The effect of cell-attachment on the group of self-equivalences of an R -localized space

Mahmoud Benkhalifa · Samuel Bruce Smith

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Abstract Let $R \subseteq \mathbb{Q}$ be a ring with least non-invertible prime p . Let $X = X^n \cup_{\alpha} (\bigcup_{j \in J} e^q)$ be a cell attachment with J finite and q small with respect to p . Let $\mathcal{E}(X_R)$ denote the group of homotopy self-equivalences of the R -localization X_R . We use DG Lie models to construct a short exact sequence

$$0 \rightarrow \bigoplus_{j \in J} \pi_q(X^n)_R \rightarrow \mathcal{E}(X_R) \rightarrow \mathcal{C}^q \rightarrow 0$$

where $\mathcal{C}^q$ is a subgroup of $\mathrm{GL}_{|J|}(R) \times \mathcal{E}(X_R^n)$. We obtain a related result for the R -localization of the nilpotent group $\mathcal{E}_*(X)$ of classes inducing the identity on homology. We deduce some explicit calculations of both groups for spaces with few cells.

Keywords Homotopy self-equivalences · Quillen model · Anick model · R -local homotopy theory Moore space · Nilpotent group

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M. Benkhalifa
Umm Al-Qura University, Mekka, Saudia Arabia
e-mail: makhalifa@uqu.edu.sa

S. B. Smith (✉)
Saint Joseph's University, Philadelphia, PA 19131, USA
e-mail: smith@sju.edu

1 Introduction

Let X be a finite, simply connected CW complex. Let $\mathcal{E}(X)$ denote the group of homotopy equivalence classes of homotopy self-equivalences of X . Let $\mathcal{E}_*(X)$ denote the subgroup represented by self-equivalences that induce the identity map on $H_*(X; \mathbb{Z})$. The study of the groups $\mathcal{E}(X)$ and $\mathcal{E}_*(X)$ by means of a cellular decomposition of X is a difficult problem with a long history. See Rutter [16, Chapter 11] for a survey.

In [11], Dror-Zabrodsky proved $\mathcal{E}_*(X)$ is a nilpotent group. Maruyama [14] then proved $\mathcal{E}_*(X)_R \cong \mathcal{E}_*(X_R)$. These results together opened the door to the use of algebraic models for studying the localization of nilpotent self-equivalence groups. The group $\mathcal{E}_*(X)$ was studied in [5]. The rationalization of the subgroup $\mathcal{E}_\#(X)$ of self-equivalences inducing the identity on homotopy groups has been studied extensively using Sullivan models (c.f. [3, 6, 12]).

The group $\mathcal{E}(X_{\mathbb{Q}})$ has emerged as a recent object of interest. Arkowitz-Lupton [4] gave the first examples of finite groups occurring as $\mathcal{E}(X_{\mathbb{Q}})$. Further examples were given by the first named author in [9]. Costoya-Viruel [10] then proved the remarkable result that every finite group G occurs as $G \cong \mathcal{E}(X_{\mathbb{Q}})$ for some finite X . Again, all this work was accomplished using Sullivan models. The purpose of this paper is to explore the use of Anick's and Quillen's DG Lie algebra models for studying the groups $\mathcal{E}(X_R)$ and $\mathcal{E}_*(X)_R$.

We briefly recall the main result of Anick's and Quillen's theories now in order to establish our overriding hypotheses. Let $R \subseteq \mathbb{Q}$ be a subring with least non-invertible prime $p \in R$. When $R = \mathbb{Q}$ set $p = +\infty$. Let $\mathbf{CW}_m^{k+1}$ denote the category of m -connected, finite CW complexes of dimension no greater than $k+1$ with m -skeleton reduced to a point. Let $\mathbf{CW}_m^{k+1}(R)$ denote the category obtained by R -localizing the spaces in $\mathbf{CW}_m^{k+1}$. By Anick [1, 2], when $k < \min(m+2p-3, mp-1)$ the homotopy category of $\mathbf{CW}_m^{k+1}(R)$ is equivalent to the homotopy category of $\mathbf{DGL}_m^k(R)$ consisting of free differential graded (DG) Lie algebras $(L(V), \partial)$ in which V is a free R -module satisfying $V_n = 0$ for $n < m$ and $n > k$. When $R = \mathbb{Q}$ the corresponding result for $m = 1$ and any k is due to Quillen [15]. Summarizing, we have:

Hypothesis 1.1 We assume that $R \subseteq \mathbb{Q}$ is a ring with least non-invertible prime p . With R fixed, we take $1 \leq m < k$ satisfying $k < \min(m+2p-3, mp-1)$. By a space X we always mean an object in $\mathbf{CW}_m^{k+1}$, an m -connected finite CW complex with top degree cells of dimension $\leq k+1$. When $R = \mathbb{Q}$ we assume $m = 1$ and k is finite.

Let X be an object in $\mathbf{CW}_m^{k+1}$. Write X^n for the n -skeleton of X . We consider the situation in which

$$X = X^q = X^n \cup_{\alpha} \left(\bigcup_{i=1}^j e_i^q \right)$$

is the space obtained by attaching q -cells to a space X^n for $n < q \leq k+1$ by a map $\alpha: \bigvee_i S^{q-1} \rightarrow X^n$.

Theorem 1 Given $R \subseteq \mathbb{Q}$ and X satisfying Hypothesis 1.1, there are short exact sequences

$$0 \rightarrow \bigoplus_{i=1}^j \pi_q(X_R^n) \rightarrow \mathcal{E}(X_R) \rightarrow \mathcal{C}^q \rightarrow 0 \quad (1)$$

$$0 \rightarrow \bigoplus_{i=1}^j \pi_q(X_R^n) \rightarrow \mathcal{E}_*(X_R) \rightarrow \mathcal{C}_*^q \rightarrow 0 \quad (2)$$

Here, $\mathcal{C}^q \subseteq \mathrm{GL}_j(R) \times \mathcal{E}(X_R^n)$. The subgroup $\mathcal{C}_*^q \subseteq \mathcal{C}^q$ is contained in $\mathrm{GL}_k(R) \times \mathcal{E}_*(X_R^n)$ with k the rank of the linking homomorphism $\pi_{n+1}(X^q, X^n)_R \rightarrow \pi_n(X^n, X^{n-1})_R$ in the long exact sequence of the triple. In particular, $\mathcal{C}_*^q \subseteq \mathcal{E}_*(X_R^n)$ when $q > n + 1$.

