On the transfer between the Dickson algebras as modules over the Steenrod algebra

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Abstract Let $D_\ell := D(\mathbb{W})$ be the Dickson–Mùi algebra of $\mathbb{W}$ for an elementary abelian p -group $\mathbb{W}$ of rank ℓ , which consists of all invariants in the mod p cohomology of $\mathbb{W}$ under the general linear group $GL(\mathbb{W})$. Hưng (Math Ann 353:827–866, 2012) determined explicitly all the homomorphisms between the Dickson–Mùi algebras (regarded as modules over the Steenrod algebra, $\mathcal{A}$). He showed that the compositions of the restrictions $res_{\ell,m}: D_\ell \rightarrow D_m$ and the transfers $tr_{m,n}: D_m \rightarrow D_n$ for $m \leq \min\{\ell, n\}$ form a basis of $\text{Hom}_{\mathcal{A}}(D_\ell, D_n)$. The restriction $res_{\ell,m}$ has explicitly been known (from Lemmas 3.3 and 3.4 for $p = 2$ and Lemma 9.1 for $p > 2$ of the cited article), while the transfer $tr_{m,n}$ has only been computed for $m = 1$ in some degrees (see Lemma 9.2 of the article). In this paper, we study $tr_{m,n}$ for $p = 2$ in general. We determine completely $tr_{1,n}$ for any n , and compute the image of $tr_{m,n}$ for arbitrary m, n on some powers of multilinear and alternating invariants. Then, we recognize some families of invariants in D_m on which the transfer $tr_{m,n}$ vanishes.

Keywords Steenrod algebra · Modular representations · Invariant theory · Dickson algebra

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1 Introduction

Let $\mathbb{V}$ be an elementary abelian 2-group rank n . Then $\mathbb{V}$ can also be regarded as an n -dimensional vector space over $\mathbb{F}_2$, the field of two elements. Let $H^*(\mathbb{V})$ denote the cohomology of the group $\mathbb{V}$. As it is well-known, $H^*(\mathbb{V}) \cong S(\mathbb{V}^*)$ where $S(\mathbb{V}^*)$ denote the symmetric algebra over the space $\mathbb{V}^*$. Throughout the paper, the coefficient ring for homology and cohomology is always $\mathbb{F}_2$. Let $x_1, \dots, x_n$ be a basis of $\mathbb{V}^*$, we have

$$H^*(\mathbb{V}) \cong \mathbb{F}_2[x_1, \dots, x_n].$$

The general linear group $GL(\mathbb{V}) \cong GL(n, \mathbb{F}_2)$ acts regularly on $\mathbb{V}$ and therefore on $H^*(\mathbb{V})$. The Dickson algebra is the algebra of all invariants of $H^*(\mathbb{V})$ under the action of $GL(\mathbb{V})$. It is explicitly determined by Dickson in [2] as follows:

$$D(\mathbb{V}) := H^*(\mathbb{V})^{GL(\mathbb{V})} \cong \mathbb{F}_2[x_1, \dots, x_n]^{GL(n, \mathbb{F}_2)} = \mathbb{F}_2[Q_{n,0}, Q_{n,1}, \dots, Q_{n,n-1}],$$

where $Q_{n,i}$ denotes the Dickson invariant of degree $2^n - 2^i$. (A precise definition for the Dickson invariants will be given in Sect. 2.)

Being the cohomology of the classifying space $B\mathbb{V}$, the group $H^*(\mathbb{V})$ is equipped with a structure of module over the mod 2 Steenrod algebra, $\mathcal{A}$. Each $\gamma \in GL(\mathbb{V})$ induces an $\mathcal{A}$ -isomorphism γ^* on $H^*(\mathbb{V})$. The map $\gamma \mapsto \gamma^*$ gives rise to the regular action of $GL(\mathbb{V})$ on $H^*(\mathbb{V})$. So, the actions of $GL(\mathbb{V})$ and $\mathcal{A}$ on $H^*(\mathbb{V})$ commute with each other. Hence, the Dickson algebra $D(\mathbb{V})$ inherits a structure of module over the Steenrod algebra $\mathcal{A}$ from $H^*(\mathbb{V})$.

Let $\mathbb{U}$ be a $\mathbb{F}_2$ -vector space of dimension m with $m \leq n$. The subject of the present paper is the transfer $tr_{\mathbb{U}, \mathbb{V}}: D(\mathbb{U}) \rightarrow D(\mathbb{V})$, which is also denoted by $tr_{m,n}: D_m \rightarrow D_n$ defined in [3]. For fixed n and ℓ , the set $\{tr_{m,n} \mid m < \min\{\ell, n\}\}$ is known as a key component forming all $\mathcal{A}$ -homomorphisms from D_ℓ to D_n in the sense as follows. Let $\mathbb{W}$ be a vector space of dimension ℓ . The restriction $res_{\ell,m}: D_\ell \rightarrow D_m$ is the homomorphism induced from an inclusion $\mathbb{U} \rightarrow \mathbb{W}$ (in [3, Lemma 3.1], Hưng showed that any inclusion $\mathbb{U} \rightarrow \mathbb{W}$ induces the same homomorphism on $H^*(\mathbb{W})^{GK(\mathbb{W})}$). Let $\bar{D}(\mathbb{V})$ or $\bar{D}_n$ be the augmentation ideal of all positive degree elements in the Dickson algebra D_n .

Theorem 1.1 [3, Theorem 1.1] *The $\mathcal{A}$ -module homomorphisms $\{tr_{m,n}res_{\ell,m}: 1 \leq m \leq \min\{\ell, n\}\}$ form a basis of the vector space $Hom_{\mathcal{A}}(\bar{D}_\ell, \bar{D}_n)$ of $\mathcal{A}$ -module homomorphisms from $\bar{D}_\ell$ to $\bar{D}_n$. In particular, $\dim Hom_{\mathcal{A}}(\bar{D}_\ell, \bar{D}_n) = \min\{\ell, n\}$.*

The restrictions $res_{\ell,m}$ are explicitly determined on each Dickson monomial in [3, Lemma 3.4]; so, in order to compute $Hom_{\mathcal{A}}(\bar{D}_\ell, \bar{D}_n)$ we need to compute the transfers $tr_{m,n}$.

First, the transfer $tr_{1,n}$ is concretely computed in [3, Lemma 4.2]. Then, it is further studied in the undergraduate thesis of Phạm H. Đăng under the guidance of Nguyễn H. V. Hưng (see [1]).

In the case $m = 1$, we have $D_m = \mathbb{F}_2[x]$ in which $\deg x = 1$. The following theorem is one of main results of the paper.

Theorem 1.2 For $k > 0$,

$$tr_{1,n}(x^k) = \sum_{\sum_{i=0}^{n-1} (2^n - 2^i) s_i = k} \frac{(\sum_{i=0}^{n-1} s_i - 1)!}{(s_0 - 1)! s_1! \dots s_{n-1}!} Q_{n,0}^{s_0} Q_{n,1}^{s_1} \dots Q_{n,n-1}^{s_{n-1}}.$$

Here $\frac{(\sum_{i=0}^{n-1} s_i - 1)!}{(s_0 - 1)! s_1! \dots s_{n-1}!} = 0$ if $s_0 = 0$.

Let $x_1, \dots, x_m$ be a basis for $\mathbb{U}^*$. Then each $f \in D_m = D(\mathbb{U})$ is a polynomial in the variables x_i ($1 \leq i \leq m$); so, we rewrite f as $f(x_1, \dots, x_m)$. We denote by f the map $\underbrace{\mathbb{V}^* \times \dots \times \mathbb{V}^*}_{m \text{ times}} \rightarrow S(\mathbb{V}^*)$, $(t_1, \dots, t_m) \mapsto f(t_1, \dots, t_m)$. We

say that $f(x_1, \dots, x_m)$ is *multilinear* on $\mathbb{V}^*$ if f is m -multilinear. The invariant $f(x_1, \dots, x_m)$ is called *alternating* on $\mathbb{V}^*$ if the map f is alternating.

The set of invariants in D_m which are multilinear and alternating forms a subspace of D_m . Furthermore, this subspace is subset of the ideal of D_m generated by $Q_{m,0}$ (see Corollary 4.7). It is infinity dimensional and can be explicitly determined as follows. For each set of m distinct non-negative numbers $\beta = \{\beta_1, \dots, \beta_m\}$, let Σ_m be the symmetric group on $\{1, \dots, m\}$ and let

$$\omega_\beta(x_1, \dots, x_m) = \sum_{\sigma \in \Sigma_m} x_1^{2^{\beta_{\sigma(1)}}} \dots x_m^{2^{\beta_{\sigma(m)}}}.$$

Proposition 1.3 The set

$$\{\omega_\beta(x_1, \dots, x_m) \mid \beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m, \quad \beta_i \neq \beta_j \text{ for all } i \neq j\}$$

forms a basis for the subspace of D_m generated by multilinear and alternating invariants.

The following theorem shows the image of the transfer $tr_{m,m+1}$ on powers of multilinear and alternating invariants in D_m . For each $Q \in D_m$ and $y_1, \dots, y_m \in \mathbb{F}[x_1, \dots, x_{m+1}]$, we denote by $Q(y_1, \dots, y_m)$ the polynomial in $\mathbb{F}[x_1, \dots, x_{m+1}]$ obtained from Q by the substitution y_j for x_j ($j = 1, \dots, m$).

Theorem 1.4 Suppose that Q is a multilinear and alternating invariant in D_m . Let $y_j = Q(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{m+1})$ for $j = 1, \dots, m+1$. Then, for $r > 0$ we have

$$tr_{m,m+1}(Q^r) = \sum \frac{(s_0 + \dots + s_m - 1)!}{(s_0 - 1)! s_1! \dots s_m!} \left(Q_{m+1,0}^{s_0} \dots Q_{m+1,m}^{s_m} \right) (y_1, \dots, y_{m+1})$$

summed over all sequences of non-negative integers $(s_0, \dots, s_m)$ such that

$$\sum_{i=0}^m (2^{m+1} - 2^i) s_i = r.$$

Since $Q_{m,0}$, $Q_{m,0}Q_{m,i}$ are multilinear and alternating invariants (see Example 4.8), we get the two following explicit formulas for image of $tr_{m,m+1}$ on some powers of these invariants.

Corollary 1.5 (i) $tr_{m,m+1}(Q_{m,0}^j) = 0$ for $0 < j < 2^{m+1} - 1$.
(ii) $tr_{m,m+1}(Q_{m,0}^{2^{m+1}-1}) = Q_{m+1,0}^{2^m-1}$.

Corollary 1.6 For $r > 0$, $tr_{2,3}(Q_{2,0}^r Q_{2,1}^r)$

$$= \sum_{7s_0+6s_1+4s_2=r} \frac{(s_0+s_1+s_2-1)!}{(s_0-1)!s_1!s_2!} (Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 + Q_{3,0}^5) ^{s_0} (Q_{3,0}^2 Q_{3,2}^4) ^{s_1} \\ \times (Q_{3,1}^2 Q_{3,2}^2 + Q_{3,0}^2 Q_{3,1}) ^{s_2}.$$

The following proposition gives some families of invariants on which the transfer is zero.

Proposition 1.7 $tr_{m,n}(Q)$ is zero for each of the following cases.

- (i) $Q \in D_m$ is a multilinear and alternating invariant;
- (ii) $Q = Q_{m,0}^{2^{s_0}} Q_{m,1}^{2^{s_1}} \dots Q_{m,m-1}^{2^{s_{m-1}}}$ where $s_0, s_1, \dots, s_{m-1}$ are nonnegative with m, n such that $n \geq \frac{3m^2+3m}{2}$;
- (iii) $m = 2, n \geq 4$ and $Q = Q_{2,0}^{2^{s_0}} Q_{2,1}^{2^{s_1}}$ for any nonnegative integers s_0, s_1 .

The statement in Proposition 1.7(iii) is no longer true for the case $n = 3$. In fact, $tr_{2,3}(Q_{2,0}^{2^{s_0}} Q_{2,1}^{2^{s_1}}) \neq 0$ for $s_0 - s_1 > 1$ or $s_0 - s_1 < 0$ (see Remark 7.3).

The paper is divided into seven sections. The introduction in Sect. 1 is followed by the preliminary in Sect. 2, where we recall definition of the transfer between the Dickson algebras. Section 3 is a proof of Theorem 1.2. The concepts of multilinear and alternating invariants are introduced in Sect. 4, where we prove Proposition 1.3. In Sect. 5, we study the transfer on powers of multilinear and alternating invariants, then prove Theorem 1.4 and Proposition 1.7(i). Section 6 deals with a method of finding invariant monomials in the image of the transfer by finding “leading elements” of it. By using this method, we prove Lemma 5.5, the most technical lemma of the paper that is used in the proofs of Corollaries 1.5 and 1.6. Finally, in Sect. 7, we prove Proposition 1.7(ii) and (iii).

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2 Preliminary

In this section, we exploit Hưng’s definition of the transfer $tr_{m,n}$.

