A. P. Santhakumaran · S. V. Ullas Chandran

The 2-edge geodetic number and graph operations

Received: 19 April 2011 / Accepted: 16 November 2011 / Published online: 5 April 2012
© The Author(s) 2012. This article is published with open access at Springerlink.com

Abstract For a connected graph $G = (V, E)$ of order $n \geq 2$, a set $S \subseteq V$ is a 2-edge geodetic set of G if each edge $e \in E - E(S)$ lies on a $u-v$ geodesic with $d(u, v) = 2$ for some vertices u and v in S . The minimum cardinality of a 2-edge geodetic set in G is the 2-edge geodetic number of G , denoted by $eg_2(G)$. It is proved that for any connected graph G , $\beta_1(G) \leq eg_2(G)$, where $\beta_1(G)$ is the matching number of G . It is shown that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the matching number and 2-edge geodetic number, respectively, of some connected graph. We determine bounds for the 2-edge geodetic number of Cartesian product of graphs. Also we determine the 2-edge geodetic number of certain classes of Cartesian product graphs. The 2-edge geodetic number of join of two graphs is obtained in terms of the 2-edge geodetic number of the factor graphs.

Mathematics Subject Classification (2010) 05C12

المخلص

ليبيان مترابط $G = (V, E)$ من الرتبة $n \geq 2$ ، تسمى المجموعة $S \subseteq V$ مجموعة متقاطعة حافة-2 إذا كانت كل حافة $e \in E - E(S)$ تقع على متقاطعة $u - v$ حيث $d(u, v) = 2$ لرأسين u و v في S . العدد/المتقاطعة حافة-2 لـ G هو أقل أصلائية لمجموعة متقاطعة حافة-2 في G ، ويرمز له بـ $eg_2(G)$. لكل بيان مترابط G ، يتم إثبات أن $\beta_1(G) \leq eg_2(G)$ حيث $\beta_1(G)$ عدد المواءمة لـ G . تم تبين أنه يمكن الحصول على أي زوج a, b من الأعداد الصحيحة، حيث $2 \leq a \leq b$ كعدد مواءمة و عدد متقاطعة حافة-2، على الترتيب، لبيان مترابط. نحدد حدوداً للعدد المتقاطعة حافة-2 للجداء الديكارتي للبيانين. نحدد أيضاً العدد المتقاطعة حافة-2 لصفوف معينة من بيانات الجداءات الديكارتيّة. يتم الحصول على العدد المتقاطعة حافة-2 لوصول بيانين بدلالة العدد المتقاطعة حافة-2 للبيانين العاملين.

A. P. Santhakumaran (✉)
Department of Mathematics,
St. Xavier's College (Autonomous),
Palayamkottai 627 002, India
E-mail: apskumar1953@yahoo.co.in

S. V. Ullas Chandran
Department of Mathematics,
Amrita Vishwa Vidyapeetham,
Amritapuri Campus, Kollam 690 525, India
E-mail: ullaschandra01@yahoo.co.in

1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. For basic graph theoretic terminology, we refer to [5, 7]. For vertices u and v in a connected graph G , the distance $d(u, v)$ is the length of a shortest u - v path in G . It is known that the distance is a metric on the vertex set of G . A u - v path of length $d(u, v)$ is called a u - v geodesic. For any vertex u of G , the eccentricity of u is $e(u) = \max\{d(u, v) : v \in V\}$. The radius $rad\ G$ and diameter $diam\ G$ are defined by $rad\ G = \min\{e(v) : v \in V\}$ and $diam\ G = \max\{e(v) : v \in V\}$, respectively. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v and $N[v] = N(v) \cup \{v\}$. Denote by $\Delta(G)$, the maximum degree of a vertices of G . A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. A vertex of degree 1 is called an *end vertex*. The *closed interval* $I[u, v]$ consists of all vertices lying on some u - v geodesic of G , while for $S \subseteq V$, $I[S] = \bigcup_{u, v \in S} I[u, v]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G . The geodetic number of a graph was introduced in [1, 3] and further studied in [2, 4, 9]. It was shown in [1] that determining the geodetic number of a graph is an NP-hard problem. A set S of vertices is an *edge geodetic set* of a graph G if each edge of G lies on a geodesic between two vertices from S , and the minimum cardinality of an edge geodetic set is the *edge geodetic number* $eg(G)$. An edge geodetic set of cardinality $eg(G)$ is called an *eg-set* of G . The edge geodetic number of a graph was studied in [11]. The edge geodetic number of the Cartesian product of two connected graphs is investigated in [13]. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle when considering geodetic sets, some of the edges may be left out. This drawback is rectified in the case of edge geodetic sets and hence considering edge geodetic sets is more advantageous to the real life application of routing problem. In particular, the edge geodetic sets are more useful than geodetic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places.