We prove Theorem 1 in Sect. 2. In Sect. 3, we deduce some consequences for R -localized spaces.

Given a finitely generated abelian group G , let $M(G, m)$ denote the Moore space. Barcus–Barrett [8] proved the homology representation $\mathcal{E}(M(G, m)) \rightarrow \mathrm{aut}(G)$ is surjective and identified the kernel as an Ext-group. It is a classical open problem to complete this calculation [13, Problem 8] with many partial results and extensions (cf. Rutter [op. cit] and Baues [7]). We obtain the following result in Anick's category of R -local spaces:

Example 1.2 Let $G_1, \dots, G_r$ be finitely generated abelian groups and $R \subseteq \mathbb{Q}$. Let

$$X = M(G_1, m + 1) \vee M(G_2, m + 3) \vee \dots \vee M(G_r, m + 2r - 1)$$

with $r \leq m/2$ and $m, k = m + 2r$ and R satisfying Hypothesis 1.1. Then

$$\mathcal{E}(X_R) \cong \prod_{i=1}^r \mathrm{aut}((G_i)_R).$$

We obtain a related calculation for the group $\mathcal{E}_*(X)_R$:

Example 1.3 Let G be a finitely generated abelian group and $R \subseteq \mathbb{Q}$. Let X in $\mathbf{CW}_m^{k+1}$ and R be as in Hypothesis 1.1 with X having a cellular decomposition

$$M(G, m + 1) = X^{m+2} \subset X^{m+3} \subset \dots \subset X^{2m} = X.$$

Assume that the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \rightarrow \pi_r(X^r, X^{r-1})_R$ vanish for $r = m + 2, \dots, 2m - 1$. Let $j = \dim_R(H_{2m}(X, X^{2m-1}; R))$. Then

$$\mathcal{E}_*(X)_R \cong \bigoplus_{i=1}^j [G_R, G_R].$$

Here, $[G_R, G_R]$ is additive group corresponding to the sub R -module of $G_R \otimes G_R$ generated by elements of the form $x \otimes y - (-1)^m y \otimes x$.

We give a complete calculation of both groups in a simple case. Let $R^* = \text{aut}(R)$ denote the group of units of R .

Example 1.4 Let $m, m + n + 1$ and $R \subseteq \mathbb{Q}$ satisfy Hypothesis 1.1. Then

$$\mathcal{E}((S^{m+1} \times S^{n+1})_R) \cong \begin{cases} R^* \times R^* & \text{for } n \neq 2m \text{ or } m \text{ even.} \\ R \oplus (R^* \times R^*) & \text{for } n = 2m \text{ and } m \text{ odd} \end{cases}$$

$$\mathcal{E}_*(S^{m+1} \times S^{n+1})_R \cong \begin{cases} 0 & \text{for } n \neq 2m \text{ or } m \text{ even.} \\ R & \text{for } n = 2m \text{ and } m \text{ odd} \end{cases}$$

We prove two results concerning the question of finiteness $\mathcal{E}(X_R)$. First:

Theorem 1.5 Let $R \subseteq \mathbb{Q}$ and X in $\mathbf{CW}_m^{k+1}$ be as in Hypothesis 1.1. Suppose $\pi_{n+1}(X^{n+1}, X^n)_R \neq 0$ and the linking homomorphism $\pi_{n+1}(X^{n+1}, X^n)_R \rightarrow \pi_n(X^n, X^{n-1})_R$ vanishes. Then $\mathcal{E}(X_R)$ finite implies $\mathcal{E}(X_R^{n+1})$ is infinite.

Finally, we give a calculation to indicate that finiteness of $\mathcal{E}(X_R)$ requires a reasonably large CW complex X .

Theorem 1.6 Let X be a simply-connected finite CW complex of dimension ≤ 5 . Let $R \subseteq \mathbb{Q}$ have least invertible prime $p \geq 7$. If p is finite, assume the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \rightarrow \pi_r(X^r, X^{r-1})_R$ vanish for $r = 2, 3, 4$ vanish. Then $\mathcal{E}(X_R)$ is infinite.

2 Homotopy self-equivalences of DG Lie algebras

Let $(L(V), \partial)$ be an object in $\mathbf{DGL}_m^k(R)$. Recall this means V is a free graded R -module concentrated in degrees n with $m \leq n \leq k$ and $L(V)$ is the free graded Lie algebra over R . Write $V_{<n} = \bigoplus_{i=m}^{n-1} V_i$. The differential ∂ is of degree -1 . For each $n \leq k$, ∂ induces a differential $\partial_{<n}$ on $L(V_{<n})$ making $(L(V_{<n}), \partial_{<n})$ a sub DG Lie algebra. We write the homology as $H_*(L(V_{<n}))$, suppressing the differential. The linear part of ∂ gives a differential d on V . The homology $H_*(V) = H_*(V, d)$ can be identified with the graded module of indecomposable generators of $(L(V), \partial)$.

Homotopies between maps $\alpha, \alpha': (L(V), \partial) \rightarrow (L(W), \delta)$ in $\mathbf{DGL}_m^k(R)$ are defined by means of the Tanré cylinder (cf. [17, Ch.II.5] and [1, pp. 425–6]). Let

$$(L(V), \partial)_I = (L(V, sV, V'), D)$$

be the DG Lie algebra with $V \cong V'$ and $(sV)_i = V_{i-1}$. Let S denote the derivation of degree $+1$ on $L(V, sV, V')$ with $S(v) = sv$ and $S(sv) = S(v') = 0$. The differential D is given by $D(v) = \partial(v)$, $D(sv) = v'$ and $D(v') = 0$. The degree zero derivation $\theta = D \circ S + S \circ D$ of $(L(V), \partial)_I$ gives rise to an automorphism e^θ of $(L(V), \partial)_I$. We note that for $v \in V$

$$e^\theta(v) = v + v' + \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v).$$

Define $\alpha \simeq \alpha'$ if there is a DG Lie morphism

$$F: (L(V, sV, V'), D) \rightarrow (L(V), \partial)$$

satisfying $F(v) = \alpha(v)$ and $F \circ e^\theta(v) = \alpha'(v)$.