Let $\mathbb{U}$ and $\mathbb{V}$ be a $\mathbb{F}_2$ -vector spaces of dimensions m and n with $m \leq n$. Let $\mathbb{K}$ be a subspace of $\mathbb{V}$ and $\pi_{\mathbb{K}}: \mathbb{V} \rightarrow \mathbb{U}$ be an epimorphism with $\ker \pi_{\mathbb{K}} = \mathbb{K}$. Suppose that $\pi: \mathbb{V} \rightarrow \mathbb{U}$ is another epimorphism whose kernel is also $\mathbb{K}$. Then

there is an isomorphism $\alpha: \mathbb{U} \rightarrow \mathbb{U}$ such that $\pi_{\mathbb{K}} = \alpha\pi$. It follows that $\pi_{\mathbb{K}}^* = \pi^*\alpha^*: H^*(\mathbb{U}) \rightarrow H^*(\mathbb{V})$. Since α^* acts identically on the invariants $H^*(\mathbb{U})^{GL(\mathbb{U})}$, $\pi_{\mathbb{K}}^*$ and π^* are the same on $H^*(\mathbb{U})^{GL(\mathbb{U})}$. Therefore, the induced homomorphism $\pi_{\mathbb{K}}^*: H^*(\mathbb{U})^{GL(\mathbb{U})} \rightarrow H^*(\mathbb{V})$ does not depend on the choice of the epimorphism $\pi_{\mathbb{K}}$. It only depends on the kernel of $\pi_{\mathbb{K}}$, $\mathbb{K}$. On the other hand, the group $GL(\mathbb{V})$ permute the subspaces $\mathbb{K}$ of dimension $n - m$ in $\mathbb{V}$. So the sum of $\pi_{\mathbb{K}}^*$ maps $H^*(\mathbb{U})^{GL(\mathbb{U})}$ to the $GL(\mathbb{V})$ -invariants.

Definition 2.1 [3, Definition 4.1] $tr_{\mathbb{U}, \mathbb{V}} = \sum_{\mathbb{K}} \pi_{\mathbb{K}}^*: H^*(\mathbb{U})^{GL(\mathbb{U})} \rightarrow H^*(\mathbb{V})^{GL(\mathbb{V})}$, where the sum runs over all the subspaces $\mathbb{K}$ of dimension $(\dim \mathbb{V} - \dim \mathbb{U})$ in $\mathbb{V}$.

As it is well known that $H^1(\mathbb{V}) \cong \mathbb{V}^*$ and $H^*(\mathbb{V}) \cong S(\mathbb{V}^*)$ where $S(\mathbb{V}^*)$ denote the symmetric algebra over the space $\mathbb{V}^*$. Let $x_1, \dots, x_n$ be a basis of $\mathbb{V}^*$, we have

$$H^*(\mathbb{V}) \cong \mathbb{F}_2[x_1, \dots, x_n].$$

Recall that the algebra of the $GL(\mathbb{V})$ -invariants $H^*(\mathbb{V})^{GL(\mathbb{V})}$ was computed by Dickson [2]. He showed in [2] that $H^*(\mathbb{V})^{GL(\mathbb{V})}$ is also a polynomial algebra and denoted it by D_n :

$$D_n := H^*(\mathbb{V})^{GL(\mathbb{V})} = \mathbb{F}_2[Q_{n,0}, Q_{n,1}, \dots, Q_{n,n-1}].$$

Here $Q_{n,0}, Q_{n,1}, \dots, Q_{n,n-1}$ are the Dickson invariants. They are inductively defined by the formula

$$Q_{n,i} = Q_{n-1,i-1}^2 + V_n Q_{n-1,i},$$

where, by convention, $Q_{n,n} = 1$, $Q_{n,i} = 0$ for $i < 0$, and

$$V_n = \prod_{c_j \in \mathbb{F}_2} (c_1 x_1 + \dots + c_{n-1} x_{n-1} + x_n)$$

is the Mui invariant under the Sylow 2-subgroup T_n of GL_n consisting of all upper triangular $n \times n$ -matrices with 1 on the main diagonal (see Mui [5]).

Since the transfer $tr_{\mathbb{U}, \mathbb{V}}$ only depends on the dimensions of $\mathbb{U}$ and $\mathbb{V}$, it is denoted by $tr_{m,n}: D_m \rightarrow D_n$, where $m = \dim \mathbb{U}$ and $n = \dim \mathbb{V}$. For each epimorphism $\pi: \mathbb{V} \rightarrow \mathbb{U}$, the induced homomorphism in cohomological degree 1, $\pi^*: H^1(\mathbb{U}) \rightarrow H^1(\mathbb{V})$ is a monomorphism. So $W := \pi^* H^1(\mathbb{U})$ is a subspace of dimension m in $H^1(\mathbb{V})$. Let $x_1, \dots, x_m$ be a basis of $H^1(\mathbb{U})$. Then $\pi^*(x_1), \dots, \pi^*(x_m)$ a basis for $\pi^* H^1(\mathbb{U})$. And for each $Q \in D_m$ we have $\pi^*(Q) = Q(\pi^*(x_1), \dots, \pi^*(x_m))$. Since Q is invariant under any invertible transformation on $\mathbb{U}$, $Q(\pi^*(x_1), \dots, \pi^*(x_m)) = Q(y_1, \dots, y_m)$ for any basis $y_1, \dots, y_m$ of W . Hence $Q(\pi^*(x_1), \dots, \pi^*(x_m))$ only depends on W ; so, we denote it by $Q(W)$ for short. Therefore, by Definition 2.1 we have

$$tr_{m,n}(Q) = \sum_W Q(W)$$

where the sum runs over all m -dimensional subspaces W in $H^1(\mathbb{V})$. For example, with $m = 1$, $n = 3$ let x_1, x_2, x_3 be a basis for $\mathbb{V}^*$ we have

$$tr_{1,3}(x^k) = x_1^k + x_2^k + x_3^k + (x_1 + x_2)^k + (x_1 + x_3)^k + (x_2 + x_3)^k + (x_1 + x_2 + x_3)^k$$

since all one-dimensional subspaces of $\mathbb{V}^*$ are

$$\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, \langle x_1 + x_2 \rangle, \langle x_1 + x_3 \rangle, \langle x_2 + x_3 \rangle, \langle x_1 + x_2 + x_3 \rangle.$$

3 The transfer from D_1 to D_n

The aim of this section is to prove the following theorem, which is also numbered as Theorem 1.2 in the introduction.

Theorem 3.1 For $k > 0$,

$$tr_{1,n}(x^k) = \sum_{\sum_{i=0}^{n-1} (2^n - 2^i) s_i = k} \frac{\left(\sum_{i=0}^{n-1} s_i - 1\right)!}{(s_0 - 1)! s_1! \dots s_{n-1}!} Q_{n,0}^{s_0} Q_{n,1}^{s_1} \dots Q_{n,n-1}^{s_{n-1}}.$$

Here $\frac{(\sum_{i=0}^{n-1} s_i - 1)!}{(s_0 - 1)! s_1! \dots s_{n-1}!} = 0$ if $s_0 = 0$.

At small degrees, $tr_{1,n}$ was explicitly determined by Hưng in [3] as follows.

Lemma 3.2 [3, Lemma 4.2]

- (i) $tr_{1,n}(x^{2^n-1}) = Q_{n,0}$.
- (ii) $tr_{1,n}(x^j)$, for $0 < j < 2^n - 1$.

In order to prove Theorem 3.1, we need Lemma 3.2 and the following inductive formula.

Lemma 3.3 $tr_{1,n}(x^k) = \sum_{i=0}^{n-1} Q_{n,i} tr_{1,n}(x^{k-2^n+2^i})$ for $k \geq 2^n - 1$.

Proof The lemma is proved by induction on n . As $k \geq 2^n - 1$, we rewrite k as the form $k = \ell + 2^n - 1$ for $\ell \geq 0$. Then the lemma is equivalent to

$$tr_{1,n}(x^{\ell+2^n-1}) = \sum_{i=0}^{n-1} Q_{n,i} tr_{1,n}(x^{\ell+2^i-1}),$$

for $\ell \geq 0$.

For $n = 1$, as $tr_{1,1}$ is the identity map, we have $tr_{1,1}(x^{\ell+1}) = x^{\ell+1} = Q_{1,0} tr_{1,1}(x^\ell)$. So the statement holds for $n = 1$. Suppose inductively that it is true for n . Let

$$U = \{c_1 x_1 + \dots + c_n x_n | c_i \in \mathbb{F}_2 \text{ for } i = 1, \dots, n\},$$

$$W = \{c_1 x_1 + \dots + c_n x_n + x_{n+1} | c_i \in \mathbb{F}_2 \text{ for } i = 1, \dots, n\}.$$

For $s \geq 0$, we set

$$R_s^U := \sum_{x \in U} x^s \quad \text{and} \quad R_s^W := \sum_{x \in W} x^s.$$

Then, by the definition of $tr_{1,n}$, we have $tr_{1,n}(x^s) = R_s^U$ and $tr_{1,n+1}(x^s) = R_s^U + R_s^W$. Furthermore, we get the two following formulas. They are proved latter.

$$\begin{aligned} (a) \quad \sum_{i=0}^{n-1} Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^{i+1}-1}) &= R_{\ell+2^{n+1}-1}^U + \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W, \\ (b) \quad V_{n+1} \sum_{i=0}^n Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^i-1}) &= R_{\ell+2^{n+1}-1}^W + \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W. \end{aligned}$$

Then, we have

$$\begin{aligned} &\sum_{i=0}^n Q_{n+1,i} tr_{1,n+1}(x^{\ell+2^i-1}) \\ &= \sum_{i=0}^n (Q_{n,i-1}^2 + Q_{n,i} V_{n+1}) tr_{1,n+1}(x^{\ell+2^i-1}) \\ &\quad (\text{as } Q_{n+1,i} = Q_{n,i-1}^2 + Q_{n,i} V_{n+1} \text{ for } i \geq 0 \text{ and } Q_{n,i} = 0 \text{ for } i < 0) \\ &= \sum_{i=1}^n Q_{n,i-1}^2 tr_{1,n+1}(x^{\ell+2^i-1}) + V_{n+1} \sum_{i=0}^n Q_{n,i} tr_{1,n+1}(x^{\ell+2^i-1}) \\ &= \sum_{i=0}^{n-1} Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^{i+1}-1}) + \left(R_{\ell+2^{n+1}-1}^W + \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W \right) \\ &\quad [\text{applying (b)}] \\ &= R_{\ell+2^{n+1}-1}^U + R_{\ell+2^{n+1}-1}^W \quad [\text{applying (a)}] \\ &= tr_{1,n+1}(x^{\ell+2^{n+1}-1}). \end{aligned}$$

So, the lemma is proved.

Now, we prove the two formulas (a) and (b) above.

Case (a)

$$\begin{aligned} tr_{1,n}(x^{\ell+2^{n+1}-1}) &= tr_{1,n}(x^{(\ell+2^n)+2^n-1}) \\ &= \sum_{i=0}^{n-1} Q_{n,i} tr_{1,n}(x^{\ell+2^n+2^i-1}) \\ &\quad (\text{by the inductive hypothesis on } tr_{1,n} \text{ for } \ell+2^n) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} Q_{n,i} Q_{n,j} tr_{1,n}(x^{(\ell+2^i)+2^j-1}) \\ &\quad [\text{applying the inductive hypothesis to each } tr_{1,n}(x^{\ell+2^n+2^i-1})]. \end{aligned}$$

However, we have

$$\begin{aligned} \sum_{i \neq j} Q_{n,i} Q_{n,j} tr_{1,n}(x^{(\ell+2^i)+2^j-1}) &= \sum_{i < j} (Q_{n,i} Q_{n,j} + Q_{n,j} Q_{n,i}) tr_{1,n}(x^{\ell+2^i+2^j-1}) \\ &= 0. \end{aligned}$$

Therefore

$$tr_{1,n}(x^{\ell+2^{n+1}-1}) = \sum_{i=0}^{n-1} Q_{n,i}^2 tr_{1,n}(x^{\ell+2^{i+1}-1}).$$

It is equivalent to

$$R_{x^{\ell+2^{n+1}-1}}^U = \sum_{i=0}^{n-1} Q_{n,i}^2 tr_{1,n}(x^{\ell+2^{i+1}-1}).$$

Then, we get

$$\begin{aligned} \sum_{i=0}^{n-1} Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^{i+1}-1}) &= \sum_{i=0}^{n-1} Q_{n,i}^2 (R_{\ell+2^{i+1}-1}^U + R_{\ell+2^{i+1}-1}^W) \\ &= \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^U + \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W \\ &= R_{\ell+2^{n+1}-1}^U + \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W. \end{aligned}$$

So, we obtain (a).