For an integer $k \geq 1$, a geodesic in a connected graph G of length k is called a k -geodesic. A vertex v is called a k -extreme vertex if v is not the internal vertex of a k -geodesic joining any pair of distinct vertices of G . Obviously, each extreme vertex of a connected graph G is a k -extreme vertex of G . In particular, each end vertex of G is a k -extreme vertex of G . A set $S \subseteq V$ is called a k -geodetic set of G if each vertex in $V - S$ lies on a k -geodesic of vertices in S . The minimum cardinality of a k -geodetic set of G is its k -geodetic number $g_k(G)$. A k -geodetic set of cardinality $g_k(G)$ is called g_k -set. The k -geodetic number of a graph was referred to as k -geodomination number and studied in [10]. The k -geodetic sets in prisms, which are the simplest Cartesian products of two graphs (products of the form $G \square K_2$), and the 2-geodetic number of Cartesian product of an arbitrary graph with K_n were studied in [6].

For any $S \subseteq V$, let $E(S)$ denote the edge set of the subgraph induced by S . A set $S \subseteq V$ is called a k -edge geodetic set of G if each edge in $E - E(S)$ lies on a k -geodesic of vertices in S . The minimum cardinality of a k -edge geodetic set of G is its k -edge geodetic number $eg_k(G)$. A k -edge geodetic set of cardinality $eg_k(G)$ is called eg_k -set of G . For the graph G given in Fig. 1, it is easy to see that the set $S = \{v_1, v_2, v_5, v_6\}$ of end vertices is a g_2 -set and so $g_2(G) = 4$. Since the edge v_3v_4 does not lie on any 2-geodesic of vertices of S , S is not a 2-edge geodetic set of G . It is easily seen that $S_1 = \{v_1, v_2, v_3, v_5, v_6\}$ is a minimum 2-edge geodetic set of G so that $eg_2(G) = 5$. Also, $S_2 = \{v_1, v_2, v_4, v_5, v_6\}$ is another eg_2 -set of G . The k -edge geodetic number of a graph was introduced and studied in [12].

Fig. 1 A graph with $g_2(G) = 4$ and $eg_2(G) = 5$

The Cartesian product of two graphs G and H , denoted by $G \square H$, has the vertex set $V(G) \times V(H)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. The mappings $\pi_G : V(G \square H) \rightarrow V(G)$ and $\pi_H : V(G \square H) \rightarrow V(H)$ defined by $\pi_G(x, y) = x$ and $\pi_H(x, y) = y$, respectively for all $(x, y) \in V(G \square H)$ are called projections. For a set $S \subseteq V(G \square H)$, we define the *projection of S on G* as $\pi_G(S) = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}$ and the *projection of S on H* as $\pi_H(S) = \{y \in V(H) : (x, y) \in S \text{ for some } x \in V(G)\}$. For any $y \in V(H)$, the subgraph of $G \square H$ induced by $\{(x, y) : x \in V(G)\}$ is isomorphic to G . We denote it by G_y and call it the *copy of G corresponding to y* . Similarly, for any $x \in V(G)$ the subgraph of $G \square H$ induced by $\{(x, y) : y \in V(H)\}$ is isomorphic to H , and we denote it by H_x and call it the *copy of H corresponding to x* . We refer to [3] for results on distance in graphs and to [8] for metric structures in Cartesian product of graphs.

The following theorems will be used in the sequel.

Theorem 1.1 [11] *Each extreme vertex of a connected graph G belongs to every edge geodetic set of G . In particular, if the set S of all extreme vertices is an edge geodetic set of G , then S is the unique eg-set of G .*

Theorem 1.2 [12] *For an integer $k \geq 1$, each k -edge geodetic set of a connected graph G with at least two vertices contains every k -extreme vertex of G . In particular, if the set W of k -extreme vertices is a k -edge geodetic set of G , then W is the unique eg_k -set of G .*

The *edge independence number* or the *matching number* $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges.

Theorem 1.3 [7] *For any graph G of order $n \geq 2$, $\beta_1(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Theorem 1.4 [13] *Let G and H be two connected graphs and S an edge geodetic set of $G \square H$. Then the projections $\pi_G(S)$ and $\pi_H(S)$ are edge geodetic sets of G and H , respectively.*

2 The edge and 2-edge geodetic numbers of a graph

A set S of vertices in a connected graph G is called a *2-edge geodetic set* of G if each edge in $E - E(S)$ lies on a 2-geodesic of vertices in S . The minimum cardinality of a 2-edge geodetic set of G is its *2-edge geodetic number* $eg_2(G)$. A 2-edge geodetic set of cardinality $eg_2(G)$ is called *eg₂-set* of G . It is straightforward to verify the following observation.