Quillen [15] and Anick [1] proved that, under Hypothesis 1.1, there is an assignment $X \mapsto (L(V), \partial)$ setting up an equivalence between the homotopy categories of $\mathbf{CW}_m^{k+1}(R)$ and $\mathbf{DGL}_m^k(R)$. The model $(L(V), \partial)$ recovers R -local homotopy invariants of X via isomorphisms (with shifts)

$$\tilde{H}_*(X; R) \cong H_{*-1}(V) \quad \text{and} \quad \pi_*(X)_R \cong H_{*-1}(L(V)).$$

As for self-equivalence groups, their results directly imply identifications:

$$\mathcal{E}(X_R) \cong \frac{\text{aut}(L(V), \partial)}{\simeq} \quad \text{and} \quad \mathcal{E}_*(X_R) \cong \frac{\text{aut}_*(L(V), \partial)}{\simeq}$$

Here, $\text{aut}(L(V), \partial)$ is the group of DG Lie homotopy self-equivalences and the subgroup $\text{aut}_*(L(V), \partial)$ consists of maps inducing the identity automorphism of $H_*(V)$.

We write

$$\mathcal{E}(L(V), \partial) = \text{aut}(L(V), \partial) / \simeq \quad \text{and} \quad \mathcal{E}_*(L(V), \partial) = \text{aut}_*(L(V), \partial) / \simeq$$

for these algebraic equivalence groups.

Now consider a cellular attachment $X = X^n \cup_\alpha (\bigcup_{i=1}^j e_i^q)$ as in Theorem 1. Let $(L(V), \partial)$ denote the DG Lie algebra model for X . Then $V = V_{q-1} \oplus V_{<n}$ where $\dim V_{q-1} = j$, the number of q -cells attached. A homotopy self-equivalence $f: X \rightarrow X$ induces a DG Lie algebra isomorphism $\alpha: (L(V), \partial) \rightarrow (L(V), \partial)$. Let $\tilde{\alpha}_{q-1}: V_{q-1} \rightarrow V_{q-1}$ denote the map induced on V_{q-1} by restricting α and then projecting to V_{q-1} . Let $\alpha_{<n}: (L(V_{<n}), \partial_{<n}) \rightarrow (L(V_{<n}), \partial_{<n})$ denote the DG Lie algebra restriction map. We then obtain a commutative square of the form:

$$\begin{array}{ccc} V_{q-1} & \xrightarrow{\tilde{\alpha}_{q-1}} & V_{q-1} \\ \downarrow B_{q-1} & & \downarrow B_{q-1} \\ H_{q-2}(L(V_{<n})) & \xrightarrow{H(\alpha_{<n})} & H_{q-2}(L(V_{<n})) \end{array}$$

Here, $B_{q-1}: V_{q-1} \rightarrow H_{q-2}(L(V_{<n}))$ is given by

$$B_{q-1}(v_{q-1}) = \{\partial_{q-1}(v_{q-1})\} \in H_{q-2}(L(V_{<n})).$$

where $\{ \}$ denotes the homology class of a cycle. We use this diagram to define the group $\mathcal{C}^q$. Given a self-equivalence α of $L(V)$ we write $[\alpha]$ for the homotopy equivalence class in $\mathcal{E}(L(V), \partial)$.

Definition 2.1 Let $\mathcal{C}^q$ the subset of pairs $(\xi, [\gamma]) \in \text{aut}(V_{q-1}) \times \mathcal{E}(L(V_{<n}), \partial_{<n})$ such that the following diagram is commutative:

$$\begin{array}{ccc} V_{q-1} & \xrightarrow{\xi} & V_{q-1} \\ \downarrow B_{q-1} & & \downarrow B_{q-1} \\ H_{q-2}(L(V_{<n})) & \xrightarrow{H(\gamma)} & H_{q-2}(L(V_{<n})) \end{array}$$

Proposition 2.2 $\mathcal{C}^q$ is a subgroup of $\text{aut}(V_{q-1}) \times \mathcal{E}(L(V_{<n}), \partial_{<n})$

Proof Straightforward. $\square$

Define $\Psi_q: \mathcal{E}(L(V_{q-1} \oplus V_{<n}), \partial) \rightarrow \mathcal{C}^q$ by setting:

$$\Psi_q([\alpha]) = (\tilde{\alpha}_{q-1}, [\alpha_{<n}])$$

Proposition 2.3 The map Ψ_q is a surjective homomorphism

Proof It is easy to see Ψ_q is a homomorphism. We prove surjectivity. Let $(\xi, [\gamma]) \in \mathcal{C}^q$. Choose $\{v_i\}_{i \in J}$ as a basis of V_{q-1} . From the hypothesized commutative diagram we obtain $(\gamma \circ \partial - \partial \circ \xi)(v_i) \in \text{im } \partial_{<n}$. Choose $u_i \in L(V_{<n})$ of degree $q-1$ with $(\gamma \circ \partial - \partial \circ \xi)(v_i) = \partial_{<n}(u_i)$. Define $\alpha: (L(V_{q-1} \oplus V_{<n}), \partial) \rightarrow (L(V_{q-1} \oplus V_{<n}), \partial)$ by setting $\alpha(v_i) = \xi(v_i) + u_i$ for $v_i \in V_{q-1}$ and $\alpha = \gamma$ on $V_{<n}$. and then extending. Then α is clearly an automorphism of $L(V)$. Observe

$$\partial \circ \alpha(v_i) = \partial(\xi(v_i)) + \partial_{<n}(u_i) = \gamma \circ \partial(v_i) = \alpha \circ \partial(v_i)$$

Thus α represents a class in $\mathcal{E}(L(V), \partial)$ satisfying $\Psi_q([\alpha]) = (\xi, [\gamma])$. $\square$

We next identify

$$\ker \Psi_q = \left\{ [\alpha] \in \mathcal{E}(L(V), \partial) \mid \tilde{\alpha}_{q-1} = \text{id}_{V_{q-1}}, \quad \alpha_{<n} \simeq \text{id}_{L(V_{<n})} \right\}.$$