Case (b). We have

$$\sum_{i=0}^n Q_{n,i} tr_{1,n+1}(x^{\ell+2^i-1}) = \sum_{i=0}^n Q_{n,i} (R_{\ell+2^i-1}^U + R_{\ell+2^i-1}^W).$$

Since the inductive hypothesis $tr_{1,n}(x^{\ell+2^n-1}) = \sum_{i=0}^{n-1} Q_{n,i} tr_{1,n}(x^{\ell+2^i-1})$ and $Q_{n,n} = 1$, it follows that

$$\sum_{i=0}^n Q_{n,i} tr_{1,n}(x^{\ell+2^i-1}) = 0.$$

In the other word

$$\sum_{i=0}^n Q_{n,i} R_{\ell+2^i-1}^U = 0.$$

So

$$\begin{aligned}
 \sum_{i=0}^n Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^i-1}) &= \sum_{i=0}^n Q_{n,i}^2 \left(R_{\ell+2^i-1}^U + R_{\ell+2^i-1}^W \right) \\
 &= \sum_{i=0}^n Q_{n,i}^2 R_{\ell+2^i-1}^U + \sum_{i=0}^n Q_{n,i}^2 R_{\ell+2^i-1}^W \\
 &= \sum_{i=0}^n Q_{n,i}^2 R_{\ell+2^i-1}^W \\
 &= \sum_{i=0}^n Q_{n,i}^2 \sum_{x \in W} x^{\ell+2^i-1} \quad \left(\text{as } R_{\ell+2^i-1}^W = \sum_{x \in W} x^{\ell+2^i-1} \right) \\
 &= \sum_{x \in W} x^{\ell-1} \sum_{i=0}^n Q_{n,i}^2 x^{2^i}.
 \end{aligned}$$

It implies that

$$V_{n+1} \sum_{i=0}^n Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^i-1}) = \sum_{x \in W} x^{\ell-1} V_{n+1} \sum_{i=0}^n Q_{n,i}^2 x^{2^i}.$$

On the other hand, $V_{n+1}(x_1, \dots, x_n, x_{n+1}) = V_{n+1}(x_1, \dots, x_n, x)$ for all $x \in W$; so, by [5, Proposition 2.6], we get

$$V_{n+1} = \sum_{i=0}^n Q_{n,i}^2 x^{2^i} \quad \text{for all } x \in W.$$

As a consequence

$$\begin{aligned}
 V_{n+1} \sum_{i=0}^n Q_{n,i}^2 tr_{1,n+1}(x^{\ell+2^i-1}) &= \sum_{x \in W} x^{\ell-1} \left(\sum_{i=0}^n Q_{n,i}^2 x^{2^i} \right)^2 \\
 &= \sum_{x \in W} x^{\ell-1} \sum_{i=0}^n Q_{n,i}^2 x^{2^{i+1}} \\
 &= \sum_{x \in W} \sum_{i=0}^n Q_{n,i}^2 x^{\ell+2^{i+1}-1} \\
 &= \sum_{i=0}^n Q_{n,i}^2 \sum_{x \in W} x^{\ell+2^{i+1}-1} \\
 &= \sum_{i=0}^n Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W \\
 &= R_{\ell+2^{n+1}-1}^W + \sum_{i=0}^{n-1} Q_{n,i}^2 R_{\ell+2^{i+1}-1}^W.
 \end{aligned}$$

Therefore, we get (b). $\square$

Proof of Theorem 3.1 for convenience, we denote

$$Q^{(s_0, s_1, \dots, s_{n-1})} = Q_{n,0}^{s_0} Q_{n,1}^{s_1} \dots Q_{n,n-1}^{s_{n-1}},$$

$$C(s_0, s_1, \dots, s_{n-1}) = \frac{\left(\sum_{i=0}^{n-1} s_i - 1\right)!}{(s_0 - 1)! s_1! \dots s_{n-1}!}.$$

Here $C(s_0, s_1, \dots, s_{n-1}) = 0$ if one of the numbers $(s_0 - 1), s_1, \dots, s_{n-1}$ is negative. We see that

$$\begin{aligned} C(s_0, s_1, \dots, s_{n-1}) &= \left(\sum_{i=0}^{n-1} s_i - 1\right) \frac{\left(\sum_{i=0}^{n-1} s_i - 2\right)!}{(s_0 - 1)! s_1! \dots s_{n-1}!} \\ &= \frac{\left(\sum_{i=0}^{n-1} s_i - 2\right)!}{(s_0 - 2)! s_1! \dots s_{n-1}!} + \sum_{i=1}^{n-1} \frac{\left(\sum_{i=0}^{n-1} s_i - 2\right)!}{(s_0 - 1)! \dots (s_i - 1) \dots s_{n-1}!} \\ &= \sum_{i=0}^{n-1} C(s_0, \dots, s_i - 1, \dots, s_{n-1}). \end{aligned}$$

For $0 \leq i \leq n - 1$, setting $d_i = \deg Q_{n,i}$, we get $d_i = 2^n - 2^i$. So, the theorem is equivalent to

$$tr_{1,n}(x^k) = \sum C(s_0, s_1, \dots, s_{n-1}) Q^{(s_0, s_1, \dots, s_{n-1})}, \quad (1)$$

where the sum is over all sequences $s_0, s_1, \dots, s_{n-1}$ such that $\sum_{i=0}^{n-1} d_i s_i = k$.

The proposition is proved by induction on k .

Case $k < 2^n - 1$. We have

$$\sum_{i=0}^{n-1} d_i s_i = k \iff (2^n - 1)s_0 + \sum_{i=1}^{n-1} s_i = k \implies s_0 = 0.$$

Then $C(s_0, s_1, \dots, s_{n-1}) = 0$ for all sequences $(s_0, \dots, s_{n-1})$ such that $\sum_{i=0}^{n-1} d_i s_i = k$. As a consequence, the right hand side of (1) is equal to zero. On the other hand, by Lemma 3.2, $tr_{1,n}(x^k) = 0$ as $k < 2^n - 1$. Therefore, we get (1) for this case.

Case $k \geq 2^n - 1$. Suppose inductively that the theorem holds at degree less than or equal k . We have

$$\begin{aligned}
 & \sum_{d_0 s_0 + \dots + d_{n-1} s_{n-1} = k+1} C(s_0, s_1, \dots, s_{n-1}) Q^{(s_0, s_1, \dots, s_{n-1})} \\
 &= \sum_{\sum d_j s_j = k+1} \sum_{i=0}^{n-1} C(s_0, \dots, s_i - 1, \dots, s_{n-1}) Q^{(s_0, s_1, \dots, s_{n-1})} \\
 &= \sum_{i=0}^{n-1} \sum_{\sum d_j s_j = k+1} C(s_0, \dots, s_i - 1, \dots, s_{n-1}) Q^{(s_0, s_1, \dots, s_{n-1})} \\
 &= \sum_{i=0}^{n-1} Q_{n,i} \sum_{\sum d_j s_j - d_i = k+1 - (2^n - 2^i)} C(s_0, \dots, s_i - 1, \dots, s_{n-1}) Q^{(s_0, \dots, s_i - 1, \dots, s_{n-1})} \\
 &= \sum_{i=0}^{n-1} Q_{n,i} tr_{1,n}(x^{k+1 - (2^n - 2^i)}) \\
 &\quad [\text{by inductive hypothesis since } k+1 - (2^n - 2^i) \leq k] \\
 &= tr_{1,n}(x^{k+1}) \quad (\text{by Lemma 3.3}).
 \end{aligned}$$

So, (1) holds at degree $k+1$. Therefore, the theorem is proved. $\square$

4 Multilinear and alternating invariants

Recall that $\mathbb{U}$ and $\mathbb{V}$ are vector spaces of dimensions m and n respectively with $m < n$, $tr_{\mathbb{U}, \mathbb{V}} = \sum_{\mathbb{K}} \pi_{\mathbb{K}}^*: H^*(\mathbb{U})^{GL(\mathbb{U})} \rightarrow H^*(\mathbb{V})^{GL(\mathbb{V})}$, where the sum runs over all the subspaces $\mathbb{K}$ of dimension $(\dim \mathbb{V} - \dim \mathbb{U})$ in $\mathbb{V}$. Let $x_1, \dots, x_m$ be a basis of $\mathbb{U}^*$. Then $H^*(\mathbb{U}) = S(\mathbb{U}^*) = \mathbb{F}_2[x_1, \dots, x_m]$. For each $f(x_1, \dots, x_m) \in \mathbb{F}_2[x_1, \dots, x_m]$, we define by f the map

$$\underbrace{\mathbb{V}^* \times \dots \times \mathbb{V}^*}_{m \text{ times}} \rightarrow S(\mathbb{V}^*), \quad (t_1, \dots, t_m) \mapsto f(t_1, \dots, t_m).$$

Definition 4.1 For $m < n$, a polynomial $f(x_1, \dots, x_m) \in \mathbb{F}_2[x_1, \dots, x_m]$ is called multilinear (alternating) on $\mathbb{V}^*$ if the map f is m -multilinear (alternating).

Lemma 4.2 If $f(x_1, \dots, x_m) \in \mathbb{F}_2[x_1, \dots, x_m]$ is multilinear and alternating, then $f(x_1, \dots, x_m)$ is a $GL(m, \mathbb{F}_2)$ -invariant.

Proof For each $\omega \in GL(m, \mathbb{F}_2)$, we have

$$\omega f(x_1, \dots, x_m) = f(\omega x_1, \dots, \omega x_m).$$

Since $f(x_1, \dots, x_m)$ is multilinear and alternating, we get

$$f(\omega x_1, \dots, \omega x_m) = (\det \omega) f(x_1, \dots, x_m).$$

As $\det \omega = 1$, we obtain $f(\omega x_1, \dots, \omega x_m) = f(x_1, \dots, x_m)$. Therefore $\omega f(x_1, \dots, x_m) = f(x_1, \dots, x_m)$. It follows that $f(x_1, \dots, x_m)$ is a $GL(m, \mathbb{F}_2)$ -invariant. $\square$

Definition 4.3 For each set of m distinct non-negative numbers $\beta = \{\beta_1, \dots, \beta_m\}$, let Σ_m be the symmetric group on $\{1, \dots, m\}$ and let

$$\omega_\beta(x_1, \dots, x_m) = \sum_{\sigma \in \Sigma_m} x_1^{2^{\beta_{\sigma(1)}}} \dots x_m^{2^{\beta_{\sigma(m)}}}.$$

This sum is invariant under Σ_m .

Lemma 4.4 (i) $\omega_\beta(x_1, \dots, x_m)$ is multilinear and alternating.

(ii) $\omega_\beta(x_1, \dots, x_m)$ is a $GL(m, \mathbb{F}_2)$ -invariant.

Note that, for each $d > 0$, in [4, Definition 2.1], Hưng and Peterson defined $\omega_m(d) = \sum x_1^{i_1} \dots x_m^{i_m}$, where the sum is over all sequence $i_1, \dots, i_m$ with i_j either 0 or a power of 2 and $\sum i_j = d$. If d is the degree of $\omega_\beta(x_1, \dots, x_m)$, let $\alpha(d)$ be the number 1's occurring in the dyadic expansion of d , then $d = 2^{\beta_1} + \dots + 2^{\beta_m}$ is the dyadic expansion of d and $\alpha(d) = m$. Therefore, in this case, $\omega_\beta(x_1, \dots, x_m)$ is actually the element $\omega_m(d)$ in [4, Definition 2.1]. However if $\alpha(d) < m$ then $\omega_m(d)$ maybe a $GL(m, \mathbb{F}_2)$ -invariant but not multilinear and alternating. For example, let $m = 2$ and $d = 2$ we get $\omega_2(d) = Q_{2,1}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ is neither multilinear nor alternating, since $Q_{2,1}(x_1 + x_3, x_2) = Q_{2,1}(x_1, x_2) + x_2x_3 + x_3^2 \neq Q_{2,1}(x_1, x_2) + Q_{2,1}(x_3, x_2)$ and $Q(x_1, x_1) = x_1^2 \neq 0$.

Proof of Lemma 4.4 The part (ii) is a corollary of [4, Theorem 2.2], where $\omega_\beta(x_1, \dots, x_m) = \omega_m(d)$ with $\alpha(d) = k$. On the other hand, the part (i) is also a corollary of the part (i) and Lemma 4.2. So, we only need to show the part (i).

We have $\omega_\beta(x_1, \dots, x_m)$ is symmetric, so it is enough to show that the map $t_1 \mapsto \omega_\beta(t_1, \dots, t_m)$ is linear and $\omega_\beta(t_1, t_2, \dots, t_m) = 0$ when $t_1 = t_2$.

If $t_1 = t_{11} + t_{12}$ then $t_1^{2^{a_1}} = t_{11}^{2^{a_1}} + t_{12}^{2^{a_1}}$. Thus

$$\begin{aligned} \omega_\beta(t_{11} + t_{12}, t_2, \dots, t_m) &= \sum (t_{11} + t_{12})^{2^{a_1}} t_2^{2^{a_2}} \dots t_m^{2^{a_m}} \\ &= \sum t_{11}^{2^{a_1}} t_2^{2^{a_2}} \dots t_m^{2^{a_m}} + \sum t_{12}^{2^{a_1}} t_2^{2^{a_2}} \dots t_m^{2^{a_m}} \\ &= \omega_\beta(t_{11}, t_2, \dots, t_m) + \omega_\beta(t_{12}, t_2, \dots, t_m). \end{aligned}$$

Being lax on notation to avoid cumbersome formulas, we rewrite $\omega_\beta(t_1, t_2, \dots, t_m)$ as

$$\omega_\beta(t_1, t_2, \dots, t_m) = \sum_{a_1 < a_2} (t_1^{2^{a_1}} t_2^{2^{a_2}} + t_1^{2^{a_2}} t_2^{2^{a_1}}) t_3^{2^{a_3}} \dots t_m^{2^{a_m}}.$$

Thus, if $t_1 = t_2$ then $(t_1^{2^{a_1}} t_2^{2^{a_2}} + t_1^{2^{a_2}} t_2^{2^{a_1}}) = 2t_1^{2^{a_1}} t_2^{2^{a_2}} = 0$. So $\omega_\beta(t_1, t_1, \dots, t_m) = 0$. The lemma is proved. $\square$

The following proposition is also numbered as Proposition 1.3 in the introduction.