Observation 2.1 Let G be a connected graph and let S be an edge geodetic set of G . Then S is a 2-edge geodetic set of G if and only if every edge in G has at least one end in S .

Definition 2.2 Let G be a connected graph. A vertex u in G is said to be a *semi-extreme* vertex if $\Delta(< N(u) >) = |N(u)| - 1$.

Each extreme vertex of a connected graph G is obviously a semi-extreme vertex of G . However, a semi-extreme vertex need not be an extreme vertex. For the graph G in Fig. 2, the vertices v_2 and v_3 are semi-extreme and not extreme. Also, For the wheel $W_{1,n} = K_1 + C_n$ ($n \geq 4$), where C_n is a cycle of order n , each vertex of C_n is a semi-extreme vertex of $W_{1,n}$.

Theorem 2.3 *Let G be a connected graph. Then each semi-extreme vertex belongs to every edge geodetic set of G .*

Fig. 2 A graph with no vertex of degree $n - 1$ and $eg_2(G) = n - 1$

Proof Let S be an edge geodetic set of G . Suppose that there exists a semi-extreme vertex u in G such that $u \notin S$. Since $\Delta(< N(u) >) = |N(u)| - 1$, there exists $v \in N(u)$ such that $\deg_{<N(u)>}(v) = |N(u)| - 1$. Since S is an edge geodetic set of G , the edge $e = uv$ lies on a $x - y$ geodesic $P : x = x_0, x_1, \dots, x_i = u, x_{i+1} = v, \dots, x_n = y$ with $x, y \in S$. Then $v \neq x, y$. Since $\deg_{<N(u)>}(v) = |N(u)| - 1$, we have that u is adjacent to x_{i+2} , which is a contradiction to the fact that P is a $x - y$ geodesic. Hence each semi-extreme vertex belongs to every edge geodetic set of G . $\square$

Since every 2-edge geodetic set of a connected graph G is an edge geodetic set of G , we have the following corollary.

Corollary 2.4 *Let G be a connected graph. Then each semi-extreme vertex belongs to every 2-edge geodetic set of G .*

In a connected graph G , if there exist vertices u and v such that $N[u] = N[v]$, then u and v are semi-extreme vertices and hence we have the following observation.

Observation 2.5 *Let G be a connected graph and $e = uv$ an edge in G . If $N[u] = N[v]$, then u and v belong to every edge geodetic set of G .*

Theorem 2.6 *If G is a graph of order n that has exactly one vertex of degree $n - 1$, then $eg_2(G) = n - 1$.*

Proof Let v be the unique vertex in G of degree $n - 1$. Then $\deg_{<N(u)>}(v) = |N(u)| - 1$, for each vertex $u \neq v$ in G and hence u is semi-extreme. Also, since v is the unique vertex of degree $n - 1$, it follows that v is not a semi-extreme vertex of G . Now, let $S = V(G) - \{v\}$. We prove that S is a 2-edge geodetic set of G . Let e be any edge in G such that $e \notin E(S)$. Then $e = uv$ for some $u \in S$. Let u' be a vertex in G such that u and u' are non-adjacent. Then the edge $e = uv$ lies on the 2-geodesic $P : u, v, u'$ with $u, u' \in S$. Thus S is a 2-edge geodetic set of G and so by Theorem 2.3, $eg_2(G) = n - 1$. $\square$

Remark 2.7 The converse of Theorem 2.6 is not true. For the graph G given in Fig. 2, it follows from Theorem 1.2 and Observation 2.5 that the set $S = \{v_1, v_2, v_3, v_5\}$ is an eg_2 -set of G . Therefore $eg_2(G) = 4 = n - 1$. However, no vertex of G has degree $n - 1$.

If a graph G of order n has more than one vertex of degree $n - 1$, then each vertex of G is a semi-extreme vertex of G and so we have the following result.

Theorem 2.8 *If G is a connected graph of order n with at least two vertices of degree $n - 1$, then $eg_2(G) = n$.*

Remark 2.9 The converse of Theorem 2.8 is not true. For the graph G in Fig. 3, $eg_2(G) = 6$ and G has no vertex of degree 5.

