, not open, not closed in $\mathbb{R}^3$.

2.4 Equilibria and co-equilibria in Hilbert spaces

We clarify the relationship between the existence of equilibria and co-equilibria for a set-valued map. For simplicity, assume in this section that the underlying space is a real Hilbert space $(E, \langle \cdot, \cdot \rangle)$ identified with its dual. The results below remain valid with a dual pair (E, E') of a normed space and its topological dual, with the suitable adaptations.

Recall that, given a subset X of E , an element $x_0 \in cl(X)$ is an equilibrium for a set-valued map $\Phi : cl(X) \rightrightarrows E$ if $0 \in \Phi(x_0)$; x_0 is a *co-equilibrium* for Φ if $0 \in \Phi(x_0) - N_X^C(x_0)$ where $N_X^C(x_0)$ is the Clarke's normal cone to X at x_0 as defined in Sect. 2.1. Clearly, an interior co-equilibrium is an equilibrium since, for such a point, $N_X^C = \{0\}$.

Observe that x_0 is a co-equilibrium for Φ if and only if the maps Φ and N_X^C coincide at x_0 , i.e., $\Phi(x_0) \cap N_X^C(x_0) \neq \emptyset$. In view of the fact that $N_X^C(x_0) = T_X^C(x_0)^-$, this coincidence implies the infsup inequality:

$$\inf_{y \in \Phi(x_0)} \sup_{v \in T_X^C(x_0)} \langle y, v \rangle \leq 0.$$

Conversely, $\inf_{y \in \Phi(x_0)} \sup_{v \in T_X^C(x_0)} \langle y, v \rangle \leq 0$ implies that x_0 is a co-equilibrium for Φ , provided $\Phi(x_0)$ is weakly compact. Indeed, the extended real valued function $y \mapsto \sup_{v \in T_X^C(x_0)} \langle y, v \rangle$ is l.s.c. and convex, hence weakly l.s.c.. Therefore, it achieves its infimum on $\Phi(x_0)$ at some y_0 verifying $\langle y_0, v \rangle \leq 0, \forall v \in T_X^C(x_0)$, i.e., $y_0 \in N_X^C(x_0)$.

By a *Hilbert space pair* we mean a pair (X, E) with E a real Hilbert space and X a closed subset of E . Denote by

$$\mathbf{H}_{wH}(X, E) = \{\Phi \in \mathbf{H}(X, E) : \Phi \text{ verifies the boundary condition } (wH) \text{ on } \partial X\}$$

Definition 2.8 Let us say that a Hilbert space pair (X, E) has *the equilibrium property* for the class $\mathbf{H}_{wH}$ if and only if any map $\Phi \in \mathbf{H}_{wH}(X, E)$ has an equilibrium in X . We write $(X, E) \in \mathcal{E}(\mathbf{H}_{wH})$.

Theorem 2.9 Let $(X, E) \in \mathcal{E}(\mathbf{H}_{wH})$ be a Hilbert space pair with X sleek. Then, any compact map $\Psi \in \mathbf{H}(X, E)$ has a co-equilibrium, i.e., $\exists x_0 \in X$ such that $0 \in \Psi(x_0) - N_X^C(x_0)$.

Proof The image $\Psi(X)$ of Ψ is contained in a closed disk D centered at the origin with radius $M > 0$ in E . Consider the set-valued map $\Phi : X \rightrightarrows E$ given by $\Phi(x) := \Psi(x) - (N_X^C(x) \cap D)$. By Proposition 2.1 and since X is sleek, the map $N_X^C : X \rightrightarrows E$ has closed graph. By Proposition 1.3 (3) and since the graph of N_X^C is also convex and the values $N_X^C(x) \cap D$ are closed, convex and bounded, hence weakly compact, it follows that the set-valued map $x \mapsto N_X^C(x) \cap D$ is u.h.c. with closed convex, and bounded values. As a linear combination of u.h.c. maps, Φ is also u.h.c. Being the sum of a compact convex set and a closed bounded convex set, $\Phi(x)$ is closed and convex for each $x \in X$, i.e., $\Phi \in \mathbf{H}(X, E)$. We need to show that Φ verifies (wH) . For any given $x \in \partial X$, since the cone $T_X^C(x)$ is closed and convex, the Moreau decomposition theorem [55] implies that any $y \in \Psi(x)$ can be decomposed as a sum $y = y_T + y_N$ with $y_T = Proj_{T_X^C(x)}(y)$ and $y_N = Proj_{N_X^C(x)}(y)$ and $\langle y_N, y_T \rangle = 0$. Hence, $0 = \langle y_N, y_T \rangle = \langle y_N, y - y_N \rangle = \langle y_N, y \rangle - \|y_N\|^2$ and by the Cauchy-Schwarz-Bunyakowsky's inequality $\|y_N\| \leq \|y\| \leq M$, which is $y_T = y - y_N \in \Psi(x) - (N_X^C(x) \cap D)$, i.e., $\Phi(x) \cap T_X^C(x) \neq \emptyset$. The fact that (X, E) has the equilibrium property for $\mathbf{H}_{wH}$ ends the proof. $\square$

3 Equilibria in convex domains

For convex domains and the classes of maps in Definition 1.4, in case E is a locally convex t.v.s. and $E \supset X$ is compact convex, an equilibrium for $\Phi : X \rightrightarrows E$ occurs in the following situations:

- (a) Browder [24]: $\Phi \in \mathbf{K}(X, E)$ and $\Phi(x) \cap S_X(x) \neq \emptyset, \forall x \in \partial X$.
- (b) Fan [35]: $\Phi \in \mathbf{D}(X, E)$ and $\Phi(x) \cap S_X(x) \neq \emptyset, \forall x \in \partial X$.
- (c) Halpern [45]: $\Phi \in \mathbf{K}(X, E)$ and $\Phi(x) \cap cl(S_X(x)) \neq \emptyset, \forall x \in \partial X$.
- (d) Cornet [29]: $\Phi \in \mathbf{H}(X, E)$ and $\Phi(x) \cap cl(S_X(x)) \neq \emptyset$ for all $x \in X$.
- (e) Fan [37]: $\Phi \in \mathbf{D}(X, E)$ and Φ verifies condition (KF) .

Remark 3.1

- (i) Situation (b) follows from the coincidence Theorem 6 in [35] with $g(x) = \{0\}$ for all $x \in \partial X$.
- (ii) Halpern (see theorem 6 in [45]) replaced the compactness hypothesis of the domain X by that of the map Φ under the stronger inwardness $\Phi(x) \cap S_X(x) \neq \emptyset, \forall x \in \partial X$, and provided the underlying space is complete. More precisely, he showed: if E is a complete locally convex t.v.s., $E \supseteq X$ closed convex, $\Phi \in \mathbf{K}(X, E)$ with $cl(\Phi(X))$ compact, and $\Phi(x) \cap S_X(x) \neq \emptyset, \forall x \in \partial X$, then Φ has an equilibrium.
- (iii) Situation (e) follows from the coincidence Theorem 9 in [37] with $g(x) = \{0\}$ for all $x \in \partial X$.
- (iv) The compactness of the domain X can be replaced by a weaker coercivity condition of “Karamardian type” (see Theorem 3.8 and Remark 3.9(3) below).

3.1 A proof based on the Browder–Ky Fan fixed point theorem

We opt to present the proof of the existence of an equilibrium, in the fully convex case, as it opens the door to further generalizations.