Proposition 4.5 *The set*

$$\{\omega_\beta(x_1, \dots, x_m) \mid \beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m, \beta_i \neq \beta_j \text{ for all } i \neq j\}$$

forms a basis for the subspace of $\mathbb{F}_2[x_1, \dots, x_m]$ generated by multilinear and alternating invariants.

Proof For each $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$ with $\beta_i \neq \beta_j$ for all $i \neq j$, the sum $2^{\beta_1} + \dots + 2^{\beta_m}$ is the dyadic expansion of $\deg \omega_\beta(x_1, \dots, x_m)$. So, in each degree d , there exists at most an element of the form $\omega_\beta(x_1, \dots, x_m)$ with $d = \sum_{j=1}^m 2^{\beta_j}$. Therefore the set of elements $\omega_\beta(x_1, \dots, x_m)$ is linearly independent.

Suppose that $Q(x_1, \dots, x_m)$ is a multilinear and alternating invariant of degree d in D_m . We prove that $Q(x_1, \dots, x_m) = \omega_\beta(x_1, \dots, x_m)$ for some $\beta = (\beta_1, \dots, \beta_m)$ such that $\beta_i \neq \beta_j$ for all $i \neq j$ and $d = 2^{\beta_1} + \dots + 2^{\beta_m}$. Let $x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$ be an arbitrary monomial of $Q(x_1, \dots, x_m)$. We write $Q(x_1, \dots, x_m)$ as

$$Q(x_1, \dots, x_m) = x_1^{i_1} x_2^{i_2} \dots x_m^{i_m} + \sum_{(j_1, j_2, \dots, j_m) \neq (i_1, i_2, \dots, i_m)} x_1^{j_1} x_2^{j_2} \dots x_m^{j_m}.$$

Suppose that $i_1 = 0$. Since $Q(x_1, \dots, x_m)$ is multilinear, by Definition 4.1, we get $Q(x_1 + x_n, \dots, x_m) = Q(x_1, \dots, x_m) + Q(x_n, \dots, x_m)$. So

$$\begin{aligned} & x_2^{i_2} \dots x_m^{i_m} \\ &= \sum_{(j_1, \dots, j_m) \neq (i_1, \dots, i_m)} \left[(x_1 + x_n)^{j_1} x_2^{j_2} \dots x_m^{j_m} + x_1^{j_1} x_2^{j_2} \dots x_m^{j_m} + x_n^{j_1} x_2^{j_2} \dots x_m^{j_m} \right] \\ &= \sum_{j_1=0} x_2^{j_2} \dots x_m^{j_m} + \sum_{j_1>0} \sum_{j=1}^{j_1-1} \binom{j_1}{j} x_1^j x_2^{j_2} \dots x_m^{j_m} x_n^{j_1-j}. \end{aligned}$$

This is a contradiction as $(j_2, \dots, j_m) \neq (i_2, \dots, i_m)$. Thus $i_1 \neq 0$. Since $Q(x_1 + x_n, \dots, x_m) = Q(x_1, \dots, x_m) + Q(x_n, \dots, x_m)$ again, we get

$$\sum_{i=1}^{i_1-i} \binom{i_1}{i} x_1^i x_2^{i_2} \dots x_m^{i_m} x_n^{i_1-i} = \sum_{j_1=0} x_2^{j_2} \dots x_m^{j_m} + \sum_{j_1>0} \sum_{j=1}^{j_1-1} \binom{j_1}{j} x_1^j x_2^{j_2} \dots x_m^{j_m} x_n^{j_1-j}.$$

If i_1 is not a power of 2, then $\binom{i_1}{i} \neq 0$ for some $1 \leq i \leq i_1$. Therefore, there exists a $(m+1)$ -tuple $(j, j_1, j_2, \dots, j_m)$ with $j_1 > 0$, $1 \leq j \leq j_1 - 1$ and $\binom{j_1}{j} \neq 0$ such that $(i, i_2, \dots, i_m, i_1 - i) = (j, j_2, \dots, j_m, j_1 - j)$. It follows that $(i_1, \dots, i_m) = (j_1, \dots, j_m)$. This is a contradiction. Hence i_1 is a power of 2. Similarly, we also get

$i_2, \dots, i_m$ are powers of 2. We write $i_1 = 2^{\beta_1}, i_2 = 2^{\beta_2}, \dots, i_m = 2^{\beta_m}$ for some non-negative integers $\beta_1, \beta_2, \dots, \beta_m$. Now, we show that $i_s \neq i_t$ for all $s \neq t$. Without loss of generality, suppose to the contrary that $i_1 = i_2$. As $Q(x_1, x_2, \dots, x_m) \in D_m$, it is invariant under the transposition of x_1 and x_2 that keeps the other variables fixed. It follows that $Q(x_1, x_2, \dots, x_m)$ has the form

$$x_1^{i_1} x_2^{i_1} x_3^{i_3} \dots x_m^{i_m} + \sum_{j_1=j_2 \neq i_1} x_1^{j_1} x_2^{j_2} x_3^{j_3} \dots x_m^{j_m} + \sum_{j_1 \neq j_2} (x_1^{j_1} x_2^{j_2} + x_1^{j_2} x_2^{j_1}) x_3^{j_3} \dots x_m^{j_m}.$$

On the other hand $Q(x_1, x_2, \dots, x_m)$ is alternating, by Definition 4.1, we get $Q(x_1, x_1, \dots, x_m) = 0$. So

$$x_1^{2i_1} x_3^{i_3} \dots x_m^{i_m} + \sum_{j_1=j_2} x_1^{j_1+j_2} x_3^{j_3} \dots x_m^{j_m} = 0.$$

Hence, there is a tuple $(j_1, j_2, \dots, j_m)$ such that $j_1 = j_2, j_1 + j_2 = 2i_1, j_k = i_k$ for $k \geq 3$. It implies that $(i_1, i_2, i_3, \dots, i_m) = (j_1, j_2, j_3, \dots, j_m)$. This is a contradiction. Therefore $i_1 \neq i_2$. Similarly, we get $i_s \neq i_t$ for all $s \neq t$.

It has shown that if $x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$ is a monomial of $Q(x_1, x_2, \dots, x_m)$ then $i_j = 2^{\beta_j}$ for $1 \leq j \leq m$ and $\beta_i \neq \beta_j$ for $i \neq j$. As $Q(x_1, x_2, \dots, x_m)$ is invariant under the symmetric group Σ_m , it contains all permutations of $x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$ as terms. Since $\beta_1, \dots, \beta_m$ are distinct, the sum of powers of $x_1, \dots, x_m$ in the monomial $x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$, $2^{\beta_1} + 2^{\beta_2} + \dots + 2^{\beta_m}$, is the dyadic expansion of $\deg Q(x_1, x_2, \dots, x_m)$. Therefore $Q(x_1, x_2, \dots, x_m)$ only contains $x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$ and its permutations as terms. Let $\beta = (\beta_1, \beta_2, \dots, \beta_m)$, we get

$$Q(x_1, x_2, \dots, x_m) = \omega_\beta(x_1, x_2, \dots, x_m).$$

So, the proposition is proved. $\square$

As we had known that $\omega_\beta(x_1, \dots, x_m)$ is $\omega_m(d)$ with $\alpha(d) = m$ in [4, Definition 2.1], where d is the degree of $\omega_\beta(x_1, \dots, x_m)$, the following is a corollary of [4, Corollary 2.7].

Corollary 4.6 Suppose that $\beta = \{\beta_1, \dots, \beta_m\}$ with $\beta_1 < \beta_2 < \dots < \beta_m$. Then $(Q_{m,0})^{2^{\beta_1}} \mid \omega_\beta(x_1, \dots, x_m)$.

From Proposition 4.5 and Corollary 4.6, we get the following result.

Corollary 4.7 If $Q \in D_m$ is multilinear and alternating, then $Q_{m,0}$ divides Q .

Example 4.8 (i) $Q_{m,0} = \omega_\beta(x_1, \dots, x_m)$ with $\beta = (0, 1, \dots, m-1)$.

(ii) For each $0 \leq i \leq m-1$, $Q_{m,0} Q_{m,i} = \omega_\beta(x_1, \dots, x_m)$ with $\beta = (0, 2, \dots, i-1, i+1, \dots, m)$.

Proof From [4, Theorem 2.2], we have $Q_{m,0} = x_1^{2^{m-1}} x_2^{2^{m-2}} \dots x_{m-1}^2 x_m + (\text{symmetried})$. Then, we obtain (i).

We have

$$\begin{aligned} Q_{m+1,0} &= Q_{m,0} V_{m+1} \\ &= Q_{m,0} \sum_{i=0}^{m-1} Q_{m,i} x_{m+1}^2 \quad (\text{by [5, Proposition 2.6]}). \end{aligned}$$

So, $Q_{m,0} Q_{m,i}$ is the coefficient of $x_{m+1}^{2^i}$ in $Q_{m+1,0}$. On the other hand, from [4, Theorem 2.2] again, we get

$$Q_{m+1,0} = x_1^{2^m} x_2^{2^{m-1}} \dots x_m^2 x_{m+1} + (\text{symmetried}).$$

So the coefficient of $x_{m+1}^{2^i}$ in $Q_{m+1,0}$ is $\sum_{(i_1, \dots, i_m)} x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$, where the sum is over all permutations of $(1, \dots, 2^{i-1}, 2^{i+1}, \dots, 2^m)$. As a consequence, we get

$$Q_{m,0} Q_{m,i} = \sum_{(i_1, \dots, i_m)} x_1^{i_1} x_2^{i_2} \dots x_m^{i_m} = \omega_\beta(x_1, \dots, x_m)$$

with $\beta = (0, \dots, 2^{i-1}, 2^{i+1}, \dots, 2^m)$. Thus, (ii) is proved. $\square$

5 Transfer on powers of multilinear and alternating invariants

In this section, we first study the transfer $tr_{m,n}$ in which $n = m + 1$. Recall that $tr_{\mathbb{U}, \mathbb{V}}: H^*(\mathbb{U})^{GL(\mathbb{U})} \rightarrow H^*(\mathbb{V})^{GL(\mathbb{V})}$, or $tr_{m,m+1}: D_m \rightarrow D_{m+1}$, is given by

$$tr_{m,m+1}(Q) = \sum_{W \leq \mathbb{V}^*, \dim W = m} Q(W).$$

Let $x_1, \dots, x_{m+1}$ be a basis of $\mathbb{V}^*$. For each $1 \leq j \leq m+1$, we denote by X_j the subspace of $\mathbb{V}^*$ generated by $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{m+1}$. Then, we have $\dim X_j = m$.

Lemma 5.1 *There is an 1–1 correspondence between the set of all m -dimensional vector spaces W of $\mathbb{V}^*$ and the set of $(m+1)$ -tuple of scalars $(c_1, \dots, c_{m+1})$ in which not all c_i are zero such that $Q(W) = c_1 Q(X_1) + \dots + c_{m+1} Q(X_{m+1})$ for all multilinear and alternating invariants Q of D_m .*

Proof Suppose that W is an m -dimensional vector space of $\mathbb{V}^*$ and $y_1, \dots, y_m$ is a basis of W . We have $(y_1 \dots y_m) = (x_1 \dots x_{m+1})C$ where C is a $(m+1) \times m$ -matrix and $\text{rank } C = m$. So $y_j = \sum_{i=1}^{m+1} c_{ij} x_i$. Setting $c_j = \sum_{i_\ell \neq j \forall \ell, i_k \neq i_\ell} c_{i_1,1} \dots c_{i_m,m}$ for $1 \leq j \leq m+1$, for arbitrary multilinear and alternating Q of D_m , we get

$$Q(W) = Q(y_1, \dots, y_m) = c_1 Q(X_1) + \dots + c_{m+1} Q(X_{m+1}).$$

Conversely, we show by induction on m that for each $(m+1)$ -tuple of scalars $(c_1, \dots, c_{m+1})$ in which not all c_i are zero, there exists exactly one m -dimensional

subspace W of $\mathbb{V}^*$ satisfying $Q(W) = c_1 Q(X_1) + \cdots + c_{m+1} Q(X_{m+1})$ for all multilinear alternating invariants Q of D_m .

For $m = 1$, let $W = \mathbb{F}_2 \langle c_1 x_2 + c_2 x_1 \rangle$ we have $\dim W = 1$ as $(c_1, c_2) \neq (0, 0)$. For a multilinear alternating invariants Q of D_1 , we get $Q(W) = Q(c_1 x_2 + c_2 x_1) = c_1 Q(x_2) + c_2 Q(x_1) = c_1 Q(X_1) + c_2 Q(X_2)$ as $X_1 = \mathbb{F}_2 \langle x_2 \rangle$, $X_2 = \mathbb{F}_2 \langle x_1 \rangle$.

Let us consider the following two cases.