Theorem 2.10 *If G is a connected graph of diameter 2, then $eg_2(G) = eg(G)$.*

Proof Let S be an edge geodetic set of G . Let $e = xy$ be any edge of G such that $e \notin E(S)$. Then e lies on a $u - v$ geodesic with $u, v \in S$. Since the diameter of G is 2, it follows that $d(u, v) = 2$. Thus, S is a 2-edge geodetic set and so $eg_2(G) \leq eg(G)$. Since every 2-edge geodetic set is also an edge geodetic set, we have $eg(G) \leq eg_2(G)$. $\square$

Remark 2.11 The converse of Theorem 2.10 is not true. For the graph G given in Fig. 3, the diameter is 3. Since $N[v_1] = N[v_5]$ and $N[v_2] = N[v_4]$, by Observation 2.5, the vertices v_1, v_2, v_4 and v_5 belong to every edge geodetic set of G . Also, by Theorem 1.1, the vertices v_3 and v_6 belong to every edge geodetic set of G . Hence $eg_2(G) = eg(G) = 6$.

Theorem 2.12 *Let G be a connected graph and S be a set of vertices that contains all semi-extreme vertices of G . Then S is a 2-edge geodetic set of G if and only if S is a vertex cover of G .*

Proof Suppose that S is a vertex cover of G . Let $e = uv$ be any edge in G such that $e \notin E(S)$. Assume, without loss of generality that $u \in S$ and $v \notin S$ and hence v is not a semi-extreme vertex of G . Thus there exists u' in $N(v)$ such that u and u' are non-adjacent and so the edge e lies on the $u - u'$ geodesic $P : u, v, u'$. Since S is a vertex cover of G and $v \notin S$, we have $u' \in S$. Thus S is a 2-edge geodetic set of G . The converse part follows from Theorem 2.1. $\square$

A *vertex cover* of a graph G is a set of vertices of G that contains at least one end vertex of every edge. The minimum cardinality of a vertex cover of G is the *covering number* $\beta(G)$ of G .

Corollary 2.13 For a connected graph G , $\beta(G) \leq eg_2(G)$.

Proof Let S be an eg_2 -set of G . Then by Theorem 2.3, S contains all the semi-extreme vertices of G and hence by Theorem 2.12, S is a vertex cover of G . Thus $\beta(G) \leq |S| = eg_2(G)$. $\square$

It is straightforward to prove the following corollary from Theorem 2.12.

Corollary 2.14 Let G be a connected graph without semi-extreme vertices. Then $\beta(G) = eg_2(G)$.

Remark 2.15 The converse of Corollary 2.14 is not true. For the graph G in Fig. 4, the set $S = \{v_1, v_2, v_4\}$ is a minimum 2-edge geodetic set of G and the set $T = \{v_2, v_3, v_4\}$ is a minimum vertex cover of G so that $eg_2(G) = \beta(G) = 3$. However, the vertex v_1 is a semi-extreme vertex of G .

Problem 2.16 Characterize the class of graphs for which $\beta(G) = eg_2(G)$.

A set M of independent edges in a graph G is called a *matching* of G . A vertex v in G is said to be *M-saturated* if v is incident with an edge in M .

Corollary 2.17 For a connected graph G , $\beta_1(G) \leq eg_2(G)$.

Proof Let S be an eg_2 -set of G and M a maximum matching of G . It follows from Observation 2.1 that $|M| \leq |S|$ and so $\beta_1(G) \leq eg_2(G)$. $\square$

Lemma 2.18 Let H be a connected graph and let v be a vertex of H such that v is M -saturated by every maximum matching M of H . If G is the graph obtained from H by adding pendant edges at v , then $\beta_1(G) = \beta_1(H)$.

Proof Let $\beta_1(H) = a$. Since every matching of H is also a matching of G , it follows that $\beta_1(G) \geq \beta_1(H)$. Suppose that $\beta_1(G) > a$. Let M be a matching of G of cardinality $a + 1$. Then, clearly $e \in M$ for some e in $E(G) - E(H)$ and $M_1 = M - \{e\}$ is a maximum matching of H such that v is M_1 -unsaturated, which is a contradiction. $\square$

Fig. 3 A graph with diameter 3 and $eg_2(G) = eg(G)$

Fig. 4 G

Fig. 5 A graph with $\beta_1(G) = a$ and $eg_2(G) = b$

The following theorem is a realization result of Corollary 2.17.

Theorem 2.19 For each pair of integers a, b with $2 \leq a \leq b$, there exists a connected graph G with $\beta_1(G) = a$ and $eg_2(G) = b$.