Let $[\alpha] \in \ker \Psi_q$. That $\tilde{\alpha}_{q-1} = \text{id}_{V_{q-1}}$ means for all $v \in V_q$ we have $\alpha(v) - v \in L^{q-1}(V_{<n})$. Here, $L^{q-1}(V_{<n})$ denotes the space of elements of $L(V_{<n})$ of degree $q-1$. Define

$$\varphi_\alpha: V_{q-1} \rightarrow L^{q-1}(V_{<n}) \text{ by } \varphi_\alpha(v) = \alpha(v) - v \in L^{q-1}(V_{<n}) \text{ for } v \in V_{q-1}.$$

We next prove that the class $[\alpha]$ has a representative β such that $\varphi_\beta(V)$ is contained in the cycles of $L^{q-1}(V_{<n})$:

Lemma 2.4 Let $[\alpha] \in \ker \Psi_q$. Then there exists $[\beta] \in \ker \Psi_q$ satisfying

- (i) $\partial(\varphi_\beta(v)) = 0$ for all $v \in V_{q-1}$
- (ii) $\beta_{<n} = \text{id}_{L(V_{<n})}$.
- (iii) $\alpha \simeq \beta$.

Proof Since, $\alpha_{<n} \simeq \text{id}_{L(V_{<n})}$ there is a homotopy $F: (L(V_{<n}), \partial_{<n})_I \rightarrow (L(V_{<n}), \partial_{<n})$ satisfying $F(v) = v$ and $F \circ e^\theta(v) = \alpha_{<n}$ for $v \in V_{<n}$. Define β by setting

$$\beta(v) = \begin{cases} v & \text{for } v \in V_{<n} \\ \alpha(v) - F\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) & \text{for } v \in V_{q-1}. \end{cases}$$

Given $v \in V_{q-1}$ we compute:

$$\begin{aligned} \partial(\varphi_\beta(v)) &= \partial\left(\alpha(v) - F\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) - v\right) \\ &= \alpha_{<n}(\partial(v)) - \partial \circ F\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) - \partial(v) \\ &= F \circ e^\theta(\partial(v)) - F \circ D\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) - \partial(v) \\ &= F \circ D \circ e^\theta(v) - F \circ D\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) - \partial(v) \\ &= F \circ D(v + v') - \partial(v) \\ &= 0. \end{aligned}$$

Thus β satisfies (i). For (ii), we define $G: (L(V), \partial)_I \rightarrow (L(V), \partial)$ by setting $G = F$ on $(L(V_{<n}), \partial_{<n})_I$ while, for $v \in V_{q-1}$, we set $G(v) = \beta(v)$ and $G(v') = G(sv) = 0$. It is easy to check that G is a DG Lie algebra map. Given $v \in V_{q-1}$, we have

$$\begin{aligned} G \circ e^\theta(v) &= G\left(v + v' + \sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) \\ &= G(v) + G\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) \\ &= \beta(v) + F\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ D)^n(v)\right) \\ &= \alpha(v). \end{aligned}$$

□

Using Lemma 2.4 (i), we define a map

$\Theta_q: \ker \Psi_q \rightarrow \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n})))$ by $\Theta_q([\beta])(v) = \{\varphi_\beta(v)\}$ for $v \in V_{q-1}$

where β is chosen as in Lemma 2.4.

Proposition 2.5 *The map*

$$\Theta_q : \ker \Psi_q \rightarrow \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n})))$$

is an isomorphism.

Proof First we prove that Θ_q is well-defined. Suppose $\beta \simeq \beta'$ satisfy the conclusion of Lemma 2.4. Since, both maps then restrict to the identity on $L(V_{<n})$, the homotopy $F : (L(V), \partial)_I \rightarrow (L(V), \partial)$ between them can be chosen so that $F(V'_{<n}) = F(sV_{<n}) = 0$. Given $v \in V_{q-1}$ suppose $\varphi_\beta(v) = \{y\}$ and $\varphi_{\beta'}(v) = \{y'\}$ for cycles $y, y' \in L^{q-1}(V_{<n})$. We then have

$$\begin{aligned} y' - y &= \beta'(v) - \beta(v) \\ &= F \circ e^\theta(v) - F(v) \\ &= F(v) + F(v') + F\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n(v)\right) - F(v) \\ &= F(D(sv)) + F\left(\sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n(v)\right) \\ &= \partial(F(sv)) \end{aligned}$$

Thus $y' - y$ is a boundary.

It is easy to check Θ_q is a homomorphism. For injectivity, suppose $\Theta_q([\beta])(v) = \Theta_q([\beta'])(v)$ in $H_{n+1}(L(V_{\leq n-1}), \partial_{<n})$ for all $v \in V_{q-1}$. Then $\text{im}\{\beta - \beta' : L(V) \rightarrow L(V)\}$ is contained in an acyclic sub DG Lie algebra of $(L(V), \partial)$. Thus $\beta \simeq \beta'$ by [17, Prop.II.5(4)].

Finally, given a homomorphism $\psi \in \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}), \partial_{<n}))$, we define $\beta : (L(V), \partial) \rightarrow (L(V), \partial)$ by:

$$\beta(v) = v + \psi(v) \text{ for } v \in V_{q-1} \text{ and } \beta = \text{id on } V_{<n}.$$

Then β is a DG Lie morphism with $\Theta_q([\beta]) = \psi$. □

Summarizing, we have proven:

Theorem 2.6 *Let $(L(V), \partial)$ be an object in $\mathbf{DGL}_m^k(R)$ with $V = V_{q-1} \oplus V_{<n}$ for $q > n$. Then, there exists a short exact sequence of groups:*

$$0 \rightarrow \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}))) \rightarrow \mathcal{E}(L(V), \partial) \rightarrow \mathcal{C}^q \rightarrow 0.$$

□

The first exact sequence in Theorem 1 is a direct consequence.