Case 1. There is index i such that $c_i = 0$. Setting $S = \{x_1, \dots, x_{m+1}\}$ and $\mathbb{V}^{(i)} = \mathbb{F}_2 \langle S \setminus \{x_i\} \rangle$. Then $\mathbb{V}^{(m+1)} = \mathbb{V}^*$ and $\dim \mathbb{V}^{(i)} = m$. For each $j \neq i$, we denote by $X_j^{(i)}$ the subspace of $\mathbb{V}^{(i)}$ generated by $S \setminus \{x_i, x_j\}$. We have $\dim X_j^{(i)} = m - 1$. Applying the inductive hypothesis to $\mathbb{V}^{(i)}$, since not all c_j are zero for $1 \leq j \leq m + 1$, $j \neq i$, there is one $(m - 1)$ -dimensional subspace $W^{(i)}$ of $\mathbb{V}^{(i)}$ such that

$$Q_{m-1}(W^{(i)}) = \sum_{1 \leq j \leq m+1, j \neq i} c_j Q_{m-1}(X_j^{(i)})$$

for all multilinear alternating invariants Q_{m-1} of $\mathbb{F}_2[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+1}]^{GL(m, \mathbb{F}_2)}$. Let W be the subspace generated by $W^{(i)} \cup \{x_i\}$. Since $W^{(i)} \leq \mathbb{F}_2[S \setminus \{x_i\}]$, we get $W = W^{(i)} \oplus \mathbb{F}_2 \langle x_i \rangle$ and W is an m -dimensional subspace of $\mathbb{V}^*$. Suppose that $(\alpha_1, \dots, \alpha_{m-1})$ is a basis for $W^{(i)}$. As $W^{(i)} \leq \mathbb{F}_2 \langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+1} \rangle$, each α_j can be written as

$$\alpha_j = \sum_{1 \leq k \leq m+1, k \neq i} c_{kj} x_k.$$

Then

$$\begin{aligned} Q_{m-1}(W^{(i)}) &= Q_{m-1}(\alpha_1, \dots, \alpha_{m-1}) \\ &= \sum_{j=1}^{m+1} \left(\sum_{\sigma \in \Sigma_{\{1, \dots, m+1\} \setminus \{i, j\}}} c_{\sigma(1), 1} \cdots c_{\sigma(m+1), m-1} \right) Q_{m-1}(X_j^{(i)}). \end{aligned}$$

From the uniqueness of the subspace $W^{(i)}$ by the inductive hypothesis, we get

$$c_j = \sum_{\sigma \in \Sigma_{\{1, \dots, m+1\} \setminus \{i, j\}}} c_{\sigma(1), 1} \cdots c_{\sigma(m+1), m-1} \quad \text{for } 1 \leq j \leq m + 1.$$

Now, for an arbitrary multilinear and alternating invariant Q_m of D_m we have

$$\begin{aligned} Q_m(W) &= Q_m(\alpha_1, \dots, \alpha_{m-1}, x_i) \\ &= Q_m \left(\sum_{k \neq i} c_{k1} x_k, \dots, \sum_{k \neq i} c_{k(m-1)} x_k, x_i \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{m+1} \left(\sum_{\sigma \in \Sigma_{\{1, \dots, m+1\} \setminus \{i, j\}}} c_{\sigma(1), 1} \dots c_{\sigma(m+1), m-1} \right) Q_m(X_j^{(i)}, x_i) \\
&= \sum_{j=1}^{m+1} c_j Q_m(X_j^{(i)}, x_i) = \sum_{j=1}^{m+1} c_j Q_m(X_j)
\end{aligned}$$

as $Q_m(X_j^{(i)}, x_i) = Q_m(X_j)$. So we get the conclusion.

Case 2. $c_i = 1$ for all $1 \leq i \leq m+1$. By the similar argumentation as in Case 1 for $(c_1, \dots, c_m)$, there is one $(m-1)$ -dimensional subspace $W^{(m+1)}$ of $\mathbb{F}_2\langle x_1, \dots, x_m \rangle$ such that

$$Q_{m-1}(W^{(m+1)}) = \sum_{1 \leq j \leq m} Q_{m-1}(X_j^{(m+1)})$$

for all multilinear alternating invariants Q_{m-1} of D_{m-1} . We see that $x_1 \notin W^{(m+1)}$. Indeed, if $x_1 \in W^{(m+1)}$, let $(\alpha_1, \alpha_2, \dots, \alpha_{m-1})$ be a basis for $W^{(m+1)}$ such that $\alpha_1 = x_1$ and $\alpha_2, \dots, \alpha_{m-1} \in \mathbb{F}_2 \langle x_2, \dots, x_m \rangle$. Then, the coefficient of $Q_{m-1}(X_1^{(m+1)})$ in the expansion of $Q_{m-1}(W^{(m+1)}) = Q(x_1, \alpha_2, \dots, \alpha_{m-1})$ is equal to 0. This contradicts to $c_1 = 1$. So $x_1 \notin W^{(m+1)}$. Similarly, we can prove that $x_i \notin W^{(m+1)}$ for all $1 \leq i \leq m$.

Now let W be the subspace of $\mathbb{V}^*$ generated by $W^{(m+1)}$ and $x_1 + x_{m+1}$. Since $x_{m+1} \notin W^{(m+1)}$ it follows that $\dim W = m$. Suppose that $(\alpha_1, \dots, \alpha_{m-1})$ is a basis of $W^{(m+1)}$ and $\alpha_j = \sum_{k=1}^m c_{kj} x_k$. By the similar argumentation as in Case 1, we get

$$c_j = \sum_{\sigma \in \Sigma_{\{1, \dots, m\} \setminus \{j\}}} c_{\sigma(1), 1} \dots c_{\sigma(m), m-1} \quad \text{for } 1 \leq j \leq m.$$

Hence, for any multilinear and alternating invariant Q_m of D_m , we obtain

$$\begin{aligned}
Q(W) &= Q(\alpha_1, \dots, \alpha_{m-1}, x_1 + x_{m+1}) \\
&= Q(\alpha_1, \dots, \alpha_{m-1}, x_1) + Q(\alpha_1, \dots, \alpha_{m-1}, x_{m+1}) \\
&= \sum_{j=1}^m c_j Q(X_j^{(m+1)}, x_1) + \sum_{j=1}^m c_j Q(X_j^{(m+1)}, x_{m+1}) \\
&= c_1 Q(X_1^{(m+1)}, x_1) + \sum_{j=1}^m c_j Q(X_j^{(m+1)}, x_{m+1}) \\
&\quad \left[\text{as } Q(X_j^{(m+1)}, x_1) = 0 \quad \text{for all } j \neq 1 \right]
\end{aligned}$$

$$\begin{aligned}
&= c_1 Q(X_{m+1}) + \sum_{j=1}^m c_j Q(X_j) \quad \left[\text{as } Q(X_1^{(m+1)}, x_1) = Q(X_{m+1}) \right] \\
&= Q(X_1) + \cdots + Q(X_{m+1}) \quad (\text{as } c_j = 1 \text{ for all } j).
\end{aligned}$$

The lemma follows. $\square$

The following theorem is also numbered as Theorem 1.4 in the introduction.

Theorem 5.2 Suppose that Q is a multilinear and alternating invariant in D_m . Let $y_j = Q(X_j)$ for $j = 1, \dots, m+1$. Then, for $r > 0$ we have

$$tr_{m,m+1}(Q^r) = \sum \frac{(s_0 + \cdots + s_m - 1)!}{(s_0 - 1)! s_1! \cdots s_m!} Q_{m+1,0}^{s_0} \cdots Q_{m+1,m}^{s_m}(y_1, \dots, y_{m+1})$$

summed over all sequences of non-negative integers $(s_0, \dots, s_m)$ such that

$$\sum_{i=0}^m (2^{m+1} - 2^i) s_i = r.$$

Proof By the definition of $tr_{m,m+1}$, we have

$$tr_{m,m+1}(Q^r) = \sum_{W \leq \mathbb{V}^*, \dim W = m} Q^r(W).$$

Following Lemma 5.1 and the definition of $tr_{1,m+1}$, it follows that

$$\begin{aligned}
tr_{m,m+1}(Q^r) &= \sum_{c_1, \dots, c_{m+1} \in \mathbb{F}_2} (c_1 Q(X_1) + \cdots + c_{m+1} Q(X_{m+1}))^r \\
&= \sum_{c_1, \dots, c_{m+1} \in \mathbb{F}_2} (c_1 y_1 + \cdots + c_{m+1} y_{m+1})^r \\
&= tr_{1,m+1} y_1^r.
\end{aligned}$$

Now, applying Theorem 3.1, we get the theorem. $\square$

The following is numbered as Corollary 1.5 in the introduction.

Corollary 5.3 (i) $tr_{m,m+1}(Q_{m,0}^j) = 0$ for $0 < j < 2^{m+1} - 1$.

(ii) $tr_{m,m+1}(Q_{m,0}^{2^{m+1}-1}) = Q_{m+1,0}^{2^m-1}$.

Proof By Example 4.8(i), $Q_{m,0}$ is multilinear and alternating. So, we apply Theorem 5.2 to compute $tr_{m,m+1}(Q_{m,0}^r)$.

(i) We consider the equation $\sum_{i=0}^m (2^{m+1} - 2^i) s_i = r$.

If r is odd then $s_0 \geq 1$. We have $\sum_{i=0}^m (2^{m+1} - 2^i) s_i \geq 2^{m+1} - 1$. But $0 < r < 2^{m+1} - 1$. It follows that the equation has no solution. Therefore

$tr_{m,m+1}(Q_{m,0}^r) = 0$. If r is even, we write $r = 2^k \cdot \ell$ where $k, \ell \in \mathbb{N}$ and ℓ odd.

We have $tr_{m,m+1}(Q_{m,0}^r) = (tr_{m,m+1}(Q_{m,0}^\ell))^{2^k}$. Since $0 < r < 2^{m+1} - 1$ we get $0 < \ell < 2^{m+1} - 1$. So, applying the above case, we obtain $tr_{m,m+1}(Q_{m,0}^\ell) = 0$. Hence $tr_{m,m+1}(Q_{m,0}^r) = 0$. Part (i) of the corollary is proved.

- (ii) We have the equation $\sum_{i=0}^m (2^{m+1} - 2^i) s_i = 2^{m+1} - 1$ has exactly one solution, namely $s_0 = 1, s_j = 0$ for $1 \leq j \leq m$. By Theorem 5.2, we get

$$tr_{m,m+1} \left(Q_{m,0}^{2^{m+1}-1} \right) = Q_{m+1,0}(y_1, \dots, y_{m+1}),$$

where $y_j = Q(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{m+1})$ for $1 \leq j \leq m+1$. Furthermore, from [4, Theorem 2.2] we have

$$Q_{m+1,0}(y_1, \dots, y_{m+1}) = y_1^{2^m} y_2^{2^{m-1}} \dots y_m^2 y_{m+1} + (\text{symmetried}).$$

We will use Lemma 6.2 to find invariant monomials of $Q_{m+1,0}(y_1, \dots, y_{m+1})$. First, we show by induction on m that the maximal monomial in variables $x_1, \dots, x_{m+1}$ of $Q_{m+1,0}(y_1, \dots, y_{m+1})$, by Definition 6.1, is

$$x_1^{2^m-1} x_2^{2(2^m-1)} \dots x_m^{2^{m-1}(2^m-1)} x_{m+1}^{2^m(2^m-1)}.$$

For $m = 1$, we have $y_1 = Q_{1,0}(x_2) = x_2, y_2 = Q_{1,0}(x_1) = x_2$. So $Q_{2,0}(y_1, y_2) = y_1^2 y_2 + y_1 y_2^2 = x_1 x_2^2 + x_1^2 x_2$. Hence, the maximal monomial of $Q_{2,0}(y_1, y_2)$ is $x_1 x_2^2$.