The Browder–Ky Fan fixed point theorem (Theorem 1 in [24] and Theorem 2 in [36]) asserts that if X is a compact convex subset of a t.v.s. E and $\Phi : X \rightrightarrows X$ is a set-valued map with convex values and open fibers, then either Φ has a maximal element $x_0 \in X$, i.e., $\Phi(x_0) = \emptyset$, or Φ has a fixed point $\hat{y} \in \Phi(\hat{y})$.⁸ The analytical expression of this result as an alternative for systems of nonlinear inequalities reads:

Proposition 3.2 ([13]) *If X is a convex compact subset of a t.v.s. E and $f : X \times X \rightarrow \mathbb{R}$ a numerical function satisfying:*

- (i) $\forall y \in X, x \mapsto f(x, y)$ is l.s.c. on X ;
- (ii) $\forall x \in X, y \mapsto f(x, y)$ is quasiconcave on X .

Then, for any given $\lambda \in \mathbb{R}$, the following nonlinear alternative holds:

- (1) $\exists x_0 \in X$ such that $f(x_0, y) \leq \lambda, \forall y \in X$; or
- (2) $\exists \hat{y} \in X$, with $f(\hat{y}, \hat{y}) > \lambda$.

Proof The map $\Phi : X \rightrightarrows X$, given by $\Phi(x) := \{y \in X : f(x, y) > \lambda\}$, has convex values and open fibers. Either it has a maximal element $\Phi(x_0) = \emptyset$, i.e., (1) holds; or it has a fixed point $\hat{y} \in \Phi(\hat{y})$ and (2) is true. $\square$

Corollary 3.3 ([9]) *Let X be a convex and compact subset of a t.v.s. E , $Y \subseteq \{\varphi : X \rightarrow \mathbb{R} : \varphi \text{ is u.s.c. and quasiconcave}\}$, $A \in \mathbf{S}(X, Y)$, a class of maps having continuous selections. Then,*

$$\exists x_0 \in X, \exists \varphi_0 \in A(x_0) \text{ with } \varphi_0(x_0) = \max_{u \in X} \varphi_0(u).$$

Proof Let $s : X \rightarrow Y$ be a continuous selection of A and set $f(x, y) = s(x)(y) - s(x)(x)$.

Clearly, $f(\cdot, y)$ is l.s.c. and $f(x, \cdot)$ is quasiconcave

Apply Proposition 3.2 with $\lambda = 0$: (2) is impossible; thus (1) holds, i.e., $\exists x_0 \in X$ with $s(x_0)(y) - s(x_0)(x_0) \leq 0, \forall y \in X$. Hence, $\varphi_0 = s(x_0) \in A(x_0)$ and $\varphi_0(y) \leq \varphi_0(x_0), \forall y \in X$. $\square$

A similar result was independently proved by Bellenger [8].

Corollary 3.4 *Let X be a convex compact subset of a t.v.s. E , $Y \subseteq \{\varphi : X \rightarrow \mathbb{R} : \varphi \text{ is u.s.c. and quasiconcave}\}$, and $f : X \times Y \rightarrow \mathbb{R}$ verifies $x \mapsto f(x, \varphi)$ l.s.c. and $\varphi \mapsto f(x, \varphi)$ quasiconcave. Then, either*

- (1) $\exists \hat{x} \in X$ with $f(\hat{x}, \varphi) \leq 0$ for all $\varphi \in Y$, or
- (2) $\exists x_0 \in X, \exists \varphi_0 \in Y$ with $\varphi_0(x_0) = \max_{u \in X} \varphi_0(u)$ and $f(x_0, \varphi_0) > 0$.

Proof The map $A : X \rightrightarrows Y$, $A(x) := \{\varphi \in Y : f(x, \varphi) > 0\}$ has convex values and open fibers. If (1) fails, A is an $\mathbf{F}^*$ -map defined on a compact set, hence selectionable (see [14]), i.e., $A \in \mathbf{S}(X, Y)$. Corollary 3.3 applies. $\square$

The following result contains all situations (a)–(e) above.

⁸ A set-valued map with non-empty convex values and open fibers was called an $\mathbf{F}^*$ -map in [14].

Theorem 3.5 Assume that X is a convex compact subset in a locally convex t.v.s. E and $\Phi \in \mathbf{H}(X, E)$ verifies condition (KF). Then Φ has an equilibrium in X .

Proof Let $Y = E'$, $f : X \times Y \rightarrow \mathbb{R}$ be $f(x, \varphi) = \inf_{y \in \Phi(x)} \langle \varphi, y \rangle$ which is l.s.c. in x (since $x \mapsto \sigma_\Phi(x, \varphi)$ is u.s.c.) and concave in φ . Condition (KF) opposes (2) of Corollary 3.4. Hence (1) holds: $\exists \hat{x} \in X$ with $\inf_{y \in \Phi(\hat{x})} \langle \varphi, y \rangle \leq 0$ for all $\varphi \in E'$.

If $0 \notin \Phi(\hat{x})$, by the Hahn-Banach separation theorem, $\exists \varphi \in E'$, $\exists \lambda \in \mathbb{R}$ with $\varphi(0) = 0 < \alpha < \varphi(y)$, $\forall y \in \Phi(\hat{x})$. This implies that $0 < \alpha \leq \inf_{y \in \Phi(\hat{x})} \langle \varphi, y \rangle \leq 0$, thus $0 \in \Phi(\hat{x})$. $\square$

Since convex sets are sleek, the image of a compact set under a compact valued u.h.c. map is compact, Theorems 2.9 and 3.5 imply:

Corollary 3.6 If X is a convex compact subset of a Hilbert space E , any compact-valued map $\Psi \in \mathbf{H}(X, E)$ has a co-equilibrium.

Using the same approach based on the Browder–Ky Fan fixed point theorem, Theorem 3.5 has been extended to surjectivity results for perturbations of $\mathbf{H}$ –maps by convex processes by Kryszewski and the author in [15] with the compactness condition on the domain replaced by a coercivity condition (thus extending results of Aubin [2]) as described below.

Let E, F be two normed spaces and let $\Lambda(E, F)$ be the normed space of all closed convex processes (i.e., set-valued maps whose graphs are closed convex cones in $E \times F$) with norm $\|\Lambda\| = \sup_{u \in E \setminus \{0\}} \inf_{v \in \Lambda(u)} \frac{\|v\|}{\|u\|}$. Let $E \supset X$ and let $\mathcal{L} : X \rightarrow \Lambda(E, F)$ be a continuous operator.

Definition 3.7 A map $\Phi : X \rightrightarrows F$ is said to verify the condition (wH) on a pair $(X_1 \subset X, X_2 \subset X)$ with respect to the operator $\mathcal{L}$ iff:

$$\forall x \in X_1, cl(\mathcal{L}(x)(S_{X_2}(x))) \cap \Phi(x) \neq \emptyset.$$

Theorem 3.8 Assume that X is a convex set in the normed space E , $\Phi \in \mathbf{H}(X, F)$ and $\mathcal{L}$ satisfies the boundedness condition:

$$\exists M > 0 \text{ such that } \forall x \in X, \forall u \in E, \|u\| = 1, \exists v \in \mathcal{L}(x)(u) \text{ with } \|v\| \leq M.$$

Assume also the existence of a compact subset K of X such that for each finite subset $N \subset X$, there exists a convex compact subset $C_N \subset X$ such that the map Φ verifies:

- (i) the condition (wH) on $(C_N \setminus K, C_N)$ with respect to $\mathcal{L}$.
- (ii) the condition (wH) on $(K \cap \partial X, X)$ with respect to $\mathcal{L}$.

Then,

- (1) Φ has an equilibrium, and
- (2) $\forall x_0 \in X$, $\Phi(\cdot) + \mathcal{L}(\cdot)(\cdot - x_0)$ has an equilibrium.

Remark 3.9

- (1) This theorem remains valid when E is a t.v.s. and F is a t.v.s. with separating dual.
- (2) In the case where for every $x \in X$, $\Lambda(x) \equiv L$ is a linear operator from E into F , and $C_N = C$ is the same compact convex set for all finite subsets N of K , this result can be found in [10] and (2) guarantees the surjectivity of the perturbation $\Phi + L$ onto $L(X)$.
- (3) In the case where X is compact and $\Lambda(x) = L(x)$ is a linear operator, this result is precisely the solvability theorem in Aubin and Frankowska [6]. By putting $K = \emptyset$ and $C_N = X$ for any N , (i) and (ii) reduce to:

Φ satisfying the condition (KF) with respect to L .