Proof Let $H = P_a \square K_2$, where P_a is a path of order a . Let H_1 and H_2 be the two copies of P_a on H . Let the vertex set of H_1 be $\{u_1, u_2, \dots, u_a\}$ and the vertex set of H_2 be $\{v_1, v_2, \dots, v_a\}$. Let G be the graph obtained from H by adding $(b - a)$ new vertices $w_1, w_2, \dots, w_{b-a}$ and joining them to v_a . The graph G is shown in Fig. 5. Let $e_i = u_i v_i$ ($1 \leq i \leq a$). Let A_i ($1 \leq i \leq a - 1$) be the subgraph of H induced by the vertex set $\{u_i, v_i, u_{i+1}, v_{i+1}\}$. By Theorem 1.3, we have $\beta_1(H) \leq a$. Also, since $M = \{e_1, e_2, \dots, e_a\}$ is a matching of H , it follows that $\beta_1(H) = a$. We claim that $\beta_1(G) = a$. By Lemma 2.18, it is enough to prove that the vertex v_a is saturated by every maximum matching of H . Suppose that there exists a maximum matching M_1 of H such that $u_a v_a \notin M_1$ and $v_a v_{a-1} \notin M_1$. Now, if $u_{a-1} u_a \notin M_1$, then $M_1 \cup \{u_a v_a\}$ is also a matching of H of cardinality greater than M_1 , which is a contradiction to the maximality of M_1 . Hence $u_{a-1} u_a \in M_1$. Thus, M_1 contains at most one edge of $A_{a-2} - e_{a-1}$. It follows that M_1 would contain at most two edges or at most one edge of $A_{a-3} - e_{a-2}$ accordingly as M_1 contains no edge or one edge of $A_{a-2} - e_{a-1}$. Continuing in this way up to $A_1 - e_2$, we see that $|M_1| \leq a - 1$, which is a contradiction to $\beta_1(H) = a$. Thus, the vertex v_a is saturated by every maximum matching of H . This shows that $\beta_1(G) = a$. Next we prove that $eg_2(G) = b$. Let S be a 2-edge geodetic set of G . By Theorem 2.1, $u_i \in S$ or $v_i \in S$ for $i = 1, 2, \dots, a$ and by Theorem 1.2, $w_i \in S$ for $i = 1, 2, \dots, b - a$. Thus, $|S| \geq a + b - a = b$ and so $eg_2(G) \geq b$. Let $S_1 = \{u_1, v_2, u_3, \dots, u_a, w_1, w_2, \dots, w_{b-a}\}$ and $S_2 = \{u_1, v_2, u_3, \dots, v_a, w_1, w_2, \dots, w_{b-a}\}$. Let $T = \begin{cases} S_1 & \text{if } a \text{ is odd} \\ S_2 & \text{if } a \text{ is even.} \end{cases}$

Then T is a 2-edge geodetic set of cardinality b of G so that $eg_2(G) \leq b$. $\square$

3 The 2-edge geodetic number and Cartesian product graphs

In this section we determine the bounds for the 2-edge geodetic number $eg_2(G \square H)$ of the Cartesian product of graphs G and H and obtain exact values of the 2-edge geodetic numbers for certain classes of Cartesian product graphs.

Theorem 3.1 Let G and H be connected graphs of order m and n , respectively, with $2 \leq m \leq n$. Then $n \leq eg_2(G \square H) \leq mn - m$.

Proof For any vertex y in H , let G_y be the copy of G corresponding to y on $G \square H$. Let S be an eg_2 -set of $G \square H$. It follows from Theorem 2.1 that $V(G_y) \cap S \neq \emptyset$ and so $n \leq |S| = eg_2(G \square H)$. Now, we prove the upper bound. Let $V(G) = \{x_1, x_2, \dots, x_m\}$ and $V(H) = \{y_1, y_2, \dots, y_n\}$. Let $S' = V(G \square H) - \bigcup_{i=1}^m \{(x_i, y_i)\}$. We prove that S' is a 2-edge geodetic set of $G \square H$. Let $e = (x_i, y_k)(x_j, y_l)$ be any edge in $G \square H$, where $1 \leq i, j \leq m$ and $1 \leq k, l \leq n$. Assume that $x_i x_j \in E(G)$. Then $y_k = y_l$. Choose a vertex y_l in H such that $y_k y_l \in E(H)$.

Case 1 $k = i$. Then $(x_i, y_l), (x_j, y_k) \in S'$. Since $(x_j, y_k)(x_i, y_k), (x_i, y_k)(x_i, y_l) \in E(G \square H)$ and $(x_j, y_k)(x_i, y_l) \notin E(G \square H)$, it follows that the edge $e = (x_i, y_k)(x_j, y_k)$ lies on the 2-geodesic of the vertices (x_i, y_l) and (x_j, y_k) of S' .