Now we consider $Q_{m+1,0}(y_1, \dots, y_{m+1})$. From the inductive definitions of the Dickson invariants and the Mui invariants in [2, 5], for $i = 1, \dots, m$, we have

$$\begin{aligned} y_i &= Q_{m,0}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+1}) \\ &= Q_{m-1,0}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) V_m(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+1}) \\ &= \sum_{j=1}^{m-1} (Q_{m-1,0} Q_{m-1,j})(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) x_{m+1}^{2^j} \\ &\quad (\text{applying [5, Proposition 2.6]}). \end{aligned}$$

Thus, the maximal monomial of each y_i is a term of the polynomial

$$Q_{m-1,0}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) x_{m+1}^{2^{m-1}}.$$

It should be noted that $y_{m+1} = Q_{m,0}(x_1, \dots, x_m)$; so, the power of x_{m+1} in y_{m+1} equals to zero. For $1 \leq i \leq m$, denoting

$$y'_i = Q_{m-1,0}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m).$$

Then, we get

$$\begin{aligned} Q_{m+1,0}(y_1, \dots, y_{m+1}) &= Q_{m,0}^2(y'_1, \dots, y'_m) y_{m+1} x_{m+1}^{2^{m-1}(2^m+2^{m-1}+\dots+2)} + (\text{smaller terms}) \\ &= Q_{m,0}^2(y'_1, \dots, y'_m) x_1 x_2^2 \dots x_m^{2^{m-1}} x_{m+1}^{2^m(2^m-1)} + (\text{smaller terms}), \end{aligned}$$

as $y_{m+1} = x_1 x_2^2 \dots x_m^{2^{m-1}} + (\text{smaller terms})$. On the other hand, by the inductive hypothesis, we have

$$Q_{m,0}(y'_1, \dots, y'_m) = x_1^{2^{m-1}-1} x_2^{2(2^{m-1}-1)} \dots x_m^{2^{m-1}(2^{m-1}-1)} + (\text{smaller terms}).$$

Therefore

$$Q_{m+1,0}(y_1, \dots, y_{m+1}) = x_1^{2^m-1} x_2^{2(2^m-1)} \dots x_m^{2^{m-1}(2^m-1)} x_{m+1}^{2^m(2^m-1)} + (\text{smaller terms}).$$

So, the maximal monomial of $Q_{m+1,0}(y_1, \dots, y_{m+1})$ is

$$x_1^{2^m-1} x_2^{2(2^m-1)} \dots x_m^{2^{m-1}(2^m-1)} x_{m+1}^{2^m(2^m-1)}.$$

Hence, by Lemma 6.2,

$$Q_{m+1,0}(y_1, \dots, y_{m+1}) = Q_{m+1,0}^{2^m-1} + (\text{others}).$$

We see that if $Q_{m+1,0}^{i_0} Q_{m+1,1}^{i_1} \dots Q_{m+1,m}^{i_m}$ is a term of $Q_{m+1,0}(y_1, \dots, y_{m+1})$ then

$$i_0(2^{m+1}-1) + i_1(2^{m+1}-2) + \dots + i_m(2^{m+1}-2^m) = (2^{m+1}-1)(2^m-1).$$

Since $2^i \geq 1$ for $i = 0, \dots, m$, we get

$$(2^{m+1}-1)(2^m-1) \leq (2^{m+1}-1)(i_0 + i_1 + \dots + i_m).$$

So

$$2^m - 1 \leq (i_0 + i_1 + \dots + i_m). \quad (2)$$

On the other hand, for $j = 1, \dots, m$ we have

$$Q_{m+1,j} = Q_{m,j-1}^2 + \sum_{s=0}^m (Q_{m,j} Q_{m,s})(x_1, \dots, x_m) x_{m+1}^{2^s}.$$

So

$$Q_{m+1,0}^{i_0} Q_{m+1,1}^{i_1} \dots Q_{m+1,m}^{i_m} = Q_{m,0}^{i_0} \dots Q_{m,m}^{i_m} x_{m+1}^{2^m(\sum_{j=0}^m i_j)} + (\text{smaller terms}).$$

From the above conclusion, the powers of x_{m+1} in the maximal monomial of $Q_{m+1,0}(y_1, \dots, y_{m+1})$ is $2^m(2^m - 1)$. It follows that $2^m(\sum_{j=0}^m i_j) \leq 2^m(2^m - 1)$. Or equivalently $\sum_{j=0}^m i_j \leq 2^m - 1$. Combining this with (2), we get $\sum_{j=0}^m i_j = 2^m - 1$. It follows that $i_1 = \dots = i_m = 0$ and $i_0 = 2^m - 1$. Then $Q_{m+1,0}^{i_0} Q_{m+1,1}^{i_1} \dots Q_{m+1,m}^{i_m} = Q_{m+1,0}^{2^m-1}$. As a consequence

$$Q_{m+1,0}(y_1, \dots, y_{m+1}) = Q_{m+1,0}^{2^m-1}.$$

So, we get the corollary. $\square$

The following is numbered as Corollary 1.6 in the introduction.

Corollary 5.4 For $r > 0$, $tr_{2,3}(Q_{2,0}^r Q_{2,1}^r)$

$$= \sum_{7s_0+6s_1+4s_2=r} \frac{(s_0 + s_1 + s_2 - 1)!}{(s_0 - 1)!s_1!s_2!} \left(Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 + Q_{3,0}^5 \right)^{s_0} \left(Q_{3,0}^2 Q_{3,2}^4 \right)^{s_1} \\ \times \left(Q_{3,1}^2 Q_{3,2}^2 + Q_{3,0}^2 Q_{3,1} \right)^{s_2}.$$

Proof Setting

$$y_1 = Q_{2,0} Q_{2,1}(x_2, x_3), y_2 = Q_{2,0} Q_{2,1}(x_1, x_3), y_3 = Q_{2,0} Q_{2,1}(x_1, x_2).$$

Since $Q_{2,0} Q_{2,1}$ is multilinear and alternating by Example 4.8, applying Theorem 5.2 we get

$$tr_{2,3}(Q_{2,0}^r Q_{2,1}^r) = \sum_{7s_0+6s_1+4s_2=r} \frac{(s_0 + s_1 + s_2 - 1)!}{(s_0 - 1)!s_1!s_2!} \left(Q_{3,0}^{s_0} Q_{3,1}^{s_1} Q_{3,2}^{s_2} \right) (y_1, y_2, y_3).$$

Now, applying Lemma 5.5, we get the corollary. $\square$

The following technical lemma will be proved in the next section.

Lemma 5.5 Let $y_1 = Q_{2,0} Q_{2,1}(x_2, x_3)$, $y_2 = Q_{2,0} Q_{2,1}(x_1, x_3)$, $y_3 = Q_{2,0} Q_{2,1}(x_1, x_2)$. Then

$$\begin{aligned} Q_{3,0}(y_1, y_2, y_3) &= Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 + Q_{3,0}^5, \\ Q_{3,1}(y_1, y_2, y_3) &= Q_{3,0}^2 Q_{3,2}^4, \\ Q_{3,2}(y_1, y_2, y_3) &= Q_{3,1}^2 Q_{3,2}^2 + Q_{3,0}^2 Q_{3,1}. \end{aligned}$$

Next, we study the image of transfer on multilinear and alternating invariants in general. The following is also numbered as Proposition 1.7(i) in the introduction.

Proposition 5.6 If $Q \in D_m$ is multilinear and alternating then $tr_{m,n}(Q) = 0$.

Proof From Proposition 4.5, it is sufficient to prove $tr_{m,n}\omega_\beta = 0$ with $\beta = (s_0, s_1, \dots, s_{m-1})$ such that $s_i \neq s_j$ for all $i \neq j$.

By the definition of $tr_{m,n}$ we have

$$tr_{m,n}\omega_\beta = \sum_{W \leq \mathbb{V}^*, \dim W=m} \omega_\beta(W).$$

Since ω_β is multilinear and alternating, for each m -dimensional subspace W of $\mathbb{V}^*$, there exist scalars $c_{i_1, \dots, i_m}$ such that

$$\omega_\beta(W) = \sum_{1 \leq i_1 < \dots < i_m \leq n} c_{i_1, \dots, i_m} \omega_\beta(x_{i_1}, \dots, x_{i_m}).$$

So

$$tr_{m,n}\omega_\beta = \sum_{1 \leq i_1 < \dots < i_m \leq n} d_{i_1, \dots, i_m} \omega_\beta(x_{i_1}, \dots, x_{i_m})$$

for some scalars $d_{i_1, \dots, i_m} \in \mathbb{F}_2$. From the definition of ω_β given in Sect. 4, for each sequence $(i_1, \dots, i_m)$ we have

$$\omega_\beta(x_{i_1}, \dots, x_{i_m}) = x_{i_1}^{2^{s_1}} \dots x_{i_m}^{2^{s_m}} + (\text{symmetried}).$$

We see that the set $\{x_{i_1}^{2^{s_1}} \dots x_{i_m}^{2^{s_m}} \mid 1 \leq i_1 < \dots < i_m \leq n\}$ is linearly independent in the vector space of polynomials in variables $x_1, \dots, x_n$. So, the set $\{\omega_\beta(x_{i_1}, \dots, x_{i_m}) \mid 1 \leq i_1 < \dots < i_m \leq n\}$ is also linearly independent in this vector space. Furthermore, each element of this set is obtained by another one under a permutation on $\{1, \dots, n\}$. On the other hand, as $tr_{m,n}\omega_\beta \in D_n$, $tr_{m,n}\omega_\beta$ is invariant under all permutations on $\{1, \dots, n\}$. It implies that all coefficients $d_{i_1, \dots, i_m}$ are equal. We denote these coefficients by d for short. Then, we get

$$tr_{m,n}\omega_\beta = d \left(\sum_{1 \leq i_1 < \dots < i_m \leq n} \omega_\beta(x_{i_1}, \dots, x_{i_m}) \right).$$

Let σ be the linear transformation which sends x_1 to $x_1 + x_2$ and keeps the other variables fixed. As ω_β is multilinear and alternating, we obtain

$$\sigma \omega_\beta(x_1, x_2, x_{i_3}, \dots, x_{i_m}) = \omega_\beta(x_1 + x_2, x_2, x_{i_3}, \dots, x_{i_m}) = \omega_\beta(x_1, x_2, x_{i_3}, \dots, x_{i_m})$$

for all sequences $(i_3, \dots, i_m)$ such that $3 \leq i_3 < \dots < i_m \leq n$;

$$\begin{aligned} \sigma \omega_\beta(x_1, x_{i_2}, \dots, x_{i_m}) &= \omega_\beta(x_1 + x_2, x_{i_2}, \dots, x_{i_m}) \\ &= \omega_\beta(x_1, x_{i_2}, \dots, x_{i_m}) + \omega_\beta(x_2, x_{i_2}, \dots, x_{i_m}) \end{aligned}$$

for all sequences $(i_2, \dots, i_m)$ such that $3 \leq i_2 < \dots < i_m \leq n$; and,

$$\sigma \omega_\beta(x_{i_1}, \dots, x_{i_m}) = \omega_\beta(x_{i_1}, \dots, x_{i_m})$$

for all sequences $(i_1, \dots, i_m)$ such that $1 \leq i_1 < \dots < i_m \leq n$. Therefore

$$\sigma tr_{m,n} \omega_\beta = tr_{m,n} \omega_\beta + d \sum_{3 \leq i_2 < \dots < i_m \leq n} \omega_\beta(x_2, x_{i_2}, \dots, x_{i_m}).$$

On the other hand $\sigma tr_{m,n} \omega_\beta = tr_{m,n} \omega_\beta$, it implies that

$$d \left(\sum_{3 \leq i_2 < \dots < i_m \leq n} \omega_\beta(x_2, x_{i_2}, \dots, x_{i_m}) \right) = 0.$$

However $\sum_{3 \leq i_2 < \dots < i_m \leq n} \omega_\beta(x_2, x_{i_2}, \dots, x_{i_m}) \neq 0$ as $n \geq m + 1$. So, we get $d = 0$. As a consequence $tr_{m,n} \omega_\beta = 0$. Thus, the proposition is proved. $\square$

6 Proof of Lemma 5.5

In this section, we prove Lemma 5.5. However, this method can be used for finding monomials in the image of $tr_{m,n}$ on a given invariant of D_m , the proof of Corollary 5.3(ii) for an example.

Recall that $x_1, \dots, x_n$ is a basis of $\mathbb{V}^*$ and for an invariant $Q \in D_m$,

$$tr_{m,n}(Q) = \sum_{W \leq \mathbb{V}^*, \dim W = m} Q(W).$$

Definition 6.1 In $\mathbb{F}_2[x_1, \dots, x_n]$, we say that $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} > x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$ if $(i_1, i_2, \dots, i_n) > (j_1, j_2, \dots, j_n)$. Here the order on the set of sequences of non-negative integers is the colexicographical one.

For $I = (i_0, \dots, i_{s-1})$ and W be an m -dimensional subspace of $\mathbb{V}^*$, we denote by $Q^I(W)$ the polynomial $(Q_{m,0}^{i_0} \dots Q_{m,m-1}^{i_{m-1}})(W)$. It should be noted that $Q^I(W)$ is a polynomial in $\mathbb{F}_2[x_1, \dots, x_n]$. Denote by $x^I(W)$ the maximal monomial of $Q^I(W)$ (with respect to the order defined in Definition 6.1).

Lemma 6.2 $x^I(\mathbb{V}^*) = x_1^{i_0} x_2^{2(i_0+i_1)} \dots x_{n-1}^{2^{n-2}(i_0+i_1+\dots+i_{n-2})} x_n^{2^{n-1}(i_0+i_1+\dots+i_{n-1})}$ and $x^I(\mathbb{V}^*) = x^J(\mathbb{V}^*)$ if and only if $Q^I(\mathbb{V}^*) = Q^J(\mathbb{V}^*)$.

Proof The lemma is proved by induction on n . We use $x^{I_n}(x_1, \dots, x_n)$ to stand for $x^I(\mathbb{V}^*)$.