As an immediate consequence of Theorem 3.8, we deduce that the existence of viable solutions in a non-compact domain implies the existence of a stationary solution⁹ for the inclusion $\Lambda(x'(t)) \subset \Phi(x(t))$, $t \in [0, T]$. More precisely, we have:

⁹ A stationary solution is a trajectory with vanishing velocity.

Corollary 3.10 ([15]) *Let X be a closed convex subset of $\mathbf{R}^n$, K a compact subset of X , C a bounded subset of X , $\Lambda : \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ a linear process,¹⁰ and $\Phi \in \mathbf{H}(X, \mathbf{R}^m)$.*

Assume that the following properties are satisfied:

- (i) *given any $x_0 \in K$, there exist $T > 0$ and a solution $x(\cdot)$ of the differential inclusion $\Lambda(x'(t)) \subset \Phi(x(t))$ on $[0, T]$ starting at x_0 such that for any $T' \in (0, T]$ there exists $t \in (0, T']$ with $x(t) \in X$;*
 - (ii) *given any $x_0 \in X \setminus K$, there exists $T > 0$ and a solution $x(\cdot)$ of the same differential inclusion on $[0, T]$ starting at x_0 such that for any $T' \in (0, T]$ there exists $t \in (0, T']$ such that $x(t) \in \text{conv}(x_0, C)$;*
- Then $\Lambda(x'(t)) \subset \Phi(x(t))$ has a stationary solution in X .*

Remark 3.11 (1) Conditions (i)–(ii) state that if a trajectory of the differential inclusion starts in K then it must first enter X , and if it starts in $X \setminus K$ then it is first attracted by C in some weak sense (the trajectory intersects the drop with vertex x_0 and base C).

- (2) When X is a compact convex viability domain of Φ , $\mathbf{R}^n = \mathbf{R}^m$ and $\Lambda = Id_{\mathbf{R}^n}$, then (i) and (ii) are obviously satisfied with $K = C = X$. This corresponds to the equilibrium theorem of [6].

4 Equilibria in non-smooth domains

Naturally, the generalization of the IVT that comes immediately to mind, in case of non-convex domains, is the existence of a zero for a tangent vector field defined on a compact smooth manifold with non-trivial Euler characteristic. We shall now discuss extensions of this result to set-valued maps and more general domains.

When considering non-convex domains, it is natural from a topological point of view to look at domains that are homeomorphic to convex sets or contractible (e.g., star shaped), or more generally absolute (neighborhood) retracts of normed spaces. From an optimization perspective, one would consider for example proximally smooth or regular sets defined by smooth or non-smooth inequalities, or subsets of normed spaces that are (locally) copies of epigraphs of Lipschitz continuous functions. Consideration of non-convex domains for extensions of the IVT necessitates adequate boundary conditions described in terms of tangent and normal cones discussed in Sect. 2.

4.1 Lipschitz regularity and L -retracts

Significant first steps toward relaxing the convexity of the domain are as follows:

- *homeomorphically compact convex* domain with stronger regularity of mapping (Bhatia–Szego [7]): if $\mathbf{R}^n \supset X$ is homeomorphic to a compact convex set, $f : X \rightarrow \mathbf{R}^n$ is Lipschitz and $\forall x \in X$, $f(x) \in T_X(x)$, then $\exists \bar{x} \in X$ with $f(\bar{x}) = 0$.
- “Lipschitz” cannot be weakened to “continuous” (Clarke–Ledyaev–Stern [28]): consider the surface

$$X := \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = z^4, 0 \leq z \leq 1\}$$

which is homeomorphic to the unit disk $x^2 + y^2 \leq 1, z = 0$ by projection onto the xy -plane.

The mapping $f : X \rightarrow \mathbf{R}^3$ given by:

$$f(x, y, z) = \begin{cases} (-yz + \frac{2x(1-z)}{(x^2+y^2)^{1/4}}, xz + \frac{2y(1-z)}{(x^2+y^2)^{1/4}}, 1-z) & (x, y) \neq (0, 0) \\ (0, 0, 1-z) & (0, 0, z) \end{cases}$$

is continuous on X but not Lipschitz at $(0, 0, 0)$.

Moreover, $\forall (x, y, z) \in X$, $f(x, y, z) \in T_X(x, y, z)$:

- at points where $0 < z < 1$, X is locally smooth and $f(x, y, z) \in T_X(x, y, z)$, the tangent plane;
- at the origin, $f(0, 0, 0) = (0, 0, 1) \in T_X(0, 0, 0) = \{(0, 0)\} \times [0, \infty)$; and
- on the crown of the surface, $f(x, y, 1) = (-y, x, 0)$ a tangent vector to X at $(x, y, 1)$.

But $f(x, y, z) \neq (0, 0, 0)$, $\forall (x, y, z) \in X$.

¹⁰ A linear process is a set-valued map whose graph is a vector subspace.

Note that although X is of course locally viable with respect to $u' = f(u)$, not all trajectories are viable in X . Indeed, the vector field $u(t) = (0, 0, 1 - e^{-t})$ is a trajectory starting at $u(0) = (0, 0, 0)$ but leaving X for $t > 0$.

- *Compact epi-Lipschitzian*¹¹ sets in $\mathbb{R}^n$ (Bonnisseau–Cornet [20]): if $\Phi \in \mathbf{K}(X, \mathbb{R}^n)$ with compact values, $\Phi(x) \cap T_X^C(x) \neq \emptyset, \forall x \in \partial X$, then $0 \in \Phi(\bar{x})$ for some $\bar{x}$ in X .
- *Compact proximate retracts*¹² in $\mathbb{R}^n$ (Plaskacz [57]): same conditions on Φ .
- *Bi-Lipschitz homeomorphic to a compact convex set in $\mathbb{R}^n$* (Clarke–Ledyaev–Stern [28]): same conditions on Φ .
- *Bi-Lipschitz homeomorphic to a compact convex set in a locally convex t.v.s.* (Cornet–Czanerski [31]): same conditions on Φ .
- *Compact L-retract of normed spaces with non-trivial Euler characteristic* (Ben-El-Mechaiekh–Kryszewski [16, 17]): same conditions on Φ .
- *Compact ANR(not necessarily Lipschitz) with non-trivial Euler characteristic* (Ben-El-Mechaiekh–Kryszewski [16, 17]): same conditions on Φ .

The next two subsections of this paper focus on the last two types of domains.

As mentioned earlier, one of the most natural classes of subsets of normed spaces to be considered in extending fixed point and equilibrium theorem to non-convex domains is the class of absolute retracts (ARs) and more generally that of absolute neighborhood retracts (ANRs) (see e.g., Borsuk’s generalizations of the Brouwer fixed point theorem [21]).

Recall that a subset X of a normed space E is said to be a (neighborhood) retract of E if there exists a mapping $r : E \rightarrow X$ ($r : U \rightarrow X$, where U is an open neighborhood of X) such that $r(x) = x$ for all $x \in X$; (the retraction r is not unique).

It is well-known that every continuous self-mapping f of a compact AR has a fixed point [21]; while if X is a compact ANR, then $f : X \rightarrow X$ has a fixed point provided its Lefschetz number $\lambda(f)$ is non-zero.

Since the Euler characteristic¹³ $\chi(X)$ of X is precisely the Lefschetz number of the identity mapping Id_X , then assuming that $\chi(X) \neq 0$ (which holds true if X is an absolute retract), it follows that a map from X into itself that is homotopic to the identity has a fixed point. It is well established that $\chi(X) = 0$ for any compact manifold of odd dimension X , and that for such domains there are non-vanishing tangent vector fields.