Case 2 $k \neq i$. We consider two subcases.

Subcase 2.1 $k = j$. Then, as in Case 1, e lies on the 2-geodesic of the vertices $(x_i, y_k), (x_j, y_l)$ of S' .

Subcase 2.2 $k \neq j$. Then $e \in E(S')$. □

Problem 3.2 Let m and n be integers such that $2 \leq m \leq n$. For any integer k with $n \leq k \leq mn - m$, are there graphs G and H of order m and n , respectively, such that $eg_2(G \square H) = k$?

The following theorem shows that the lower bound in Theorem 3.1 is sharp for any connected bipartite graph G and $H = K_2$.

Theorem 3.3 If G is a connected bipartite graph of order n , then $eg_2(G \square K_2) = n$.

Proof By Theorem 3.1, we have $n \leq eg_2(G \square K_2)$. Let $V(K_2) = \{v_1, v_2\}$. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be a bipartition of G . Let G_i be the copy of G corresponding to v_i on $G \square K_2$ for $i = 1, 2$. Let $S = \{(x_i, v_1) : 1 \leq i \leq r\} \cup \{(y_j, v_2) : 1 \leq j \leq s\}$. Then $|S| = n$. We claim that S is a 2-edge geodetic set of $G \square K_2$. Let $e = (g_1, h_1)(g_2, h_2) \in E(G \square K_2) - E(S)$. We consider two cases.

Case 1 $h_1 = h_2$ and $g_1 g_2 \in E(G)$. Assume that $h_1 = h_2 = v_1$ and $g_1 = x_i$ for some $1 \leq i \leq r$ and also $g_2 = y_j$ for some $1 \leq j \leq s$. It follows that the edge $e = (x_i, v_1)(y_j, v_1)$ lies on the $(x_i, v_1), (y_j, v_1), (y_j, v_2)$ geodesic of length 2 with $(x_i, v_1), (y_j, v_2) \in S$.

Case 2 $g_1 = g_2$ and $h_1 h_2 \in E(K_2)$. Assume that $h_1 = v_1$, and $h_2 = v_2$ and $g_1 = g_2 = x_i$ for some $1 \leq i \leq r$. Choose j ($1 \leq j \leq s$) such that $x_i y_j \in E(G)$. Since the edge $e = (x_i, v_1)(x_i, v_2)$ lies on the 2-geodesic $(x_i, v_1), (x_i, v_2), (y_j, v_2)$ with $(x_i, v_1), (y_j, v_2) \in S$, it follows that S is a 2-edge geodetic set of $G \square K_2$. □

Remark 3.4 Theorem 3.3 fails if G is not a connected bipartite graph. For the graph $K_3 \square K_2$, we have $eg_2(K_3 \square K_2) = 4 > 3$ (see Theorem 3.6).

Problem 3.5 Characterize graphs G of order n for which $eg_2(G \square K_2) = n$.

The following theorem shows that the upper bound in Theorem 3.1 is sharp for $G = K_m$ and $H = K_n$.

Theorem 3.6 For integers m, n with $2 \leq m \leq n$, $eg_2(K_m \square K_n) = mn - m$.

Proof Let $V(K_m) = \{x_1, x_2, \dots, x_m\}$ and let G_i be the copy of K_n corresponding to x_i on $K_m \square K_n$ for $i = 1, 2, \dots, m$. Let S be any eg_2 -set of $K_m \square K_n$. We claim that $|S \cap V(G_i)| \geq n - 1$. Suppose that $|S \cap V(G_i)| < n - 1$. Then we can choose vertices $(x_i, y_j), (x_i, y_k)$ in $V(G_i)$ ($1 \leq j \neq k \leq n$) such that $(x_i, y_j), (x_i, y_k) \notin S$. Since each $G_i \cong K_n$ ($1 \leq i \leq m$), it follows that $e = (x_i, y_j)(x_i, y_k)$ is an edge of $K_m \square K_n$. By Theorem 2.1, e has at least one end in S , which is a contradiction. Hence $|S \cap V(G_i)| \geq n - 1$ for all $i = 1, 2, \dots, m$. Thus, $eg_2(K_m \square K_n) = |S| \geq m(n - 1) = mn - m$. Now, the result follows from Theorem 3.1. □

Theorem 3.7 Let G be a connected graph of order $n \geq 2$ and let $m \geq 2$ be an integer. Then $eg_2(G \square K_m) = mn - n$ for $n \leq m$ and $mn - n \leq eg_2(G \square K_m) \leq mn - m$ for $m < n$.

