Brouwer degree of polynomial maps

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Dedicated to Hvedri Inassaridze on his 80th birthday

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Abstract Let $\mathbb{S}^n$ be the n -sphere. It is well known that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$, and if n is even, $\pi_{2n-1}(\mathbb{S}^n) = \mathbb{Z} \oplus G$, where $\mathbb{Z}$ is the infinite cyclic group and G a finite abelian group. This allows us to associate an integer to any continuous map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$, its Brouwer degree $B(f)$, and to any continuous map $g : \mathbb{S}^{2n-1} \rightarrow \mathbb{S}^n$ for even n its Hopf degree $h(g)$. The purpose of this paper is to define Brouwer and Hopf degree for polynomial maps on spheres over a field of characteristic zero.

Keywords Brouwer degree · de Rham cohomology · Hopf degree · Polynomial map · Regular (smooth) function

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1 Introduction

If we take $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ smooth and use de Rham differential forms and de Rham cohomology, the induced map f^* in $H^n(\mathbb{S}^n)$ is multiplication by the Brouwer degree $B(f)$. Also, if we choose the usual n -form α_n , representing the positive orientation on $\mathbb{S}^n$ whose integral over the sphere is one, then the Hopf degree $h(g)$ of a smooth map $g : \mathbb{S}^{2n-1} \rightarrow \mathbb{S}^n$ is given by the formula

$$[\beta_{n-1} g^*(\alpha_n)] = h(g)[\alpha_{2n-1}],$$

where $d\beta_{n-1} = g^*(\alpha_n)$, and $[-]$ means a cohomology class.

Recall that a polynomial map $p = (p_0, \dots, p_n) : \mathbb{S}^m \rightarrow \mathbb{S}^n$ is called a *form* of degree k if each polynomial $p_0, \dots, p_n$ over the reals $\mathbb{R}$ is homogeneous of degree k . The result presented in [9, Theorem 1] states: *If n is odd then the element of the infinite cyclic group $\pi_n(\mathbb{S}^n)$ corresponding to the integer k can be represented by a form of degree $|k|$ mapping $\mathbb{S}^n$ to $\mathbb{S}^n$.* Further, the well-known Hopf maps $\mathbb{S}^3 \rightarrow \mathbb{S}^2$, $\mathbb{S}^7 \rightarrow \mathbb{S}^4$ and $\mathbb{S}^{15} \rightarrow \mathbb{S}^8$ are given by forms of degree 2. In particular, elements of the infinite cyclic group $\pi_3(\mathbb{S}^2)$ are represented by forms.

If $\mathbb{C}$ is the field of complex numbers, then the complex n -sphere

$$\mathbb{S}^n(\mathbb{C}) = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1}; z_0^2 + \dots + z_n^2 = 1 \right\}$$

has the homotopy type of $\mathbb{S}^n$ as diffeomorphic to the tangent bundle $T\mathbb{S}^n$ and by [10, Theorem 2]: *Every element in $\pi_n(\mathbb{S}^n)$ can be represented by a complex polynomial map of $\mathbb{S}^n(\mathbb{C})$ for all positive integers n .*

This inspires us to present the notions above in a more algebraic setting. We replace the reals $\mathbb{R}$ by a field $\mathbb{F}$ of characteristic zero; the algebra of smooth functions $C^\infty(\mathbb{B}^n)$ on the n -ball $\mathbb{B}^n$, by a commutative $\mathbb{F}$ -algebra $\mathcal{A}$ such that $\mathbb{F}[X_1, \dots, X_n] \subseteq \mathcal{A}$; the $C^\infty(\mathbb{B}^n)$ -free module $\Omega^1(\mathbb{B}^n)$ of smooth 1-forms on $\mathbb{B}^n$, by an $\mathcal{A}$ -free module $\Omega^1(\mathcal{A})$ together with an $\mathbb{F}$ -linear map $d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A})$ such that $d(ab) = adb + bda$ for all $a, b \in \mathcal{A}$ and with $dX_1, \dots, dX_n$ being a basis. Finally, the algebra of real smooth functions $C^\infty(\mathbb{S}^n)$ is replaced by $\mathcal{B} = \mathcal{A}[X_0, \dots, X_n]/(\sum_{j=0}^n X_j^2 - 1)$.