Proof of Theorem 1 Part (1). The result follows from Theorem 2.6, the isomorphisms $\mathcal{E}(X_R) \cong \mathcal{E}(L(V), \partial)$, $\pi_q(X^n)_R \cong H_{q-1}(L(V_{<n}))$ and the identification of V_{q-1} with the free R -module with generators corresponding to the q -cells of X . See [1, Theorem 8.5]. □

We now focus on the group $\mathcal{E}_*(L(V), \partial)$ and the proof of Theorem Part (2). Again, we take $(L(V), \partial)$ to be an object in $\mathbf{DGL}_m^k(R)$ with $V = V_{q-1} \oplus V_{<n}$ for $m \leq n < q \leq k$. When $q = n + 1$ we must take into account the linear differential $d_n: V_n \rightarrow V_{n-1}$. Since V_n is a free R -module, we may choose a subspace W_n of V_n complementary to the d_n -cycles in V_n giving $V_n = (\ker d_n) \oplus W_n$. When $q > n + 1$ we set $W_{q-1} = V_{q-1}$.

Let $\alpha \in \text{aut}_*(L(V), \partial)$ be given and, as usual, let $\tilde{\alpha}_{q-1}: V_{q-1} \rightarrow V_{q-1}$ denote the induced map. Since, α induces the identity on $H_*(V)$, we see $\tilde{\alpha}_{q-1}$ fixes $\ker d_{q-1} = H_{q-1}(V)$. It follows that α induces a map $\tilde{\alpha}'_{q-1}: W_{q-1} \rightarrow W_{q-1}$.

Definition 2.7 Let $\mathcal{C}_*^q$ denote the subset of pairs $(\chi, [\eta]) \in \text{aut}(W_{q-1}) \times \mathcal{E}_*(L(V_{<n}), \partial_{<n})$ such that the following diagram is commutative:

$$\begin{array}{ccc} W_{q-1} & \xrightarrow{\chi} & W_{q-1} \\ \downarrow B_{q-1} & & \downarrow B_{q-1} \\ H_{q-2}(L(V_{<n})) & \xrightarrow{H(\eta)} & H_{q-2}(L(V_{<n})) \end{array}$$

Remark 2.8 If $q > n + 1$ or if $d_n = 0$ then $W_n = \{0\}$. In this case

$$\mathcal{C}_*^q = \mathcal{C}^q \cap \mathcal{E}_*(L(V_{<n}), \partial_{<n}).$$

We prove:

Theorem 2.9 Let $(L(V), \partial)$ be an object in $\mathbf{DGL}_m^k(R)$ with $V = V_{q-1} \oplus V_{<n}$ for $q > n$. Then, there is a short exact sequence:

$$0 \rightarrow \text{Hom}(V_{q-1}, H_{q-1}(L(V_{<n}))) \rightarrow \mathcal{E}_*(L(V), \partial) \rightarrow \mathcal{C}_*^q \rightarrow 0$$

Proof Define $\Gamma_q: \mathcal{E}_*(L(V), \partial) \rightarrow \mathcal{C}_*^q$ by $\Gamma_q(\alpha) = (\tilde{\alpha}'_{q-1}, [\alpha_{<n}])$. We claim Γ_q is surjective. For given $(\chi, [\eta]) \in \mathcal{C}_*^q$ as in Definition 2.7, we can extend χ to a map $\xi: V_{q-1} \rightarrow V_{q-1}$ by setting $\xi = \text{id}$ on $\ker d_{q-1}$. Then the pair $(\xi, [\eta]) \in \mathcal{C}^q$ and so, by Proposition 2.3, there exists $[\alpha] \in \mathcal{E}(L(V), \partial)$ with $\Psi([\alpha]) = (\xi, [\eta])$ and, further, α may be chosen in $\text{aut}_*(L(V), \partial)$ since ξ and η fix $H_*(V)$. Thus $\Gamma_q([\alpha]) = (\chi, [\eta])$. Finally, observe that $\ker \Gamma_q = \ker \Psi_q$ and the result follows from Proposition 2.5. $\square$

We deduce:

Proof of Theorem 1 Part (2). The result follows again from the Quillen-Anick identifications as in the proof of Theorem 1, above. In this case, we note that the linear differential $d_n: V_n \rightarrow V_{n-1}$ corresponds linking homomorphism $\pi_{n+1}(X^{n+1}, X^n)_R \rightarrow \pi_n(X^n, X^{n-1})_R$ in the long exact sequence of the triple. $\square$

Remark 2.10 The exact sequences in Theorems 2.6 and 2.9 do not split in general. One simple criterion for splitting occurs when $\partial_{<n} = 0$.

3 Self-equivalences of R -local spaces

We begin with a result on the full group $\mathcal{E}(X_R)$. The following was stated in the introduction as Example 1.2.

Theorem 3.1 *Let $G_1, \dots, G_r$ be finitely generated abelian groups. Given $R \subseteq \mathbb{Q}$, suppose $X = M(G_1, m+1) \vee M(G_2, m+3) \vee \dots \vee M(G_r, m+2r-1)$ for $r \leq m/2$ and X an object of $\mathcal{CW}_m^{k+1}$ satisfying Hypothesis 1.1. Then*

$$\mathcal{E}(X_R) \cong \prod_{i=1}^r \text{aut}((G_i)_R).$$

Proof Recall $M(G, i)$ is $(i-1)$ -connected and of dimension $\leq i+1$. It follows that, in the Anick model $(L(V), \partial)$ for X , we have $V = V_m \oplus V_{m+1} \oplus \dots \oplus V_{m+2r-1}$ with $\partial = d$ purely linear and taking the form

$$\begin{aligned} V_{m+2r-1} &\xrightarrow{d_{m+2r-1}} V_{m+2r-2} \xrightarrow{0} V_{m+2r-3} \xrightarrow{d_{m+2r-3}} V_{m+2r-4} \xrightarrow{0} \dots \\ \dots &\xrightarrow{0} V_{m+3} \xrightarrow{d_{m+3}} V_{m+2} \xrightarrow{0} V_{m+1} \xrightarrow{d_m} V_m \longrightarrow 0. \end{aligned}$$

Here,

$$(G_i)_R \cong \frac{V_{m+2i-2}}{\text{im } d_{m+2i-1}}.$$

Note $\mathcal{C}^{m+2}$ consists of pairs $(\xi_{m+1}, \lambda_m) \in \text{aut}(V_{m+1}) \times \text{aut}(V_m)$ with $d_{m+1} \circ \xi_{m+1} = \lambda_m \circ d_{m+1}$. Since $(G_1)_R \cong V_m / \text{Im } d_{m+1}$ we deduce that $\mathcal{C}^{m+2} \cong \text{aut}(G_R)$. Since, $H_{m+1}(L(V_{\leq m})) = 0$, invoking Theorem 2.6 we deduce $\mathcal{E}(L(V_{\leq m+1})) \cong \text{aut}(G_R)$.