Proof Let G_x be the copy of K_m corresponding to $x \in V(G)$ on $G \square K_m$. Let S be an eg_2 -set of $G \square K_m$. Since $G_x \cong K_m$ for every $x \in V(G)$, it follows as in the proof of Theorem 3.6 that $|S \cap V(G_x)| \geq m - 1$. Hence $eg_2(G \square K_m) = |S| \geq n(m - 1) = mn - n$. If $n \leq m$, then by Theorem 3.1, $eg_2(G \square K_m) \leq mn - n$. If $m < n$, then, since $eg_2(G \square K_m) = eg_2(K_m \square G)$, it follows from Theorem 3.1 that $eg_2(G \square K_m) \leq mn - m$. □

The following theorem gives a condition sufficient for a sharpening of the upper bound.

Theorem 3.8 Let G and H be connected graphs of order m and n with $2 \leq m \leq n$ such that H has an independent set of $n - m + 1$ vertices. Then $eg_2(G \square H) \leq mn - n$.

Proof If $m = n$, then the result follows from Theorem 3.1. Let $m < n$. Let $V(G) = \{x_1, x_2, \dots, x_m\}$. Let $V(H) = \{y_1, y_2, \dots, y_n\}$ be such that $\{y_m, y_{m+1}, \dots, y_n\}$ is an independent set in H . Let $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m), (x_m, y_{m+1}), \dots, (x_m, y_n)\}$. Since the set $\{y_m, y_{m+1}, \dots, y_n\}$ is independent in H , it follows that S is independent in $G \square H$. Let $U = V(G \square H) - S$. Then $|U| = mn - n$. We show that U is a 2-edge geodetic set of $G \square H$. Let $e = (x_i, y_k)(x_j, y_l)$, where $1 \leq i, j \leq m$ and $1 \leq k, l \leq n$, be any edge in $G \square H$ such that $e \notin E(U)$.

Case 1 $x_i x_j \in E(G)$. Then $y_k = y_l$. Since $e \notin E(U)$, at least one end of e , say $(x_i, y_k) \notin U$. Since S is independent, it follows that $(x_j, y_k) \in U$. Choose $t \neq k$ such that $y_k y_t \in E(H)$. Now, $(x_i, y_k)(x_i, y_t) \in E(G \square H)$. Since $(x_i, y_k) \in S$ and S is independent, we have $(x_i, y_t) \in U$. Then it follows that e lies on the 2-geodesic $(x_j, y_k), (x_i, y_k), (x_i, y_t)$ of vertices in U .

Case 2 $y_k y_l \in E(H)$. This is similar to Case 1. Hence U is a 2-edge geodetic set of $G \square H$ and so $eg_2(G \square H) \leq |U| = mn - n$. $\square$

Corollary 3.9 Let G be a connected graph of order n and let m be any integer such that $2 \leq m \leq n$. If G has an independent set of $n - m + 1$ vertices, then $eg_2(G \square K_m) = mn - n$.

Proof This follows from Theorems 3.7 and 3.8. $\square$

4 The 2-edge geodetic number and join of two graphs

The join $G + H$ of two vertex disjoint graphs G and H consists of $G \cup H$ and all the edges joining a vertex of G and a vertex of H . In this section, we determine the 2-edge geodetic number of join of two graphs in terms of the 2-edge geodetic numbers of its factor graphs.

Corollary 4.1 If G and H are connected graphs of order n and m , respectively, such that either G or H contains a vertex of full degree, then $eg_2(G + H) = m + n$ or $eg_2(G + H) = m + n - 1$.

Proof This follows from Theorems 2.6 and 2.8. $\square$

The following lemma is used to prove Theorem 4.3.

Lemma 4.2 Let G and H be connected graphs and S an eg_2 -set of $G + H$. Then either $V(G) \subseteq S$ or $V(H) \subseteq S$.

Proof Suppose that $V(G) \not\subseteq S$ and $V(H) \not\subseteq S$. Then there exists $u \in V(G)$ such that $u \notin S$ and there exists $v \in V(H)$ such that $v \notin S$. Since $uv \in E(G + H)$ and S is an eg_2 -set of $G + H$, it follows from Observation 2.1 that either $u \in S$ or $v \in S$. This is a contradiction. $\square$

Theorem 4.3 Let G and H be connected graphs of order n and m , respectively, such that neither G nor H contains a full degree vertex. Then $eg_2(G + H) = \min\{n + eg_2(H), m + eg_2(G)\}$.