Thus, $\chi(X) \neq 0$ is a standing assumption when considering domains that are not absolute retracts.

In what follows we shall assume that:

$$(A) \left\{ \begin{array}{l} X \text{ is a compact ANR of a normed space } E \\ \text{with a given retraction } r : U \rightarrow X \text{ defined on a neighborhood } U \text{ of } X. \end{array} \right.$$

Since X is compact, we may assume with no loss of generality that such a neighborhood U is a uniform neighborhood $B(X, \beta) = \{x \in E : d(x, X) < \beta\}$ of X .

Let us mention that the class of sets satisfying (A) is quite substantial. For instance, if X is compact convex, then X satisfies (A) (a by-product of the Dugundji’s Extension Theorem) in addition to having non-trivial Euler characteristic. Furthermore, since the property of being a compact neighborhood retract and the Euler characteristic are both topologically invariant, it follows that any subset Y of E , homeomorphic to X , satisfies (A) and has non-trivial Euler characteristic, provided X has both properties.

For an extensive study of retracts, the reader is referred to the monographs by Borsuk [21] and Hu [47].

Definition 4.1 Let (E, d) be a metric space; $E \supset X$ is an L -retract (in E) if there exists an open neighborhood U of X in E and a continuous retraction $r : U \rightarrow X$ and a constant $L > 0$ such that:

$$d(r(x), x) \leq Ld_X(x), \forall x \in U.$$

¹¹ A set $\mathbb{R}^n \supset X$ is *epi-Lipschitzian* if the Clarke’s normal cone $N_X^C(x)$ is pointed, i.e., $N_X^C(x) \cap (-N_X^C(x)) = \{0\}$. If X is epi-Lipschitzian, then $X = cl(Int(X))$, the map $x \mapsto T_X^C(x)$ is l.s.c. (thus N_X^C has closed graph), i.e., X is sleek.

¹² X is a *proximate retract* in $\mathbb{R}^n$ if there exists a continuous retraction $r : \mathbb{R}^n \rightarrow X$ such that $\|r(u) - u\| = d_X(u), \forall u \in \mathbb{R}^n$.

¹³ If X is a smooth submanifold of $\mathbb{R}^n$ then John Milnor defines $\chi(X)$ as the Brouwer degree of the Gauss mapping $G_X(x) =$ the unit outward normal vector to X at $x \in \partial X$. For non-smooth sets, e.g., X is a compact epi-Lipschitzian subset of $\mathbb{R}^n$, Cornet [30] defines the Gauss mapping in terms of proximal normal vectors: $G_X(x) = conv(N_X^C(x) \cap S^{n-1})$ and $\chi(X) = \deg(G_X, int(X), 0)$, the Cellina-Lasota degree. More generally, if X is a compact topological space, the singular cohomology $\{H^q(X)\}$ is a graded linear space of finite type. Denote $\dim_{\mathbb{Q}}(H^q(X)) = \beta_q(X)$ (qth-Betti number) and define: $\chi(X) := \sum_q (-1)^q \beta_q(X)$. It turns out that $\chi(X) = \lambda(Id_X)$ the Lefschetz number of the identity mapping on X .

Observe that if X is a neighborhood retract of a metric space E with a Lipschitz continuous neighborhood retraction $r : U \rightarrow X$ with modulus $k > 0$, where U is an open subset of E containing X , then $d(r(x), x) \leq Ld_X(x)$, $\forall x \in U$ with $L = k + 1$. Moreover, if X is compact, it suffices that the neighborhood retraction is locally Lipschitz continuous.

Clearly, closed convex subsets of Hilbert spaces come immediately to mind with the retraction consisting of the metric projection $Proj_X$, which is single valued and locally Lipschitz continuous. These properties of $Proj_X$ hold true for more general types of sets such as, e.g., Federer's sets with so-called local positive reach in Euclidean spaces, proximally smooth subsets of Hilbert spaces¹⁴ of Clarke–Stern–Wolenski, sets with the Shapiro property¹⁵ and more generally prox-regular subsets of Hilbert spaces in the sense of Poliquin–Rockafellar–Thibault (see [59] and references there). More precisely (see Sect. 2 above), recalling that the normal cone $N_X(x)$ to a closed set X at $x \in X$ is the limiting proximal normal cone $N_X(x) = \{v \in H : \exists(x_k, v_k) \in X \times N_X^P(x_k) \text{ with } (x_k, v_k) \rightarrow (x, v)\}$ and $N_X^P(x) = \{\lambda(x - y) : \lambda \geq 0 \text{ and } y \in Proj_X(x)\}$ being the proximal normal cone to X at x , we have:

Definition 4.2 ([59]) A closed subset X of a Hilbert space H is uniformly prox-regular with constant $\rho > 0$ if whenever $x \in X$ and $v \in N_X(x)$ with $\|v\| < 1$, then x is the unique nearest point of X to $x + \frac{1}{\rho}v$.

Uniform prox-regularity with constant $1/\rho'$ for every $0 < \rho' < \rho$ is equivalent to the Fréchet (or Gâteaux) differentiability of the distance function $d_X(\cdot)$ on a tubular neighborhood $B(X; \rho) = \{x \in H, d_X(x) < \rho\}$ of X in H . It is also equivalent to the fact that every non-zero proximal normal to X at any $x \in X$ can be realized by a ρ -ball, i.e., $\forall x \in X, \forall 0 \neq v \in N_X^P(x)$, $\langle \frac{v}{\|v\|}, x' - x \rangle \leq \frac{1}{2\rho} \|x' - x\|^2$, $\forall x' \in X$. Theorem 4.1 and Lemma 4.2 in [59] assert that a necessary and sufficient condition for X to be uniformly prox-regular with constant $\rho > 0$ is that the projection $Proj_X$ be single valued and monotone on the tubular neighborhood $B(X; \rho)$ and Lipschitz continuous on $B(X, \rho')$ with modulus $\frac{\rho}{\rho - \rho'}$ for any $0 < \rho' < \rho$.

A convex set is uniformly prox-regular with constant $\rho = +\infty$. Moreover, since the normal cone $N_X(x)$ to a proximally smooth set X at any given point $x \in X$ is closed and convex, it follows that $N_X(x) = N_X^P(x)$ for such sets, i.e., every normal is actually a proximal normal. Hence, a proximally smooth closed subset of a Hilbert space is uniformly prox-regular.

Proposition 4.3 Each of the following is an L -retract:

- (1) A proximate retract of Plaskacz.
- (2) A uniformly prox-regular closed subset X of a Hilbert space H (see also [22]).
- (3) A bi-Lipschitz homeomorphic to a convex compact subset of a normed space.

Proof (1) is obvious as a proximate retract is an L -retract with $L = 1$. To show (2) observe that for any given $\rho' \in (0, \rho)$ where $\rho > 0$ is the constant of uniform prox-regularity of X , the projection of H onto X , $Proj_X = r : B(X, \rho') \rightarrow X$, $r(x) = (x)$ is single valued and continuous with $\|x - r(x)\| = d_X(x)$ (see Theorem 4.1 and Lemma 4.2 in [59]). Hence X is a neighborhood proximate retract and thus an L -retract (with constant 1).

To show (3), let X be a closed subset of E that is bi-Lipschitz homeomorphic to a closed convex subset Y of a normed space $(F, \|\cdot\|)$ (comp. with [28]), i.e., there exists a Lipschitz homeomorphism $h : X \rightarrow Y$ with Lipschitz inverse $g = h^{-1}$. We establish that X is an L -retract. To do this, consider a system $\{U_i, a_i\}_{i \in I}$ for $E \setminus X$ such that:

- (i) $U_i \subset E \setminus X$, $a_i \in X$, $i \in I$;
- (ii) $\{U_i\}_{i \in I}$ is a locally finite open cover of $E \setminus X$;
- (iii) if $x \in U_i$, then $d(x, a_i) \leq 2d_X(x)$, $i \in I$.