Now proceed by induction. Assume $r > 1$ and $r \leq m/2$, as hypothesized. The latter assumption ensures $H_{m+2r-1}(L(V_{\leq m+2r-2})) = H_{m+2r-2}(L(V_{\leq m+2r-3})) = 0$. Since, $d_{m+2r-2} = 0$ we have

$$\mathcal{C}^{m+2r-1} = \text{aut}(V_{m+2r-2}) \times \mathcal{E}(L(V_{\leq m+2r-3}, \partial_{\leq m+2r-3})).$$

Since, $H_{m+2r-2}(L(V_{\leq m+2r-3})) = 0$, by Theorem 2.6 and the induction hypothesis we obtain

$$\mathcal{E}(L(V_{\leq m+2r-2}, \partial_{\leq m+2r-2})) \cong \text{aut}(V_{m+2r-2}) \times \prod_{i=1}^{r-1} \text{aut}((G_i)_R),$$

Finally, note $\mathcal{C}^{m+2r}$ is the set of triples

$$(\xi_{m+2r-1}, \lambda_{m+2r-2}, \alpha) \in \text{aut}(V_{m+2r-1}) \times \text{aut}(V_{m+2r-2}) \times \prod_{i=1}^{r-1} \text{aut}((G_i)_R)$$

such that the pair $(\xi_{m+2r-1}, \lambda_{m+2r-2}) \in \text{aut}(V_{m+2r-1}) \times \text{aut}(V_{m+2r-2})$ gives a commutative diagram:

$$\begin{array}{ccc} V_{m+2r-1} & \xrightarrow{\xi_{m+2r-1}} & V_{m+2r-1} \\ \downarrow d_{m+2r-1} & & \downarrow d_{m+2r-1} \\ V_{m+2r-2} & \xrightarrow{\lambda_{m+2r-2}} & V_{m+2r-2} \end{array}$$

We conclude $\mathcal{C}^{m+2r} \cong \text{aut}((G_r)_R) \times \prod_{i=1}^{r-1} \text{aut}((G_i)_R)$. Theorem 2.6 and the fact that $H_{m+2r-1}(L(V_{\leq m+2r-2})) = 0$ now completes the induction and the proof. $\square$

We deduce the following direct consequence of Theorem 1 (2).

Corollary 3.2 *Let $R \subseteq \mathbb{Q}$ and $X = X^q = X^n \cup_{\alpha} \left(\bigcup_{i=1}^j e_i^q \right)$ satisfy Hypothesis 1.1. When $q = n+1$ suppose further that the linking homomorphism $\pi_{n+1}(X^{n+1}, X^n)_R \rightarrow \pi_n(X^n, X^{n-1})_R$ vanishes. Then $\mathcal{E}_*(X^n)_R = 0$ implies $\mathcal{E}_*(X^q)_R \cong \bigoplus_{i=1}^j \pi_q(X^n)_R$.* $\square$

We apply this result to give a calculation of $\mathcal{E}_*(X)_R$. Given R -modules H_i, H_j , let $[H_i, H_j] \subseteq (H_i \otimes H_j) \oplus (H_j \otimes H_i)$ denote the R -submodule given by:

$$[H_i, H_j] = R\langle x_i \otimes x_j - (-1)^{(i-1)(j-1)} x_j \otimes x_i \mid x_i \in H_i, x_j \in H_j \rangle$$

Here, we write $R\langle \rangle$ to denote a free R -module on given generators. Given subspaces $V, W \subseteq L$ with L a graded Lie algebra we similarly write

$$[V, W] = R\langle [v, w] \mid v \in V, w \in W \rangle.$$

We prove

Theorem 3.3 *Let $R \subseteq \mathbb{Q}$ and X in $\mathbf{CW}_m^{k+1}$ satisfy Hypothesis 1.1. Suppose X has a cellular decomposition of the form*

$$X^{m+1} \subset X^{m+2} \subset \dots \subset X^{2m} = X$$

such that the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \rightarrow \pi_r(X^r, X^{r-1})_R$ vanish for $r = m+2, \dots, 2m-1$. Suppose $\mathcal{E}_(X^{m+1})_R = 0$. Then*

$$\mathcal{E}_*(X^i)_R = 0, \text{ for } i \leq 2m-1 \text{ and } \mathcal{E}_*(X^{2m})_R \cong \bigoplus_{i=1}^j [H_{m+1}(X; R), H_{m+1}(X; R)],$$

where $j = \dim_R(H_{2m}(X, X_{2m-1}; R))$.

Proof By the freeness of the Anick model as DG Lie algebra over R we obtain

$$H_{m+1}(L(V_{\leq m})) = H_{m+2}(L(V_{\leq m+1})) = \dots = H_{2m-1}(L(V_{\leq 2m-2})) = 0.$$

Since, by hypothesis, $\mathcal{E}_*(X^{m+1})_R = 0$, applying Corollary 3.2 repeatedly gives

$$\mathcal{E}_*(X^{m+2})_R = \mathcal{E}_*(X^{m+3})_R = \cdots = \mathcal{E}_*(X^{2m-1})_R = 0.$$

Applying this result again then gives

$$\mathcal{E}_*(X^{2m})_R \cong \bigoplus_{i=1}^j \pi_{2m+1}(X)_R.$$

Using the Anick model, we compute

$$\pi_{2m+1}(X)_R \cong [V_m, V_m] \cong [H_{m+1}(X; R), H_{m+1}(X; R)].$$

□

The following result was stated as Example 1.3 in the introduction.