In the second section, we give several examples and show in particular that

$$C^\infty(\mathbb{B}^n)[X_0]/\left(\sum_{j=0}^n X_j^2 - 1\right) \xrightarrow{\cong} C^\infty(\mathbb{S}^n)$$

is an isomorphism.

We associate the “de Rham cohomology” to $\mathcal{B}$ and give sufficient conditions so that $H^k(\mathcal{B}) \neq 0$ for $k = 0, n$ and both $H^0(\mathcal{B})$ and $H^n(\mathcal{B})$ are isomorphic to $\mathbb{F}$.

Then, we define the Brouwer degree $B(p)$ of a polynomial map $p : \mathbb{S}^n(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$, and prove that $B(p)$ is an integer.

The third section imitates the classical case to define a Hopf degree $h(q)$ for any polynomial map $q : \mathbb{S}^{2n-1}(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$ for even n and then we show that $h(q)$ is an integer.

2 Brouwer degree

Let $\mathbb{F}$ be a field of characteristic zero and $\mathcal{A}$ a commutative $\mathbb{F}$ -algebra such that the polynomial ring $\mathbb{F}[X_1, \dots, X_n] \subseteq \mathcal{A}$. Assume that there is also given an $\mathcal{A}$ -free module $\Omega^1(\mathcal{A})$, together with an $\mathbb{F}$ -linear map $d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A})$ such that $d(ab) = adb + bda$ for $a, b \in \mathcal{A}$ and with $dX_1, \dots, dX_n$ being a basis of $\Omega^1(\mathcal{A})$.

Given $a \in \mathcal{A}$, define $\frac{\partial a}{\partial X_i}$ for $i = 1, \dots, n$ by the formula $da = \sum_{j=1}^n \frac{\partial a}{\partial X_j} dX_j$. We further suppose that $\frac{\partial}{\partial X_i} \frac{\partial a}{\partial X_j} = \frac{\partial}{\partial X_j} \frac{\partial a}{\partial X_i}$ for $1 \leq i < j \leq n$. This implies, in particular, that d can be extended to a differential on the exterior powers $\Omega^k(\mathcal{A})$ of $\Omega^1(\mathcal{A})$ for $k \geq 1$.

Example 2.1 1. For $\mathcal{A} = \mathcal{A}_n = \mathbb{F}[X_1, \dots, X_n]$, take $\Omega^1(\mathcal{A}_n)$ as the $\mathcal{A}_n$ -module of Kähler differentials.

2. For the field of reals $\mathbb{R}$, take $\mathcal{A} = C^\infty(\mathbb{R}^n)$, the ring of smooth functions on $\mathbb{R}^n$ or $\mathcal{A} = C^\infty(\mathbb{B}^n)$, the ring of smooth functions on the n -ball $\mathbb{B}^n$ and $\Omega^1(\mathcal{A})$ the module of smooth 1-forms on $\mathbb{R}^n$ or $\mathbb{B}^n$ respectively and d the exterior derivative.
3. The ring $\mathcal{A}$ of regular functions on $\mathbb{F}^n$ and $\Omega^1(\mathcal{A})$ the $\mathcal{A}$ -module of Kähler differentials.

Then, consider the $\mathbb{F}$ -algebra $\mathcal{B} = \mathcal{A}[X_0]/(\sum_{j=0}^n X_j^2 - 1)$. Notice that there is an inclusion $\mathcal{A} \subseteq \mathcal{B}$ and elements of $\mathcal{B}$ are uniquely of the form $\alpha + X_0\beta$ with $\alpha, \beta \in \mathcal{A}$ and

$$(\alpha + X_0\beta)(\alpha' + X_0\beta') = \alpha\alpha' + \beta\beta' \left(1 - \sum_{j=1}^n X_j^2\right) + X_0(\alpha\beta' + \beta\alpha')$$

in the ring $\mathcal{B}$.