Such a system always exists. Let $f : X \rightarrow B$ be the continuous extension of the map h to the entire space E , defined by the formula:

$$f(x) = \begin{cases} h(x) & \text{for } x \in X, \\ \sum_{i \in I} \lambda_i(x) h(a_i) & \text{for } x \in E \setminus X, \end{cases}$$

where $\{\lambda_i\}_{i \in I}$ is a locally finite partition of unity subordinated to $\{U_i\}_{i \in I}$.

¹⁴ A closed set X of a Hilbert space is *proximally smooth* if the function $d_X(\cdot)$ is continuously differentiable on a set $B(X, \rho) \setminus X$.

¹⁵ A set X in a normed space has the *Shapiro property* at $x \in X$ if there exists an open neighborhood U of x and a constant $\rho > 0$ such that $d_{T_X(x)}(x' - x) \leq \rho \|x' - x\|^2$, $\forall x' \in U$, where $T_X(x)$ is the Bouligand–Severi cone.

Let $r : E \longrightarrow X$ be given by:

$$r(x) = (g \circ f)(x), x \in E.$$

We show that for each $x \in E$, $d(r(x), x) \leq Ld_X(x)$ where $L = 3L_gL_h + 1$, L_g and L_h the Lipschitz constants of h and g respectively.

Indeed, for any given $x \in E$, if $I(x) = \{i \in I; x \in U_i\}$ then $f(x) = \sum_{i \in I(x)} \lambda_i(x)h(a_i)$. Take $i \in I(x)$, $\epsilon > 0$ arbitrary and $a \in X$ such that:

$$d(x; X) \leq d(x, a) < d_X(x) + \epsilon.$$

Since $d(a_i, a) \leq d(a_i, x) + d(x, a) \leq 3d(x; X) + \epsilon$, it follows that:

$$\|f(x) - f(a)\| \leq \sum \lambda_i(x) \|h(a_i) - h(a)\| \leq L_h(3d_X(x) + \epsilon),$$

and

$$d(r(x), x) \leq d((g \circ f)(x), (g \circ f)(a)) + d(a, x) \leq L_gL_h(3d_X(x) + \epsilon) + d_X(x) + \epsilon.$$

□

The property of being an L -retract is intimately linked to a universal extension property for locally Lipschitzian mappings.

Definition 4.4 A metric space X has NLEP (w.r.t. a metric space E) iff: $\forall A \xrightarrow{\text{closed}} E, \forall f : A \xrightarrow{\text{Lip}_{loc}} X, \exists A \subset U \xrightarrow{\text{open}} E, \exists \hat{f} : A \xrightarrow{\text{Lip}_{loc}} X$ extension of f to U .

Proposition 4.5 (i) *NLEP is invariant under bi-Lipschitz homeomorphisms.*

(ii) *NLEP is stable for open subsets.*

(iii) *NLEP is stable for unions of open sets.*

(iv) *NLEP is hereditary, i.e., it goes from local to global in the presence of compactness; hence a compact metric space that has NLEP locally is an L -retract.*

The following definition by [17] extends the concept of an epi-Lipschitz subset of a Euclidean space (due to Rockafellar) to subsets of normed spaces of arbitrary dimension.

Definition 4.6 A subset X of a normed space E is an epi-Lipschitz set if each boundary point y of X has a neighborhood U (in X) for which there exists a normed space F , an open set C of F , a Lipschitz continuous function $g : C \longrightarrow \mathbf{R}$, a point $z = (x, g(x))$, $x \in C$, a neighborhood V of z , and a bi-Lipschitz homeomorphism $h : V \cap \text{Epi graph}(g) \longrightarrow U$ such that $h(z) = y$. Roughly speaking, such a set is locally the epigraph of a Lipschitz continuous function.

Example 4.7 ([17]) A compact epi-Lipschitz sets of a normed space has NLEP, and hence is a compact L -retract.

4.2 Equilibria in L -retracts

Theorem 4.8 Let X be a compact L -retract in a normed space E with $\chi(X) \neq 0$, and let $\Phi \in \mathbf{H}(X, E)$ verifying $\Phi(x) \cap T_X^C(x) \neq \emptyset, \forall x \in \partial X$. Then Φ has an equilibrium.

The proof of Theorem 4.8 is based on the following approximation under constraint result, a sort of hybrid between the celebrated E. Michael's selection theorem [52] and a crucial theorem of A. Cellina [27] on the graph approximation of u.s.c. maps with convex values.¹⁶

¹⁶ If X is paracompact, then $\mathbf{K}(X, E) \subset \mathbf{A}(X, E)$ the class of approachable maps defined by: $\Phi \in \mathbf{A}(X, E) \iff \Phi$ is u.s.c. and $\forall U, V \in N_E(0), \exists s : X \longrightarrow E$ continuous, such that $s(x) \in \Phi((x+U) \cap X) + V, \forall x \in X$ (see [11] for a detailed discussion on the approximation of u.s.c. set-valued maps).

Lemma 4.9 Let (X, d) be a metric space and $(E, \|\cdot\|)$ be a normed space. Let $\Psi : X \rightrightarrows E$ be a lower semicontinuous map with convex values, and $\Phi \in \mathbf{K}(X, E)$ be such that $\Phi(x) \cap \Psi(x) \neq \emptyset$ for each $x \in X$. Then for any $\delta > 0$, there is a continuous map $f : X \rightarrow E$ such that for every $x \in X$:

- (i) $B(f(x), \delta) \cap \Psi(x) \neq \emptyset$, and
- (ii) $\|f(x) - y\| < \delta$ for some $y \in \Phi(\bar{x})$, $d(x, \bar{x}) < \delta$.

Proof Let us put $U(x) = B_X(x, \frac{\delta}{2}) \cap \{x' \in X; \Phi(x') \subset B_E(\Phi(x), \frac{\delta}{2})\}$, $x \in X$. Let $\mathcal{V} = \{V\}$ be an open star refinement of the open cover $\mathcal{U} = \{U(x)\}_{x \in X}$, i.e., for any $V \in \mathcal{V}$ there is $\bar{x} \in X$ with $st(V, \mathcal{V}) \subset U(\bar{x})$.

For any $x \in X$, choose $z_x \in \Phi(x) \cap \Psi(x)$ and consider the open cover $\mathcal{O} = \{O_V(x)\}_{V \in \mathcal{V}, x \in V}$ of X , where $O_V(x) = \{x' \in V; \Phi(x') \cap B(z_x, \frac{\delta}{2})\}$. Let $\{\lambda_i\}_{i \in I}$ be a locally finite partition of unity subordinated to $\mathcal{O}$. Hence, for each $i \in I$, there are $V_i \in \mathcal{V}$, $x_i \in V_i$, with $\lambda_i(x') = 0$ for $x' \notin O_{V_i}(x_i)$.

The map $f : X \rightarrow E$ defined by:

$$f(x') = \sum_{i \in I} \lambda_i(x) z_i, \quad x \in X,$$

where $z_i = z_{x_i}$ is clearly continuous. Moreover, for each $x \in X$ and each index i in the finite set of essential indices $I(x) = \{i \in I; \lambda_i(x) \neq 0\}$, there exists $z'_i \in \Psi(x)$ such that $\|z'_i - z_i\| < \delta$ because $x \in O_{V_i}(x_i)$. Thus, by convexity of $\Psi(x)$,

$$\sum_{i \in I(x)} \lambda_i(x) z'_i \in \Psi(x), \text{ and } \|\sum_{i \in I(x)} \lambda_i(x) z'_i - f(x)\| \leq \sum_{i \in I(x)} \lambda_i(x) \|z'_i - z_i\| < \delta.$$

In other words, $B(f(x), \delta) \cap \Psi(x) \neq \emptyset$ for every $x \in X$.