The conclusion is trivial for $n = 1$, as $Q^I(\mathbb{V}^*) = Q_{1,0}^{i_0} = x_1^{i_0}$. Suppose inductively that the lemma holds for $n - 1$. Then we have

$$x^{I_{n-1}}(x_1, \dots, x_{n-1}) = x_1^{i_0} x_2^{2 \cdot (i_0+i_1)} \dots x_{n-1}^{2^{n-2}(i_0+i_1+\dots+i_{n-2})}.$$

Using the inductive formula $Q_{n,0} = Q_{n-1,0}V_n$, $Q_{n,i} = Q_{n-1,i-1}^2 + Q_{n-1,i}V_n$ ($1 \leq i \leq n-1$) for $i \leq 1 \leq n-1$ and $V_n(x_1, \dots, x_{n-1}, x_n) = \sum_{j=0}^{n-1} Q_{n-1,j}x_n^{2^j}$, we get

$$Q_{n,0} = \sum_{j=0}^{n-1} (Q_{n-1,0}Q_{n-1,j})x_n^{2^j}.$$

$$Q_{n-1,i} = Q_{n-1,i-1}^2 + \sum_{j=0}^{n-1} (Q_{n-1,i}Q_{n-1,j})x_n^{2^j} \quad (1 \leq i \leq n-1).$$

Therefore

$$\begin{aligned} Q^I(\mathbb{V}^*) &= \left(Q_{n-1,0}^{i_0} \cdots Q_{n-1,n-2}^{i_{n-2}} \right) x_n^{2^{n-1}(i_0+i_1+\dots+i_{n-1})} \\ &\quad + (\text{other monomials in which the powers of } x_n \text{ are smaller}) \\ &= x^{I_{n-1}}(x_1, \dots, x_{n-1}) x_n^{2^{n-1}(i_0+i_1+\dots+i_{n-1})} + (\text{smaller monomials}) \\ &= x_1^{i_0} x_2^{2(i_0+i_1)} \cdots x_{n-1}^{2^{n-2}(i_0+i_1+\dots+i_{n-2})} x_n^{2^{n-1}(i_0+i_1+\dots+i_{n-1})} \\ &\quad + (\text{smaller monomials}), \end{aligned}$$

as $x^{I_{n-1}}(x_1, \dots, x_{n-1}) = x_1^{i_0} x_2^{2(i_0+i_1)} \cdots x_{n-1}^{2^{n-2}(i_0+i_1+\dots+i_{n-2})}$ by the inductive hypothesis. So

$$x^{I_n}(x_1, \dots, x_n) = x_1^{i_0} x_2^{2(i_0+i_1)} \cdots x_{n-1}^{2^{n-2}(i_0+i_1+\dots+i_{n-2})} x_n^{2^{n-1}(i_0+i_1+\dots+i_{n-1})}.$$

Then we get the first part of the lemma.

Suppose that $I = (i_0, \dots, i_n)$ and $J = (j_0, \dots, j_n)$. We see that $x^I(\mathbb{V}^*) = x^J(\mathbb{V}^*)$ if and only if

$$\begin{cases} i_0 = j_0 \\ 2(i_0 + i_1) = 2(j_0 + j_1) \\ \dots \\ 2^{n-1}(i_0 + i_1 + \dots + i_{n-1}) = 2^{n-1}(j_0 + j_1 + \dots + j_{n-1}) \end{cases} \iff \begin{cases} i_0 = j_0 \\ i_1 = j_1 \\ \dots \\ i_n = j_n. \end{cases}$$

So, we obtain the remaining part of the lemma. $\square$

Proof of Lemma 5.5 By direct computation the Dickson invariants $Q_{2,0}, Q_{2,1}$ in terms of variables, we get

$$\begin{aligned} y_1 &= Q_{2,0}Q_{2,1}(x_2, x_3) = x_2x_3^4 + x_2^4x_3, \\ y_2 &= Q_{2,0}Q_{2,1}(x_1, x_3) = x_1x_3^4 + x_1^4x_3, \\ y_3 &= Q_{2,0}Q_{2,1}(x_1, x_2) = x_1x_2^4 + x_1^4x_2. \end{aligned}$$

By the definition of $Q_{3,0}$, we get

$$Q_{3,0}(y_1, y_2, y_3) = y_1^4 y_2^2 y_3 + (\text{symmetried terms in } y_1, y_2, y_3).$$

And, by the definition of y_1, y_2, y_3 , we have

$$\begin{aligned} y_1^4 y_2^2 y_3 &= (x_1^3 x_2^8 + x_1^6 x_2^5) x_3^{24} + (x_1^9 x_2^8 + x_1^{12} x_2^5) x_3^{18} + (x_1^3 x_2^{20} + x_1^6 x_2^{17}) x_3^{12} \\ &\quad + (x_1^9 x_2^{10} + x_1^{12} x_2^{17}) x_3^6. \end{aligned}$$

We see that if σ is a permutation on the set $\{1, 2, 3\}$, then $x_1^a x_2^b x_3^c$ is a term of $y_1^4 y_2^2 y_3$ iff $x_{\sigma(1)}^a x_{\sigma(2)}^b x_{\sigma(3)}^c$ is a term of the symmetried one $y_{\sigma(1)} y_{\sigma(2)}^2 y_{\sigma(3)}^4$. So, as $x_1^3 x_2^8 x_3^{24}$ is the maximal monomial of $y_1 y_2^2 y_3^4$, it is also the maximal monomial of $Q_{3,0}(y_1, y_2, y_3)$. Therefore, by Lemma 6.2, $Q_{3,0}(y_1, y_2, y_3)$ contains $Q_{3,0}^{i_0} Q_{3,1}^{i_1} Q_{3,2}^{i_2}$ as a term in which

$$\begin{cases} 4(i_0 + i_1 + i_2) = 24 \\ 2(i_0 + i_1) = 8 \\ i_0 = 3 \end{cases} \iff \begin{cases} i_0 = 3 \\ i_1 = 1 \\ i_2 = 2. \end{cases}$$

So

$$Q_{3,0}(y_1, y_2, y_3) = Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 + (\text{other terms}).$$

We get

$$\begin{aligned} Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 &= (Q_{3,0} Q_{3,1})(Q_{3,0} Q_{3,2})^2 \\ &= (x_1 x_2^4 x_3^8 + (\text{symmetried})) (x_1 x_2^2 x_3^8 + (\text{symmetried}))^2 \\ &= (x_1 x_2^4 x_3^8 + (\text{symmetried})) (x_1^2 x_2^4 x_3^{16} + (\text{symmetried})) \\ &= (x_1^4 x_2^8 + x_1^6 x_2^7) x_3^{24} + (x_1^6 x_2^{12} + x_1^5 x_2^{10}) x_3^{20} \\ &\quad + (x_1^6 x_2^{12} + x_1^8 x_2^{10}) x_3^{17} + (\text{symmetried}). \end{aligned}$$

Therefore

$$\begin{aligned} Q_{3,0}(y_1, y_2, y_3) + Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 &= x_1^5 x_2^{10} x_3^{20} + x_1^5 x_2^{12} x_3^{18} + x_1^8 x_2^9 x_3^{18} \\ &\quad + x_1^6 x_2^{12} x_3^{17} + x_1^8 x_2^{10} x_3^{17} + (\text{symmetried}). \end{aligned}$$

The right hand side is an invariant whose the maximal monomial is $x_1^5 x_2^{10} x_3^{20}$. So, by Lemma 6.2 again, it contains $Q_{3,0}^3$ as a term. Furthermore, by direct computation, it is equal to $Q_{3,0}^3$. Hence

$$Q_{3,0}(y_1, y_2, y_3) = Q_{3,0}^3 Q_{3,1} Q_{3,2}^2 + Q_{3,0}^3.$$

Similarly as in the proof of the first part, we have

$$\begin{aligned} Q_{3,1}(y_1, y_2, y_3) &= y_1^4 y_2^2 + y_1^4 y_2 y_3 + y_1^2 y_2^2 y_3^2 + (\text{symmetried}) \\ &= x_1^2 x_2^4 x_3^{24} + x_1^2 x_2^8 x_3^{20} + x_1^4 x_2^8 x_3^{18} \\ &\quad + \left(x_1^2 x_2^{12} + x_1^4 x_2^{10} + x_1^6 x_2^8 \right) x_3^{16} + (\text{symmetried}). \end{aligned}$$

The right hand side is an invariant whose the maximal monomial is $x_1^2 x_2^4 x_3^{24}$. So, by Lemma 6.2, it contains $Q_{3,0}^2 Q_{3,2}^4$ as a term. Furthermore it is equal to $Q_{3,0}^2 Q_{3,2}^4$; so,

$$Q_{3,1}(y_1, y_2, y_3) = Q_{3,0}^2 Q_{3,2}^4.$$

So we get the second part of the lemma. Similarly, we have

$$\begin{aligned} Q_{3,2}(y_1, y_2, y_3) &= y_1^4 + y_1^2 y_2 y_3 + y_1^2 y_2^2 + (\text{symmetried}) \\ &= x_2^4 x_3^{16} + x_1^2 x_2^2 x_3^{16} + x_1^2 x_2^8 x_3^{10} + (x_1^2 x_2^6 + x_1^3 x_2^5) x_3^{12} \\ &\quad + \left(x_1^5 x_2^6 + x_1^3 x_2^8 \right) x_3^9 + x_1^4 x_2^8 x_3^8 + (\text{symmetried}). \end{aligned}$$

The right hand side is an invariant whose the maximal monomial is $x_2^4 x_3^{16}$. So, by Lemma 6.2, it contains $Q_{3,1}^2 Q_{3,2}^2$ as a term. We see that

$$\begin{aligned} Q_{3,2}(y_1, y_2, y_3) + Q_{3,1}^2 Q_{3,2}^2 &= (x_1^2 x_2^6 + x_1^3 x_2^5) x_3^{12} + (x_1^3 x_2^8 + x_1^5 x_2^6) x_3^9 \\ &\quad + (\text{symmetried}). \end{aligned}$$

The maximal monomial of this invariant is $x_1^2 x_2^6 x_3^{12}$. So, by Lemma 6.2, it contains $Q_{3,0}^2 Q_{3,1}$ as a term. Furthermore, by direct computation, we obtain

$$Q_{3,2}(y_1, y_2, y_3) + Q_{3,1}^2 Q_{3,2}^2 = Q_{3,0}^2 Q_{3,1}.$$

So

$$Q_{3,2}(y_1, y_2, y_3) = Q_{3,1}^2 Q_{3,2}^2 + Q_{3,0}^2 Q_{3,1}.$$

Hence, the lemma is completely proved. $\square$

7 Some invariants on which the transfer vanishes

The goal of this section is to prove the two following propositions, which are numbered as Proposition 1.7(ii) and (iii) respectively.

Proposition 7.1 $tr_{m,n}(Q_{m,0}^{2^{s_0}} Q_{m,1}^{2^{s_1}} \cdots Q_{m,m-1}^{2^{s_{m-1}}}) = 0$ for any nonnegative integers $s_0, s_1, \dots, s_{m-1}$ and positive integers m, n such that $n \geq \frac{3m^2+3m}{2}$.

Proof Setting $s = \min \{s_0, s_1, \dots, s_{m-1}\}$ and suppose that $s = s_i$ for some $0 \leq i \leq m-1$. For $0 \leq j \leq m-1$, let $t_j = s_j - s$. So, we have $t_i = 0$ and

$$tr_{m,n} \left(Q_{m,0}^{2^{s_0}} Q_{m,1}^{2^{s_1}} \cdots Q_{m,m-1}^{2^{s_{m-1}}} \right) = \left(tr_{m,n} \left(Q_{m,0}^{2^{t_0}} Q_{m,1}^{2^{t_1}} \cdots Q_{m,m-1}^{2^{t_{m-1}}} \right) \right)^{2^s}.$$

Denote $Q = tr_{m,n}(Q_{m,0}^{2^{t_0}} Q_{m,1}^{2^{t_1}} \cdots Q_{m,m-1}^{2^{t_{m-1}}})$. We will show that $Q = 0$. Suppose to the contrary that $Q \neq 0$, by the definition of $tr_{m,n}$, we get

$$Q = \sum Q_{n,0}^{i_0} Q_{n,1}^{i_1} \cdots Q_{n,n-1}^{i_{n-1}}$$

for some sequences of nonnegative integers $(i_1, i_1, \dots, i_{n-1})$. We consider the maximal monomial in variables $x_1, \dots, x_n$ of Q , denoted by $x(Q)$.

First, from Lemma 6.2, there is exactly one invariant monomial $Q_{n,0}^{i_0} Q_{n,1}^{i_1} \cdots Q_{n,n-1}^{i_{n-1}}$ of Q containing $x(Q)$; furthermore,

$$x(Q) = x_1^{i_0} x_2^{2(i_0+i_1)} \cdots x_n^{2^{n-1}(i_0+i_1+\cdots+i_{n-1})}.$$

Let h be the maximal index such that $i_h \neq 0$. Then $i_0 = \cdots = i_{h-1} = 0$. So

$$\begin{aligned} x(Q) &= x_{h+1}^{2^h \cdot i_h} x_{h+2}^{2^{h+1}(i_h+i_{h+1})} \cdots x_n^{2^{n-1}(i_h+\cdots+i_{n-1})}, \\ \implies \deg(x(Q)) &= 2^h \cdot i_h + 2^{h+1}(i_h + i_{h+1}) + \cdots + 2^{n-1}(i_h + \cdots + i_{n-1}). \end{aligned}$$

It implies that 2^h divides $\deg(x(Q))$. On the other hand $\deg(x(Q)) = \sum_{j=0}^m (2^m - 2^j)2^{t_j}$; so, 2^h divides $\sum_{j=0}^m (2^m - 2^j)2^{t_j}$.