For $\mathcal{A} = \mathcal{A}_n$, the ring $\mathcal{B} = \mathcal{A}[X_0]/(\sum_{j=0}^n X_j^2 - 1)$ coincides with polynomial functions

$$\mathbb{F}[X_0, \dots, X_n]/\left(\sum_{j=0}^n X_j^2 - 1\right)$$

on the n -sphere

$$\mathbb{S}^n(\mathbb{F}) = \{(x_0, \dots, x_n) \in \mathbb{F}^{n+1}; x_0^2 + \dots + x_n^2 = 1\}$$

over $\mathbb{F}$. However, for the ring $\mathcal{A}$ of regular functions on $\mathbb{F}^n$, the ring $\mathcal{B} = \mathcal{A}[X_0]/(\sum_{j=0}^n X_j^2 - 1)$ coincides with regular functions on $\mathbb{S}^n(\mathbb{F})$.

Now, write $C^\infty(\mathbb{S}^n)$ for the ring of smooth functions on the n -sphere $\mathbb{S}^n$. Because of the inclusions $C^\infty(\mathbb{R}^n)[X_0] \subseteq C^\infty(\mathbb{R}^{n+1})$ and $C^\infty(\mathbb{B}^n)[X_0] \subseteq C^\infty(\mathbb{B}^{n+1})$, the restriction maps

$$C^\infty(\mathbb{R}^n)[X_0] \rightarrow C^\infty(\mathbb{S}^n), \quad \text{and} \quad C^\infty(\mathbb{B}^n)[X_0] \rightarrow C^\infty(\mathbb{S}^n)$$

lead to monomorphisms

$$C^\infty(\mathbb{R}^n)[X_0]/I \longrightarrow C^\infty(\mathbb{S}^n) \quad \text{and} \quad C^\infty(\mathbb{B}^n)[X_0]/\left(\sum_{j=0}^n X_j^2 - 1\right) \longrightarrow C^\infty(\mathbb{S}^n)$$

with a factorisation

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n)[X_0]/(\sum_{j=0}^n X_j^2 - 1) & \longrightarrow & C^\infty(\mathbb{S}^n) \\ \downarrow & \nearrow & \\ C^\infty(\mathbb{R}^n)[X_0]/I, & & \end{array}$$

where the ideal $I \subseteq C^\infty(\mathbb{R}^n)[X_0]$ is generated by functions in $C^\infty(\mathbb{R}^n)$ vanishing on $\mathbb{B}^n$.

To show the surjectivity of $C^\infty(\mathbb{B}^n)[X_0]/(\sum_{j=1}^n X_j^2 - 1) \longrightarrow C^\infty(\mathbb{S}^n)$, we recall:

Lemma 2.2 ([2, Chapter VI, Theorem 5.1]) *Let $f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$. If the function*

$$\tilde{f} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

given by $\tilde{f}(x_0, x_1, \dots, x_n) = f(x_0^2, x_1, \dots, x_n)$ for $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ is smooth then f is smooth as well.

Now, we are ready to show:

Proposition 2.3 *If $f : \mathbb{S}^n \rightarrow \mathbb{R}$ is a smooth function, then there exist unique real smooth functions $f_0, f_1 : \mathbb{B}^n \rightarrow \mathbb{R}$ such that*

$$f(x_0, \dots, x_n) = f_0(x_1, \dots, x_n) + x_0 f_1(x_1, \dots, x_n)$$

for $(x_0, \dots, x_n) \in \mathbb{S}^n$.

Proof Notice that $(\pm\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n) \in \mathbb{S}^n$ for any $(x_1, \dots, x_n) \in \mathbb{B}^n$.

Then, the uniqueness of smooth functions $f_0, f_1 : \mathbb{B}^n \rightarrow \mathbb{R}$ is clear because

$$f\left(\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n\right) = f_0(x_1, \dots, x_n) + \sqrt{1 - \sum_{j=1}^n x_j^2} f_1(x_1, \dots, x_n)$$

and

$$f\left(-\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n\right) = f_0(x_1, \dots, x_n) - \sqrt{1 - \sum_{j=1}^n x_j^2} f_1(x_1, \dots, x_n)$$

for all $(x_1, \dots, x_n) \in \mathbb{B}^n$.