On the other hand, given $x \in X$, $i \in I(x)$, it follows that $x \in O_{V_i}(x_i) \subset V_i$ where $x_i \in V_i$. Since $\mathcal{V}$ is a star refinement of $\mathcal{U}$, there is $\bar{x} \in X$ such that $x, x_i \in U(\bar{x})$. Therefore, $z_i \in \Phi(x_i) \subset B(\Phi(\bar{x}), \delta)$ and $\|x - \bar{x}\| < \delta$. The set $B(\Phi(\bar{x}), \delta)$ being convex, it follows that $f(x) \in B(\Phi(\bar{x}), \delta)$. $\square$

Remark 4.10 This lemma guarantees the existence of a δ -approximate selection for Ψ which is also a δ -approximation of the graph of Φ . As in the Michael's selection theorem [52], assuming that in addition the values of Ψ are closed and E is a Banach space, we conclude that there exists a selection of Ψ which is also a δ -approximation of the graph of Φ .

Given a compact L -retract X in a normed space $(E, \|\cdot\|)$, let us assume (with (A) in mind):

$$\exists \beta > 0, \exists r : B(X, \beta) \rightarrow X \text{ a retraction, } \exists L > 0 \text{ such that } \|r(x) - x\| \leq L d_X(x) \text{ for } x \in B(X, \beta).$$

It follows that:

$$(A') \quad \exists \eta > 0, \eta < \frac{\beta}{2}, \text{ such that } \|r(x) - x\| < \eta \text{ for all } x \in B(X, \eta).$$

Recall the characterization of the Clarke circatangent cone $T_X^C(x)$, $x \in X$, as being the set $T_X^C(x) = \{v \in E; d_X^0(x)(v) = 0\}$, where $d_X^0(x)(v)$ is u.s.c. on $X \times E$ and convex in v (see Sect. 2.1).

With these properties at hand, we can present the proof of Theorem 4.8.

Proof of Theorem 4.8 Given $\epsilon > 0$ arbitrary, let $\delta = \frac{\epsilon}{2L+1}$. Due to the above properties of d_X^0 , the map $\Psi : X \rightrightarrows E$ defined by the formula:

$$\Psi(x) = \{v \in E; d_X^0(x)(v) < \delta\}, \quad x \in X,$$

has convex values and its graph is open; it is hence lower semicontinuous. By hypothesis, $\Phi(x) \cap \Psi(x) \neq \emptyset$ for all $x \in X$. In view of Lemma 33, there is a continuous δ -approximation f of the graph of Φ such that:

$$B(f(x), \delta) \cap \Psi(x) \neq \emptyset \quad \text{for all } x \in X.$$

Since f is continuous, it is bounded on X , say $\|f(x)\| \leq M$ for some $M > 0$. Choose $\tau > 0$ with $M\tau < \eta$ where η is given by (A'), and a sequence $(t_n)_{n \in \mathbb{N}}$ in $(0, \tau]$, $t_n \downarrow 0^+$. For each $n \in \mathbb{N}$, the map $g_n : X \rightarrow X$ given by:

$$g_n(x) = r(x + t_n f(x)), \quad x \in X,$$

is well defined, since for each $x \in X$, $d_X(x + t_n f(x)) < \eta$.

For each $n \in \mathbf{N}$, the homotopy $h_n : X \times [0, 1] \longrightarrow X$, defined by:

$$h(x, \lambda) = r(x + \lambda t_n f(x)), (x, \lambda) \in X \times [0, 1],$$

joins g_n to the identity on X . Therefore, g_n has a fixed point $x_n \in X$. Observe now that:

$$\begin{aligned} t_n \|f(x_n)\| &= \|t_n f(x_n)\| = \|x_n + t_n f(x_n) - g_n(x_n)\| \\ &= \|x_n + t_n f(x_n) - r(x_n + t_n f(x_n))\| \leq L d_X(x_n + t_n f(x_n)). \end{aligned}$$

Since X is compact, a subsequence of (x_n) (again denoted by (x_n)) converges to some $\bar{x} \in X$. Hence, there exists $\bar{v} \in \Psi(\bar{x})$ such that $\|\bar{v} - f(\bar{x})\| < \delta$. It follows,

$$\begin{aligned} d_X(x_n + t_n f(x_n)) &\leq d_X(x_n + t_n \bar{v}) + t_n \|\bar{v} - f(x_n)\| \\ &\leq d_X(x_n + t_n \bar{v}) + t_n \|\bar{v} - f(\bar{x})\| + t_n \|f(\bar{x}) - f(x_n)\|. \end{aligned}$$

Therefore, for any n ,

$$\|f(x_n)\| \leq L \left(\frac{d_X(x_n + t_n \bar{v})}{t_n} + \|\bar{v} - f(\bar{x})\| + \|f(\bar{x}) - f(x_n)\| \right).$$

Letting $n \rightarrow \infty$, we obtain:

$$\begin{aligned} \|f(\bar{x})\| &= \lim_{n \rightarrow \infty} \|f(x_n)\| \leq L \left(\limsup_{n \rightarrow \infty} \frac{d_X(x_n + t_n \bar{v})}{t_n} + \|\bar{v} - f(\bar{x})\| \right) \\ &\leq L(c(\bar{x}, \bar{v}) + \|\bar{v} - f(\bar{x})\|) < 2L\delta. \end{aligned}$$

Taking into account that f is a δ -approximation of Φ , we infer that there exist $x' \in B_X(\bar{x}, \delta)$, $y \in \Phi(x')$, such that $\|y - f(\bar{x})\| < \delta$. Hence, $\|y\| < (2L + 1)\delta = \epsilon$. Since ϵ is arbitrary and X is compact, it follows that Φ has a zero. $\square$

The following general variational inequality follows at once from Theorems 2.9 and 4.8:

Corollary 4.11 *If X is a compact L -retract in a Hilbert space H with $\chi(X) \neq 0$, and $\Psi \in \mathbf{H}(X, E)$ is a compact-valued map, then Ψ has a co-equilibrium.*

Remark 4.12

- (1) For X compact L -retract, the following are equivalent: $\chi(X) \neq 0 \iff [(\Phi(x) \cap T_X^C(x) \neq \emptyset, \forall x \in \partial X) \implies \Phi \text{ has an equilibrium}] \iff [(\Phi(x) \cap \{x + T_X^C(x)\} \neq \emptyset, \forall x \in \partial X) \implies \Phi \text{ has a fixed point}]$.
- (2) Theorem 4.8 is in general false with the Bouligand–Severi cone in place of Clarke's cone: the subset of $\mathbb{R}^2$:

$$\begin{aligned} X &= S_1^+ \cup S_2^+, \text{ where} \\ S_1^+ &:= \{(x, \sqrt{1 - (x + 1)^2}) : -2 \leq x \leq 0\}, \text{ and} \\ S_2^+ &:= \{(x, \sqrt{1 - (x - 1)^2}) : 0 \leq x \leq 2\}, \end{aligned}$$

is bi-Lipschitz homeomorphic to the line segment $[-2, 2] \times \{0\}$, hence a compact L -retract. The mapping $f : X \longrightarrow \mathbb{R}^2$ given by:

$$f(x, y) = \begin{cases} (y, 1 - x) & \text{on } S_1^+ \\ (-y, 1 + x) & \text{on } S_2^+ \end{cases}$$

verifies $f(0, 0) = (0, 1) \in T_X(0, 0) = \{0\} \times [0, \infty)$, $f(0, 0) \notin C_X(0, 0) = \{(0, 0)\}$ and $f(x, y) \neq (0, 0)$, $\forall (x, y) \in X$.