Proof Assume that $n + eg_2(H) \leq m + eg_2(G)$. Let S be an eg_2 -set of H . Let $S' = S \cup V(G)$. Then $|S'| = n + eg_2(H)$. We claim that S' is a 2-edge geodetic set of $G + H$. Let $e = uv$ be an edge in $G + H$ such that $e \notin E(S')$. Then $e \notin E(G)$.

Case 1 $e \in E(H)$. Then at most one vertex of u, v belongs to S . Since S is an eg_2 -set of H , it follows that e lies on an $x - y$ geodesic P in H of length 2 with $x, y \in S$. Hence P is also an $x - y$ geodesic of length 2 in $G + H$ with $x, y \in S$.

Case 2 $u \in V(G)$ and $v \in V(H)$. Then $v \notin S$. Since $d_G(u) < n - 1$, there exists $u' \in V(G)$ such that $uu' \notin E(G)$ and so the edge e lies on the geodesic $Q : u, v, u'$ in $G + H$ of length 2 with $u, u' \in S'$. Thus, S' is a 2-edge geodetic set of $G + H$. Therefore, $eg_2(G + H) \leq |S'| = n + eg_2(H)$.

Now, suppose that $eg_2(G + H) < n + eg_2(H)$. Let T be an eg_2 -set of $G + H$. Then $|T| < n + eg_2(H)$. By Lemma 4.2, either $V(G) \subseteq T$ or $V(H) \subseteq T$.

If $V(G) \subseteq T$, let $T' = T - V(G)$. Then $T' \subseteq V(H)$ and $|T'| = |T| - n < n + eg_2(H) - n = eg_2(H)$. This implies that T' is not a 2-edge geodetic set of H and so there exists an edge $e = uv$ in H such that e does

not lie on any 2-geodesic of vertices in T' . Since $e \in E(G + H)$ and T is an eg_2 -set of $G + H$, e lies on a $x - y$ geodesic P of length 2 in $G + H$ with $x, y \in T$ and so it follows that both $x, y \in V(G)$. Since the length of P is 2, it follows that e is not an edge of H , which is a contradiction. If $V(H) \subseteq T$, let $T'' = T - V(H)$. Then $T'' \subseteq V(G)$ and $|T''| < n + eg_2(H) - m \leq m + eg_2(G) - m$. Thus $|T''| < eg_2(G)$. Then, as in Case 1, we obtain a contradiction. $\square$

Corollary 4.4 *Let G and H be two connected graphs of order n and m , respectively, such that neither G nor H contains a full degree vertex. Then $eg(G + H) = \min\{n + eg_2(H), m + eg_2(G)\}$.*

Proof Since $diam(G + H) = 2$, the result follows from Theorems 2.10 and 4.3. $\square$

Acknowledgments Research supported by DST Project No. SR/S4/MS: 319/06.

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Attici, M.: Computational complexity of geodetic sets. *Int. J. Comput. Math.* **79**, 587–591 (2002)
2. Brešar, B.; Klavžar, S.; Horvat, A.T.: On the geodetic number and related metric sets in Cartesian product graphs. *Discret. Math.* **308**, 5555–5561 (2008)
3. Buckley, F.; Harary, F.: *Distance in Graphs*. Addison-Wesley, Redwood City (1990)
4. Chartrand, G.; Harary, F.; Zhang, P.: On the geodetic number of a graph. *Networks* **39**(1), 1–6 (2002)
5. Chartrand, G.; Zhang, P.: *Introduction to Graph Theory*. Tata McGraw-Hill Edition, New Delhi (2006)
6. Gera, R.; Zhang, P.: On k -geodomination in Cartesian products. *Congr. Numer.* **158**, 163–178 (2002)
7. Harary, F.: *Graph Theory*. Addison-Wesley, Redwood City (1969)
8. Imrich, W.; Klavžar, S.: *Product Graphs: Structure and Recognition*. Wiley-Interscience, New York (2000)
9. Muntean, R.; Zhang, P.: On geodomination in graphs. *Congr. Numer.* **143**, 161–174 (2000)
10. Muntean, R.; Zhang, P.: k -Geodomination in graphs. *ARS Combin.* **63**, 33–47 (2002)
11. Santhakumaran, A.P.; John, J.: Edge geodetic number of a graph. *J. Discret Math. Sci. Crypt.* **10**(3), 415–432 (2007)
12. Santhakumaran, A.P.; Ullas Chandran, S.V.: The k -edge geodetic number of a graph. *Util. Math.* (2012)
13. Santhakumaran, A.P.; Ullas Chandran, S.V.: The edge geodetic number and Cartesian product of graphs. *Discuss. Math. Graph Theory* **30**(1), 55–73 (2010)