Corollary 3.4 *Let G be a finitely generated abelian group and $R \subseteq \mathbb{Q}$ with least invertible prime p . Let X be in $\mathbf{CW}_m^{k+1}$ as in Hypothesis 1.1 with cellular decomposition of the form*

$$M(G, m+1) = X^{m+2} \subset X^{m+3} \subset \cdots \subset X^{2m} = X.$$

Assume that the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \rightarrow \pi_r(X^r, X^{r-1})_R$ vanish for $r = m+2, \dots, 2m-1$ and that $\dim_R(H_{2m}(X, X^{2m-1}; R)) = j$. Then

$$\mathcal{E}_*(X)_R \cong \bigoplus_{i=1}^j [G_R, G_R]$$

where $G_R = H_{m+1}(M(G, m+1); R)$ is of degree $m+1$.

Proof By [5, Theorem 3.2], $\mathcal{E}_*(M(G, m+1)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$. By Hypothesis 1.1, the prime $p = 2$ is invertible in R . Thus $\mathcal{E}_*(M(G, m+1))_R = 0$. The result now follows from Theorem 3.3. □

We can easily compute both groups for a tame product of spheres. The following was stated as Example 1.4 in the introduction.

Theorem 3.5 *Let $R \subseteq \mathbb{Q}$ be a ring with least invertible prime p . Let $m \leq n$ be chosen so that $m, k = m+n+1$ and p satisfy Hypothesis 1.1. Then*

$$\mathcal{E}((S^{m+1} \times S^{n+1})_R) \cong \begin{cases} R^* \times R^* & \text{for } n \neq 2m \text{ or } m \text{ even.} \\ R \oplus (R^* \times R^*) & \text{for } n = 2m \text{ and } m \text{ odd} \end{cases}$$

$$\mathcal{E}_*(S^{m+1} \times S^{n+1})_R \cong \begin{cases} 0 & \text{for } n \neq 2m \text{ or } m \text{ even.} \\ R & \text{for } n = 2m \text{ and } m \text{ odd.} \end{cases}$$

Proof Let $X = S^{m+1} \times S^{n+1}$. We can write the Anick model as $(L(u, v, w), \partial)$ where $|u| = m$, $|v| = n$, $|w| = n + m + 1$ with $\partial(u) = \partial(v) = 0$ and $\partial(w) = [u, v]$. Theorem 2.6 gives a short exact sequence

$$0 \rightarrow H_{m+n+1}(L(u, v)) \rightarrow \mathcal{E}(L(u, v, w), \partial) \rightarrow \mathcal{C}^{m+n+2} \rightarrow 0.$$

For degree reasons, $H_{m+n+1}(L(u, v)) = 0$. Thus $\mathcal{E}(X_R) \cong \mathcal{C}^{m+n+2}$ and we compute the latter group.

The group $\mathcal{C}^{m+n+2}$ consists of pairs $(\xi, [\alpha]) \in \text{aut}(R\langle w \rangle) \times \mathcal{E}(L(u, v), 0)$ such that the following diagram commutes:

$$\begin{array}{ccc} R\langle w \rangle & \xrightarrow{\xi} & R\langle w \rangle \\ B_{m+n+1} \downarrow & & \downarrow B_{m+n+1} \\ R\langle [u, v] \rangle & \xrightarrow{H_{m+n}(\alpha)} & R\langle [u, v] \rangle \end{array}$$

The map ξ is determined by $H_{m+n}(\alpha)$. Thus $\mathcal{C}^{m+n+2} \cong \mathcal{E}(L(u, v), 0)$. Applying Theorem 2.6 again gives a short exact sequence:

$$0 \rightarrow H_n(L(u)) \rightarrow \mathcal{E}(L(u, v), 0) \rightarrow \mathcal{C}^{n+1} \rightarrow 0.$$

The sequence splits since the differential vanishes. As above, $\mathcal{C}^{n+1} \cong \text{aut}(L(u, v)) \cong R^* \times R^*$. If $n \neq 2m$ or m odd then $H_n(L(u)) = 0$. Otherwise, for $n = 2m$ and m even, $H_n(L(u)) = R$. The result for $\mathcal{E}(X_R)$ follows.

The proof for $\mathcal{E}_*(X)_R$ is similar. Theorem 2.9 gives $\mathcal{E}_*(L(u, v, w), \partial) \cong \mathcal{C}_*^{m+n+2}$. As above, $\mathcal{C}_*^{m+n+2} \cong \mathcal{E}_*(L(u, v); 0)$. Applying Theorem 2.9 again gives $\mathcal{E}_*(L(u, v); 0) \cong H_n(L(u))$ and the result follows. $\square$

We next give a general result showing that finiteness of $\mathcal{E}(X_R)$ is not preserved by cell attachments in consecutive degrees:

Theorem 3.6 *Let $R \subseteq \mathbb{Q}$ and X in $\mathbf{CW}_m^{k+1}$ be as in Theorem 1. Suppose $X = X^n \cup (\bigcup_{i=1}^j e^{n+1})$ for $j > 0$. Assume that the linking homomorphism $\pi_{n+1}(X^{n+1}, X^n)_R \rightarrow \pi_n(X^n, X^{n-1})_R$ vanishes. Then $\mathcal{E}(X_R^n)$ finite implies $\mathcal{E}(X_R^{n+1})$ is infinite.*

Proof Our hypothesis on the linking homomorphism ensures $d_n: V_n \rightarrow V_{n-1}$ is zero. Thus given $v \in V_n$, $\partial(v) \in L^{n-1}(V_{<n-1})$. Since, $\mathcal{E}(X_R^n) \cong \mathcal{E}(L(V_{<n}), \partial_{<n})$ is finite, applying Theorem 2.6 gives $H_{n-1}(L(V_{<n-1})) = 0$. It follows that the map $B_n = 0: V_n \rightarrow H_{n-1}(L(V_{<n}))$. Given $v \in V_n$, set $\gamma = \text{id}: (L(V_{<n}), \partial) \rightarrow (L(V_{<n}), \partial)$ and $\xi^a \in \text{aut}(V_n)$, $a \in R$ with $\xi^a(v) = av$ for $v \in V_n$. The following diagram is obviously commutative:

$$\begin{array}{ccc} V_n & \xrightarrow{\xi^a} & V_n \\ B_n=0 \downarrow & & \downarrow B_n=0 \\ H_{n-1}(L(V_{<n})) & \xrightarrow{H(\alpha)=\text{id}} & H_{n-1}(L(V_{<n})). \end{array}$$