- (3) Theorem 4.8 is, in general, false if the convexity of the values of Φ is weakened to contractibility. Indeed, consider $X = \{(x, y) \in \mathbb{R}^2 : (x + 1)^2 + y^2 \leq 1\}$ be a closed disk (hence, convex and compact) and let

$$f(x, y) = \begin{cases} \{-\sqrt{1 - y^2}, y\} : -1 \leq y \leq 1 & \text{on } \partial X \\ (-1, 0) & \text{on } \text{int}(X) \end{cases}.$$

Then, $f(x, y) \in T_X^C(x) = cl(S_X(x))$, $\forall x \in X$ but $f(x, y) \neq (0, 0)$, $\forall (x, y) \in X$.

Another recent counter-example was provided by J. Mederski who considers a mapping with star shaped values [51].

The convexity of the values of the map Φ can be dropped only under a stronger tangency condition:

Theorem 4.13 ([16]) *Let X be a compact L -retract in a normed space E with $\chi(X) \neq 0$, and let $\Phi \in \mathbf{V}(X, E) := \{\text{usco's with acyclic values}\}$ verifying $\Phi(x) \subset T_X^C(x), \forall x \in \partial X$. Then Φ has an equilibrium.*

Problem 4.14

- (1) A natural question to ask is: for what type of domains is Theorem 4.8 true with the Bouligand–Severi cone in lieu of the Clarke's cone?
- (2) Short of the excessively strong condition $\Phi(x) \subset T_X^C(x), \forall x \in \partial X$, what additional assumptions on non-convex valued maps Φ are required for Theorem 4.8 to hold? Is there any characterization of such classes?

For sets defined by locally Lipschitz inequalities and relaxed regularity and compactness conditions, co-equilibria results for H -maps with weak*compact and convex values can be found in Cwiszewski–Kryszewski [32]. A remarkable theorem on the existence of stationary solutions for the Cauchy problem $x'(t) = f(x(t)), t \geq 0, x(0) = x \in X \subset \mathbb{R}^n$, from Kamenskii–Quincampoix [49], is worth bringing to the attention of the reader. Assume that f is Lipschitzian continuous and define $X_0(f)$ as the set of those points $x \in \partial X$ such that the solution of the Cauchy problem starting at x leaves X immediately. Recall from Sect. 2.3 that Nagumo's theorem asserts that for X locally closed: $(f(x) \in T_X(x), \forall x \in X \implies X_0 = \emptyset)$.

Theorem 4.15 ([49]) *Let $\mathbb{R}^n \supset X$ be epi-Lipschitz and compact and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a (locally) Lipschitz mapping. Assume that X_0 is closed and $\chi(X_0)$ is well defined. If $\chi(X_0) \neq \chi(X_0)$, then f has an equilibrium in X .*

Related results in the context of Hilbert spaces and for condensing set-valued maps can be found in the more recent paper by Gudovich–Kamenskii–Quincampoix [40].

4.3 Non-Lipschitz and non-sleek domains

The last section is devoted to non-Lipschitzian compact neighborhood retracts of normed spaces. The following two examples show that tangency conditions involving the Bouligand–Severi cone $T_X(x)$ or the Clarke cone $T_X^C(x)$ (which are equivalent for tangentially regular sets) are inadequate for the study of equilibria in ANRs without Lipschitz regularity.

Example 4.16

- (1) The set $X = X_1 \cup X_2$ where $X_1 = \{(x, y) \in \mathbf{R}^2 : (x-1)^2 + y^2 = 1\}$ and $X_2 = \{(x, y) \in \mathbf{R}^2 : (x+1)^2 + y^2 = 1\}$ is a compact neighborhood retract of $\mathbf{R}^2$ with non-trivial Euler characteristic. The mapping $f : X \longrightarrow \mathbf{R}^2$ given by the formula $f(x, y) = \begin{cases} (y, 1-x) & \text{if } (x, y) \in X_1, \\ (-y, 1+x) & \text{if } (x, y) \in X_2. \end{cases}$ verifies $f(x, y) \in T_X(x, y) = C_X(x, y)$ for each $x \in X$, but has no zeros.
- (2) Being homotopy equivalent to the set X above, the set $Y = Y_1 \cup Y_2$ where $Y_1 = \{(x, y) \in \mathbf{R}^2 : \frac{1}{2} \leq (x-1)^2 + y^2 \leq 1\}$ and $Y_2 = \{(x, y) \in \mathbf{R}^2 : \frac{1}{2} \leq (x+1)^2 + y^2 \leq 1\}$ is also a compact neighborhood retract of $\mathbf{R}^2$ with non-trivial Euler characteristic. Furthermore, it is non-sleek as $T_Y(x, y) = C_Y(x, y)$ for $(x, y) \neq (0, 0)$ but $T_Y(0, 0) = \mathbf{R}^2 \supset \{0\} \times \mathbf{R} = C_Y(0, 0)$. One can easily extend the function f above to a function $g : Y \longrightarrow \mathbf{R}^2$ having no zeros and satisfying tangency conditions with Clarke's cone.

These examples motivate the definition of a notion of *retraction normal cone* $N_X^r(x)$ to a compact neighborhood retract X .

Denote by D^* the closed unit ball in the dual E' of E equipped with the strong topology. Given $X \subset E, x \in X, \epsilon > 0$, we have:

$$D_X(x, \epsilon) := \{y \in X : \|x - y\| \leq \epsilon\}, B_X(x, \epsilon) := \{y \in X : \|x - y\| < \epsilon\}.$$

Let $X \subset E$ be a set satisfying (A). Then, as seen before, the following holds:

$$(A') \quad \exists \eta > 0, \eta < \frac{1}{2}\beta, \forall x \in B(X, \eta), \|x - r(x)\| < \eta.$$

Definition 4.17 For any $x \in X$ and $0 < \epsilon < \eta$, let:

$$M_X^r(x; \epsilon) := \left\{ p \in D^* : x \in r[\overline{\text{conv}}\{y \in D_X(x, \epsilon) : \sup_{z \in B_X(x, \epsilon)} \langle p, z - y \rangle \leq 0\}] \right\}.$$

In other words, $p \in M_X^r(x; \epsilon)$ if and only if x is a fixed point for the map $X \ni x \mapsto r[\overline{\text{conv}}\{G_\epsilon(x, p)\}]$ where $G_\epsilon : X \times D^* \rightrightarrows X$ is the compact-valued map given by:

$$G_\epsilon(x, p) := \left\{ y \in D_X(x, \epsilon) : \sup_{z \in B_X(x, \epsilon)} \langle p, z - y \rangle \leq 0 \right\}.$$

We are ready now to define the retraction normal cone.

Definition 4.18 Assume that X satisfies (A).

- (i) The retraction normal pre-cone at $x \in X$ is defined to be the set: $M_X^r(x) = \limsup_{\substack{y \rightarrow x, y \in X \\ \epsilon \downarrow 0}} M(y; \epsilon)$.
- (ii) The retraction normal cone at $x \in X$ is the cone $N_X^r(x)$ spanned by $M_X^r(x)$.

The following properties are important.

Proposition 4.19 (i) $N_X^r(x)$ is a closed cone in E' ;
(ii) the maps $M_X^r, N_X^r : X \rightrightarrows E'$ have closed graphs.

Definition 4.20 The retraction tangent cone at $x \in X$ is defined by the formula:

$$T_X^r(x) := [N_X^r(x)]^- = \{v \in E : \langle p, v \rangle \leq 0, \forall p \in N_X^r(x)\}.$$

As expected, $T_X^r(x)$ is a closed convex cone in E' . Moreover, the map $T_X^r : X \rightrightarrows E'$ is lower semicontinuous provided the normal cone $N_X^r(x)$ is convex for each $x \in X$ (which is not true in general).