Second, we see that each invariant $Q_{m,j}$ with $0 \leq j \leq m-1$ is a polynomial in m variables. Thus, if W is an m -dimensional subspace of $\mathbb{V}^*$ then $(Q_{m,0}^{2^{t_0}} Q_{m,1}^{2^{t_1}} \cdots Q_{m,m-1}^{2^{t_{m-1}}})(W)$ is a polynomial of at most m^2 distinct variables. Thus, by the definition of $tr_{m,n}$, $Q = \sum_{W \leq \mathbb{V}^*, \dim W = m} (Q_{m,0}^{2^{t_0}} Q_{m,1}^{2^{t_1}} \cdots Q_{m,m-1}^{2^{t_{m-1}}})(W)$ is a polynomial of at most m^2 distinct variables. As $x(Q)$ is a monomial of Q , from the formula of $x(Q)$ given above, we get the number of distinct variables of $x(Q)$, $n-h$, is less than or equal to m^2 . Combining with the hypothesis $n \geq \frac{3m^2+3m}{2}$, we obtain

$$h \geq n - m^2 \geq \frac{3m^2+3m}{2} - m^2 = \frac{m^2+3m}{2}.$$

So $2^{\frac{m^2+3m}{2}}$ divides 2^h . As a consequence $2^{\frac{m^2+3m}{2}}$ divides $\sum_{j=0}^m (2^m - 2^j)2^{t_j}$. Let $2^{\alpha_1} + \cdots + 2^{\alpha_k}$ be the dyadic expansion of $\sum_{j=0}^m (2^m - 2^j)2^{t_j}$ with $\alpha_1 < \cdots < \alpha_k$. It implies that

$$\alpha_i > 2^{\frac{m^2+3m}{2}} \quad \text{for all } i \geq 1. \quad (3)$$

Third, since $t_i = 0$ we get

$$\sum_{j=0}^m (2^m - 2^j) 2^{t_j} = (2^{m-1} + \cdots + 1) 2^{t_1} + \cdots + (2^{m-1} + \cdots + 2^i) \\ + \cdots + 2^{m-1} \cdot 2^{t_{m-1}}.$$

Now, we reduce this sum to the dyadic expansion form $2^{\alpha_1} + \cdots + 2^{\alpha_k}$ given above. If 2^i is a term in the dyadic expansion form, then there exists an index j such that $i = \alpha_j$. As $i \leq m-1$, we get

$$\alpha_j < m < \frac{m^2 + 3m}{2}.$$

This contradicts the inequality (3). If 2^i does not appear in the the dyadic expansion form, then there exists a positive integer k and an index j such that

$$2^i + 2^i + 2^{i+1} + \cdots + 2^{i+k} = 2^{i+k} = 2^{\alpha_j}.$$

It should be noted that the number of powers of 2 in the sum $\sum_{j=0}^m (2^m - 2^j) 2^{t_j}$ is $(m-1) + (m-2) + \cdots + 1 = \frac{m^2+m}{2}$. So $k \leq \frac{m^2+m}{2}$. Therefore

$$\alpha_j = k + i \leq \frac{m^2 + m}{2} + i < \frac{m^2 + m}{2} + m = \frac{m^2 + 3m}{2},$$

(as $i < m$). This contradicts the inequality (3). As a consequence, we get $Q = tr_{m,n}(Q_{m,0}^{2^{t_0}} Q_{m,1}^{2^{t_1}} \cdots Q_{m,m-1}^{2^{t_{m-1}}}) = 0$. So

$$tr_{m,n} \left(Q_{m,0}^{2^{s_0}} Q_{m,1}^{2^{s_1}} \cdots Q_{m,m-1}^{2^{s_{m-1}}} \right) = 0.$$

Hence, the proposition is completely proved. $\square$

For case $m = 2$, we get the following.

Proposition 7.2 $tr_{2,n}(Q_{2,0}^{2^{s_0}} Q_{2,1}^{2^{s_1}}) = 0$ for any nonnegative intergers s_0, s_1 and $n \geq 4$.

Remark 7.3 The statement in Proposition 7.2 is no longer true for $n = 3$. In fact, we have

- (i) if $s_0 - s_1 > 1$ then $tr_{2,3}(Q_{2,0}^{2^{s_0}} Q_{2,1}^{2^{s_1}}) = Q_{3,0}^{2^{s_1}} Q_{3,1}^{2^{s_0-1}-2^{s_1}} + (\text{others})$,
- (ii) if $s_0 - s_1 = 0$ or 1 then $tr_{2,3}(Q_{2,0}^{2^{s_0}} Q_{2,1}^{2^{s_1}}) = 0$,
- (iii) if $s_0 - s_1 < 0$ then $tr_{2,3}(Q_{2,0}^{2^{s_0}} Q_{2,1}^{2^{s_1}}) = Q_{3,0}^{2^{s_0}} Q_{3,2}^{2^{s_1-1}-2^{s_0}} + (\text{others})$.

Proof Case 1. $s_0 > s_1$. We have $tr_{2,3}(Q_{2,0}^{2s_0} Q_{2,1}^{2s_1}) = (tr_{2,3}(Q_{2,0}^{2s_0-s_1} Q_{2,1}))^{2s_1}$. Setting $a = s_0 - s_1$. By the definition of $tr_{2,3}$, we get

$$\begin{aligned} tr_{2,3}(Q_{2,0}^{2s_0-s_1} Q_{2,1}) &= tr_{2,3}(Q_{2,0}^{2a} Q_{2,1}) \\ &= (Q_{2,0}^{2a} Q_{2,1})(x_1, x_2) + (Q_{2,0}^{2a} Q_{2,1})(x_1, x_3) \\ &\quad + (Q_{2,0}^{2a} Q_{2,1})(x_1, x_3) + (Q_{2,0}^{2a} Q_{2,1})(x_1 + x_2, x_3) \\ &\quad + (Q_{2,0}^{2a} Q_{2,1})(x_1, x_2 + x_3) + (Q_{2,0}^{2a} Q_{2,1})(x_1 + x_3, x_2 + x_3). \end{aligned}$$

By direct computation, we have

$$\begin{aligned} (Q_{2,0}^{2a} Q_{2,1})(x_1, x_2) &= x_1^{2a} x_2^{2a+1+2} + x_1^{2a+1+1} x_2^{2a+1} + x_1^{2a+1} x_2^{2a+2} \\ &\quad + (\text{symmetried}), \\ (Q_{2,0}^{2a} Q_{2,1})(x_1 + x_2, x_3) &= x_2^{2a} x_3^{2a+1+2} + x_2^{2a+1} x_3^{2a+1+1} + x_2^{2a+2} x_3^{2a+1} + x_1^{2a} x_3^{2a+1+2} \\ &\quad + x_1^{2a+1} x_3^{2a+1+1} + x_1^{2a+2} x_3^{2a+1} + x_1 x_2^{2a} x_3^{2a+1+1} \\ &\quad + x_1^2 x_2^{2a} x_3^{2a+1} + x_1 x_2^{a+1} x_3^{2a+1} + x_1^2 x_2^{a+1} x_3^{2a} \\ &\quad + (\text{symmetried}), \end{aligned}$$

$$\begin{aligned} (Q_{2,0}^{2a} Q_{2,1})(x_1 + x_2, x_2 + x_3) &= (x_1^2 x_2^{2a} x_3^{2a+1} + x_1 x_2^{2a+1} x_3^{2a+1}) + (\text{symmetried}) \\ &\quad + \sum_{T \in S_3} x_{T(1)}^{2a+1+2} x_{T(2)}^{2a} + \sum_{T \in S_3} x_{T(1)}^{2a+1+1} x_{T(2)}^{2a+1} \\ &\quad + \sum_{T \in S_3} x_{T(1)}^{2a+1} x_{T(2)}^{2a+2}. \end{aligned}$$

We see that if $a = s_0 - s_1 = 1$ then $tr_{2,3}(Q_{2,0}^{2a} Q_{2,1}) = 0$; otherwise, if $a > 1$ then

$$tr_{2,3}(Q_{2,0}^{2a} Q_{2,1}) = x_1^2 x_2^{2a} x_3^{2a+1} + (\text{symmetried}).$$

In this case $x_1^2 x_2^{2a} x_3^{2a+1}$ is the maximal monomial of $tr_{2,3}(Q_{2,0}^{2a} Q_{2,1})$. So, by Lemma 6.2, $tr_{2,3}(Q_{2,0}^{2a} Q_{2,1})$ contains the invariant $Q_{3,0}^{i_0} Q_{3,1}^{i_1} Q_{3,2}^{i_2}$ as a term in which

$$\begin{cases} i_0 = 2 \\ 2(i_0 + i_1) = 2^a \\ 4(i_0 + i_1 + i_2) = 2^{a+1} \end{cases} \iff \begin{cases} i_0 = 2 \\ i_1 = 2^{a-1} - 2 \\ i_2 = 0. \end{cases}$$

Therefore $tr_{2,3}(Q_{2,0}^{2a}Q_{2,1}) = Q_{3,0}Q_{3,1}^{2^{a-1}-1} + (\text{others})$. Hence

$$tr_{2,3}\left(Q_{2,0}^{2s_0}Q_{2,1}^{2s_1}\right) = \left(Q_{3,0}Q_{3,1}^{2^{s_1-1}-1} + (\text{others})\right)^{2^{s_1}} = Q_{3,0}^{2^{s_1}}Q_{3,1}^{2^{s_0-1}-2^{s_1}} + (\text{others}).$$

Case 2. $s_0 = s_1$. From Example 4.8, we see that $Q_{2,0}Q_{2,1}$ is multilinear and alternating. So, by Proposition 5.6, $tr_{2,3}(Q_{2,0}Q_{2,1}) = 0$. Hence $tr_{2,3}(Q_{2,0}^{2s_0}Q_{2,1}^{2s_1}) = (tr_{2,3}(Q_{2,0}Q_{2,1}))^{2^{s_0}} = 0$.

Case 3. $s_0 < s_1$. Similarly as proof of Case 1, setting $b = 2^{s_1} - 2^{s_0}$ we have $Q_{2,0}^{2s_0}Q_{2,1}^{2s_1} = (Q_{2,0}Q_{2,1}^b)^{2^{s_0}}$. And

$$Q_{2,0}Q_{2,1}^b = x_1x_2^2x_3^{2^{b+1}} + (\text{symmetried}).$$

The maximal monomial of $Q_{2,0}Q_{2,1}^b$ is $x_1x_2^2x_3^{2^{b+1}}$. So, by Lemma 6.2, $Q_{2,0}Q_{2,1}^b$ contains $Q_{3,0}Q_{3,2}^{2^{b-1}-1}$ as a term. Therefore

$$\begin{aligned} tr_{2,3}\left(Q_{2,0}^{2s_0}Q_{2,1}^{2s_1}\right) &= \left(tr_{2,3}(Q_{2,0}Q_{2,1}^b)\right)^{2^{s_0}} \\ &= (Q_{3,0}Q_{3,2}^{2^{b-1}-1} + (\text{others}))^{2^{s_0}} \\ &= Q_{3,0}^{2^{s_0}}Q_{3,2}^{2^{s_1-1}-2^{s_0}} + (\text{others}). \end{aligned}$$

□

Proof of Proposition 7.2 Since $tr_{2,n} = tr_{3,n}tr_{2,3}$ by [3, Lemma 2.4], we have

$$tr_{2,n}\left(Q_{2,0}^{2s_0}Q_{2,1}^{2s_1}\right) = tr_{3,n}tr_{2,3}\left(Q_{2,0}^{2s_0}Q_{2,1}^{2s_1}\right).$$

Setting $Q = tr_{2,3}(Q_{2,0}^{2s_0}Q_{2,1}^{2s_1})$. We need to prove that $tr_{3,n}(Q) = 0$.

Case 1. $s_0 - s_1 > 1$. We have

$$\begin{aligned} Q &= \left(tr_{2,3}\left(Q_{2,0}^{2s_0-s_1}Q_{2,1}\right)\right)^{2^{s_1}} \\ &= \left(x_1^2x_2^{2s_0-s_1}x_3^{2^{s_0-s_1}+1} + (\text{symmetried})\right)^{2^{s_1}} \\ &\quad [\text{as shown in the proof of Remark 7.3(i)}] \\ &= x_1^{2^{s_1}+1}x_2^{2^{s_0}}x_3^{2^{s_0}+1} + (\text{symmetried}). \end{aligned}$$

From Proposition 4.5, we see that Q is the multilinear and alternating invariant ω_β with $\beta = (s_1 + 1, s_0, s_0 + 1)$. So, by Proposition 5.6, we get $tr_{3,n}(Q) = 0$.

Case 2. $s_0 - s_1 = 1$ or 0. Following Remark 7.3(ii), we get $Q = 0$. Thus $tr_{3,n}(Q) = 0$.

Case 3. $s_0 - s_1 < 0$. We have

$$\begin{aligned} Q &= \left(tr_{2,3} \left(Q_{2,0} Q_{2,1}^{2s_1-s_0} \right) \right)^{2s_0} \\ &= \left(x_1 x_2^2 x_3^{2s_1-s_0+1} + (\text{symmetried}) \right)^{2s_0} \\ &\quad [\text{as shown in the proof of Remark 7.3(ii)}] \\ &= x_1^{2s_0} x_2^{2s_0+1} x_3^{2s_1+1} + (\text{symmetried}). \end{aligned}$$

Similarly, from Proposition 4.5, we obtain that Q is the multilinear and alternating invariant ω_β with $\beta = (s_0, s_0+1, s_1+1)$. So, by Proposition 5.6, we get $tr_{3,n}(Q) = 0$.

The proposition is proved. $\square$

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