To show an existence, given a smooth $f : \mathbb{S}^n \rightarrow \mathbb{R}$, we define:

$$f_0(x_1, \dots, x_n) = \frac{1}{2} \left(f \left(\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n \right) + f \left(-\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n \right) \right)$$

for any $(x_1, \dots, x_n) \in \mathbb{B}^n$ and

$$f_1(x_1, \dots, x_n) = \frac{1}{2\sqrt{1 - \sum_{j=1}^n x_j^2}} \left(f \left(\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n \right) - f \left(-\sqrt{1 - \sum_{j=1}^n x_j^2}, x_1, \dots, x_n \right) \right)$$

for any $(x_1, \dots, x_n) \in \mathring{\mathbb{B}}^n$, the interior of $\mathbb{B}^n$.

Observe that $f_0(x_1, \dots, x_n) = f(0, x_1, \dots, x_n)$ whenever $\sum_{j=1}^n x_j^2 = 1$ and it is obvious that f_0 and f_1 are smooth in the interior of $\mathbb{B}^n$. It remains to show that f_1 extend continuously to the frontier of $\mathbb{B}^n$ and both f_0 and f_1 are also smooth on this frontier.

Now, in view of Tubular Neighborhood Theorem [7, Chapter 4, Theorem 5.1], $f : \mathbb{S}^n \rightarrow \mathbb{R}$ extends to a smooth function $\tilde{f} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. Next, denote by $\mathbb{R}^+$ the set of nonnegative reals and define $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$g(x_0, \dots, x_n) = \frac{1}{2} (\tilde{f}(\sqrt{x_0}, x_1, \dots, x_n) + \tilde{f}(-\sqrt{x_0}, x_1, \dots, x_n)),$$

for $(x_0, \dots, x_n) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $\tilde{g} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by

$$\tilde{g}(x_0, \dots, x_n) = \frac{1}{2} (\tilde{f}(x_0, x_1, \dots, x_n) + \tilde{f}(-x_0, x_1, \dots, x_n)) = g(x_0^2, x_1, \dots, x_n)$$

for $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$.

Because $\tilde{g}$ is smooth, Lemma 2.2 yields that g is smooth as well. Then,

$$f_0(x_1, \dots, x_n) = g \left(1 - \sum_{j=1}^n x_j^2, x_1, \dots, x_n \right)$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$ yields that $f_0 : \mathbb{B}^n \rightarrow \mathbb{R}$ is also a smooth function.

Next, define a smooth function $\tilde{h} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by

$$\tilde{h}(x_0, \dots, x_n) = \frac{1}{2}(\tilde{f}(x_0, x_1, \dots, x_n) - \tilde{f}(-x_0, x_1, \dots, x_n))$$

for $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$.

Notice that

$$\tilde{f} = \tilde{g} + \tilde{h}$$

and

$$\tilde{h}(x_0, \dots, x_n) = x_0 f_1(x_1, \dots, x_n)$$

provided $\sum_{j=0}^n x_j = 1$ and $x_0 \neq 0$.

Furthermore, we have

$$\frac{\partial \tilde{h}}{\partial x_0}(0, x_1, \dots, x_n) = \lim_{x_0 \rightarrow 0} \frac{\tilde{h}(x_0, \dots, x_n)}{x_0}.$$

It is clear that the function $\bar{h} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ given by

$$\bar{h}(x_0, \dots, x_n) = \begin{cases} \frac{\tilde{h}(x_0, \dots, x_n)}{x_0}, & \text{if } x_0 \neq 0 \\ \frac{\partial \tilde{h}}{\partial x_0}(0, x_1, \dots, x_n) & \text{if } x_0 = 0 \end{cases}$$

for $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ is smooth and

$$\tilde{h}(x_0, \dots, x_n) = x_0 \bar{h}(x_0, \dots, x_n)$$

for all $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$.

Observe that $\bar{h}(x_0, \dots, x_n) = f_1(x_1, \dots, x_n)$ provided $\sum_{j=0}^n x_j^2 = 1$ and $x_0 \neq 0$. Therefore f_1 extends continuously to the frontier of $\mathbb{B}^n$ by defining $f_1(x_1, \dots, x_n) = \bar{h}(0, x_1, \dots, x_n)$ whenever $\sum_{j=1}^n x_j^2 = 1$.