Therefore, there exists an infinity of pairs $(\xi^a, [\text{id}]) \in \mathcal{C}_{n+1}$. Since, $\mathcal{C}_{n+1}$ is infinite, $\mathcal{E}(X_R^{n+1}) \cong \mathcal{E}(L(V_{\leq n}, \partial_{\leq n}))$ is infinite by Theorem 2.6. $\square$

When $R = \mathbb{Q}$ the result becomes:

Corollary 3.7 *Let X be a finite, simply connected CW complex. Suppose that $\pi_{n+1}(X^{n+1}, X^n)_{\mathbb{Q}} \neq 0$. Then $\mathcal{E}(X_{\mathbb{Q}}^n)$ finite implies $\mathcal{E}(X_{\mathbb{Q}}^{n+1})$ is infinite.* $\square$

Finally, we show $\mathcal{E}(X_R)$ is infinite for CW complexes of small dimension.

Theorem 3.8 *Let X be a simply-connected finite CW complex of dimension ≤ 5 . Let $R \subseteq \mathbb{Q}$ have least invertible prime $p \geq 7$. If p is finite, assume the linking homomorphisms $\pi_{r+1}(X^{r+1}, X^r)_R \rightarrow \pi_r(X^r, X^{r-1})_R$ vanish for $r = 2, 3, 4$. Then $\mathcal{E}(X_R)$ is infinite.*

Proof Let $(L(V), \partial)$ denote the Anick model for X . Our hypothesis on the linking homomorphism implies $d = 0$. Since, X_R is not contractible, $V \neq 0$.

When $X = X^3$ then for degree reasons $\partial = 0$. It follows that $\mathcal{E}(X_R^3) \cong \text{aut}(V)$ which is infinite since $V \neq 0$.

Next suppose $X = X^4$. By Theorem 1 (1), it suffices to show that $\mathcal{C}^4$ is infinite. Here, $V = V_3 \oplus V_2 \oplus V_1$ with $\partial(V_3) \subseteq [V_1, V_1]$ and at least one $V_i \neq 0$. The group $\mathcal{C}^4$ consists of pairs (ξ, α) where $\alpha: L(V_2 \oplus V_1) \rightarrow L(V_2 \oplus V_1)$ is an automorphism and $\xi: V_3 \rightarrow V_3$ makes the diagram commute:

$$\begin{array}{ccc} V_3 & \xrightarrow{\xi} & V_3 \\ \partial_3 \downarrow & & \downarrow \partial_3 \\ [V_1, V_1] & \xrightarrow{\alpha} & [V_1, V_1] \end{array}$$

Given $a \neq 0$, define $\alpha^a(v) = av$ for $V_{\leq 2}$. Define ξ^{a^2} by $\xi^{a^2}(v) = a^2v$ for $v \in V_3$. This gives an infinity of distinct pairs (α^a, ξ^{a^2}) in $\mathcal{C}^4$.

For the case $X = X^5$ we have $V = V_{\leq 4}$. Again $\partial_3(V_3) \subseteq [V_1, V_1]$ by minimality. We identify $H_4(L(V_{\leq 3}))$ as vector space:

$$H_4(L(V_{\leq 3})) = [\ker \partial_3, V_1] \oplus [V_2, V_2] \oplus \frac{[V_2, [V_1, V_1]]}{[\partial_3(V_3), V_2]} \oplus \frac{[V_1, [V_1, [V_1, V_1]]]}{[\partial_3(V_3), [V_1, V_1]]}.$$

By Theorem 2.6, if $H_4(L(V_{\leq 3})) \neq 0$ then $\mathcal{E}(X_{\mathbb{Q}}^5)$ is infinite. Thus we may assume $H_4(L(V_{\leq 3})) = 0$ which forces $\dim V_2 \leq 1$. If $V_1 = 0$ then $\partial = 0$ and $\mathcal{E}(X_R) \cong \text{aut}(V)$ is infinite. So assume $V_1 \neq 0$. Then we must have $\ker \partial_3 = 0$ and $\partial_3(V_3) = [V_1, V_1]$. Again, it suffices to show $\mathcal{C}^5$ is infinite. We note that

$$H_3(L(V_{\leq 3})) = \ker \partial_3 \oplus [V_2, V_1] \oplus \frac{[V_1, [V_1, V_1]]}{[\partial_3(V_3), V_1]} \cong [V_2, V_1]$$

Then $\mathcal{C}^5$ is the set of pairs $(\xi, [\alpha])$ with $\xi \in \text{aut}(\mathbb{Q}\langle w \rangle)$ and $\alpha \in \text{aut}(L(V_{\leq 3}), \partial_{\leq 3})$ making the diagram commute:

$$\begin{array}{ccc} V_4 & \xrightarrow{\xi} & V_4 \\ \partial_4 \downarrow & & \downarrow \partial_4 \\ [V_2, V_1] & \xrightarrow{H(\alpha)} & [V_2, V_1]. \end{array}$$

Given $a \in R^*$ define a DG Lie map $\alpha^a: (L(V_{\leq 3}), \partial_{\leq 3}) \rightarrow (L(V_{\leq 3}), \partial_{\leq 3})$ by setting

$$\alpha(v) = av \text{ for } v \in V_1 \quad \text{and} \quad \alpha(u) = a^2u \text{ for } u \in V_3 \oplus V_2,$$

and extending. We then obtain an infinity of pairs $(\xi^{a^3}, [\alpha^a]) \in \mathcal{C}^5$ where $\alpha^{a^3}(x) = a^3x$ for $x \in V_4$. $\square$

We conclude by proposing a problem. By Costoya–Viruel [10], every finite group G occurs as $\mathcal{E}(X_{\mathbb{Q}})$. The construction of X for a given G requires cells (cohomology classes) in a wide range of dimensions. This suggests the following:

Problem 3.9 *Given $R \subseteq \mathbb{Q}$, find the smallest $n \geq 1$ such that there exists a simply connected CW complex X with $\dim X \leq n$, X_R non-contractible and $\mathcal{E}(X_R)$ is finite.*

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