Example 4.21 Assume that X is locally convex at $x \in X$. Then,

- (i) for sufficiently small $\epsilon > 0$, ($\epsilon < \eta$), we have: $M_X^r(x; \epsilon) = D^* \cap N_X(x)$, where $N_X(x) = cl(S_X(x))^-$ is the normal cone of convex analysis to X at x . In particular, if $x \in \text{int}(X)$, then $M_X^r(x) = \{0\}$.
- (ii) $N_X^r(x) = N_X(x)$ and $T_X^r(x) = cl(S_X(x))$, the tangent cone of convex analysis.

It is important to note that these notions of retraction cones depend upon the choice of the retraction r and on that of the norm in E .

The proof presented here of the equilibrium theorem for arbitrary compact ANRs is based on a fixed point property for so-called approachable maps (see e.g., [11] and references there). Recall that given the topological spaces X, Y endowed with compatible uniformities $\mathcal{U}$ and $\mathcal{V}$, respectively, and entourages $U \in \mathcal{U}$ and $V \in \mathcal{V}$, a (U, V) -approximate selection for a map $\Psi : X \rightrightarrows Y$ is a single-valued map $s : X \rightarrow Y$ verifying $s(x) \in V[\Psi(U[x])]$ for all $x \in X$. Denote by $\mathbf{a}(\Psi; U, V)$ the set of all continuous (U, V) -approximate selections for Φ .

Definition 4.22 The map Ψ is said to be approachable if and only if it is upper semicontinuous and $\mathbf{a}(\Phi; U, V)$ is non-empty for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The class of approachable maps from X into Y is denoted by $\mathcal{A}(X, Y)$ ($\mathcal{A}(X)$ denotes $\mathcal{A}(X, X)$).

The class $\mathcal{A}$ of approachable maps is quite broad. The well-known approximation theorem of Cellina asserts that if X is a paracompact topological space equipped with a compatible uniformity $\mathcal{U}$ and Y is a convex subset of a locally convex topological vector space, then every upper semicontinuous map $\Psi : X \rightrightarrows Y$ with non-empty convex values is approachable [27].¹⁷

¹⁷ For non-convex maps: if X and Y are ANRs with X compact, then every u.s.c. map $\Psi : X \rightrightarrows Y$ with compact contractible values is approachable. More generally, if X is an approximative absolute neighborhood extension space for compact spaces, and Y is a uniform space, then every u.s.c. map on X with non-empty compact ∞ -proximally connected values in Y is approachable (see [11]).

The class $\mathcal{A}$ enjoys a number of stability properties (again, see [11] and references there). One of them, the closedness under composition products, is crucial for the sequel:

$$\Psi_1 \in \mathcal{A}(X, Y), \Psi_2 \in \mathcal{A}(Y, Z) \implies \Psi_2 \Psi_1 \in \mathcal{A}(X, Z) \text{ provided } X \text{ is compact.}^{18}$$

Theorem 4.23 Assume that X satisfies (A) and that $\chi(X) \neq 0$ and let $\Psi \in \mathcal{A}(X)$ be a map with non-empty closed values such that, for all $x \in X$, $\Psi(x) \subset B(x, 2\eta)$, where η is given by (A'). Then Ψ has a fixed point.

Proof Consider a sequence $\epsilon_n \downarrow 0^+$ so that $2\epsilon_n + 2\eta < \delta$ for all n . Every ϵ_n -approximative selection s_n of Ψ is homotopic to the identity Id_X through the homotopy $h_n(x, t) = r(tx + (1-t)s_n(x))$, $(x, t) \in X \times [0, 1]$ (keep in mind (A')). Hence, s_n has a fixed point x_n which is an ϵ_n -approximative fixed point for Ψ . The compactness of X together with the closedness of the graph of Ψ end the argument. $\square$

We are ready to state and prove our second main theorem.

Theorem 4.24 Assume that X is a compact ANR in a normed space E with $\chi(X) \neq 0$. If $\Phi \in \mathbf{H}(X, E)$ satisfies the $(KF)_r$ condition:

$$\text{for all } x \in X \text{ and all } p \in N_X^r(x), \inf_{y \in \Phi(x)} \langle \varphi, y \rangle \leq 0.^{19}$$

Then Φ has an equilibrium.

Proof Suppose that $0 \notin \Phi(x)$ for each $x \in X$. By the Hahn-Banach separation theorem, for each $x \in X$ there is $p_x \in E'$ such that $\inf\{\langle p_x, y \rangle : y \in \Phi(x)\} > 0$, i.e., $\sup_{y \in \Phi(x)} \langle -p_x, y \rangle < 0$. Since Φ is u.h.c., the set:

$$U(x) := \left\{ z \in X : \sup_{y \in \Phi(z)} \langle -p_x, y \rangle < 0 \right\},$$

is an open neighborhood of x , and the collection $\mathcal{U} = \{U(x)\}_{x \in X}$ constitutes an open covering of X . Let $\{\lambda_x\}_{x \in X}$ be a locally finite partition of unity subordinated to $\mathcal{U}$. Let us define a continuous mapping $f : X \rightarrow E'$ by the formula:

$$f(z) := \sum_{x \in X} \lambda_x(z) p_x, \text{ for } z \in X.$$

Then, for any $z \in X$, we have $\sup_{y \in \Phi(z)} \langle -f(z), y \rangle < 0$, hence, $f(z) \neq 0$. Indeed, let $\{x_1, \dots, x_k\} = \{x \in X : z \in U(x)\}$. Then $f(z) = \sum_{i=1}^k \lambda_i(z) p_i$ where $\lambda_i = \lambda_{x_i}$, $p_i = p_{x_i}$, $i = 1, \dots, k$. Since $z \in U(x_i)$, it follows that $\sup_{y \in \Phi(z)} \langle -p_i, y \rangle < 0$ and we are done.

We prove the existence of an element $\bar{x} \in X$ such that $f(\bar{x}) \in N_X^r(\bar{x})$. This together with condition (KF) will yield to a contradiction.

Let $n \in \mathbb{N}$ be such that $0 < \frac{1}{n} < \eta$, where η is given by (A'). Consider the map $\Psi_n : X \rightrightarrows X$ defined by:

$$\Psi_n(x) := r \left[\overline{\text{conv}} \left\{ G_{\frac{1}{n}} \left(x, \frac{f(x)}{\|f(x)\|} \right) \right\} \right], \text{ for } x \in X.$$

One easily verifies that $\Psi_n(x) \subset B(x, 2\eta)$ for all $x \in X$ and all n . Since the map Ψ_n is the composition of the continuous mapping r and the approachable map $\overline{\text{conv}}\{G_{\frac{1}{n}}(\cdot, \frac{f(\cdot)}{\|f(\cdot)\|})\}$, it is also approachable, i.e., $\Psi_n \in \mathcal{A}(X)$.

Theorem 49 implies that there exists $x_n \in \Psi_n(x_n)$. In other words $\frac{f(x_n)}{\|f(x_n)\|} \in M_X^r(x_n; \frac{1}{n})$. The compactness of X implies (without loss of generality) that $x_n \xrightarrow{n \rightarrow \infty} \bar{x} \in X$. Hence $f(x_n) \rightarrow f(\bar{x}) \in N_X^r(\bar{x})$. $\square$

Remark 4.25

- (i) In case X is convex and compact, we recover the classical convex case in Theorem 3.5.
- (ii) Decisive relaxation of the compactness of the domain (e.g., by that of the map) is still largely open for investigation.

¹⁸ It is still an open problem (as far as I know) as to whether this remains true if the compactness of X can be relaxed.

¹⁹ Observe that condition $(KF)_r$ implies the tangency condition $\Phi(x) \cap T_X^r(x) \neq \emptyset$, for all $x \in X$ (compare with Proposition 2.3 above).

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