At the end, defining $h : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$h(x_0, \dots, x_n) = \bar{h}(\sqrt{x_0}, x_1, \dots, x_n) = \bar{h}(-\sqrt{x_0}, x_1, \dots, x_n)$$

for $(x_0, \dots, x_n) \in \mathbb{R}^+ \times \mathbb{R}^n$, we have

$$h(x_0^2, x_1, \dots, x_n) = \bar{h}(x_0, x_1, \dots, x_n)$$

for all $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$. Then, Lemma 2.2 implies that h is smooth. Because $f_1(x_1, \dots, x_n) = h(1 - \sum_{j=1}^n x_j^2, x_1, \dots, x_n)$ for $(x_1, \dots, x_n) \in \mathbb{B}^n$, we deduce that the function f_1 is smooth and the proof is complete. $\square$

Because $\mathbb{B}^n$ is compact, by means of the smooth Tietze–Urysohn extension theorem, the restriction map $C^\infty(\mathbb{R}^n)[X_0] \rightarrow C^\infty(\mathbb{B}^n)[X_0]$ is an epimorphism; we conclude:

Corollary 2.4 *The restriction maps $C^\infty(\mathbb{B}^n)[X_0] \rightarrow C^\infty(\mathbb{S}^n)$ and $C^\infty(\mathbb{R}^n)[X_0] \rightarrow C^\infty(\mathbb{S}^n)$ lead to isomorphisms*

$$C^\infty(\mathbb{B}^n)[X_0]/\left(\sum_{j=0}^n X_j^2 - 1\right) \xrightarrow{\cong} C^\infty(\mathbb{S}^n) \quad \text{and} \quad C^\infty(\mathbb{R}^n)[X_0]/I \xrightarrow{\cong} C^\infty(\mathbb{S}^n),$$

and an epimorphism

$$C^\infty(\mathbb{R}^n)[X_0]/\left(\sum_{j=0}^n X_j^2 - 1\right) \rightarrow C^\infty(\mathbb{S}^n)$$

with a factorisation

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n)[X_0]/(\sum_{j=0}^n X_j^2 - 1) & \longrightarrow & C^\infty(\mathbb{S}^n) \\ \downarrow & \nearrow \cong & \\ C^\infty(\mathbb{R}^n)[X_0]/I, & & \end{array}$$

where the ideal $I \subseteq C^\infty(\mathbb{R}^n)[X_0]$ is generated by functions in $C^\infty(\mathbb{R}^n)$ vanishing on $\mathbb{B}^n$.

Let now $\Omega^1(\mathcal{B}) = \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{B}$ be the $\mathcal{B}$ -module which, as an $\mathcal{A}$ -module, is free with the basis

$$dX_0, \dots, dX_n, X_0 dX_1, \dots, X_0 dX_n$$

and the obvious $\mathcal{B}$ -module structure, using the relations

$$\sum_{j=0}^n X_j^2 = 1 \quad \text{and} \quad \sum_{j=0}^n X_j dX_j = 0.$$

Notice that $\Omega^1(\mathcal{B}) = \Omega(\mathcal{B})$, the $\mathcal{B}$ -module of Kähler differentials for $\mathcal{A} = \mathcal{A}_n = \mathbb{F}[X_1, \dots, X_n]$ and $\Omega^1(\mathcal{B})$ is the $\mathcal{B}$ -module of real smooth 1-forms on $\mathbb{S}^n$ for $\mathcal{A} = C^\infty(\mathbb{B}^n)$.

We extend d in the natural way to $d : \mathcal{B} \rightarrow \Omega^1(\mathcal{B})$ and then to the exterior powers $\Omega^k(\mathcal{B})$ of $\Omega^1(\mathcal{B})$. Then, $\Omega^{n+1}(\mathcal{B}) = 0$ as the $\mathcal{B}$ -module generated by $dX_0 \cdots dX_n = 0$ and certainly $\Omega^k(\mathcal{B}) = 0$ for $k > n + 1$. In fact,

$$X_j dX_0 \cdots dX_n = dX_0 \cdots (-X_0 dX_0 - \cdots - \widehat{X_j dX_j} - \cdots - X_n dX_n) \cdots dX_n = 0$$

$$\text{and so } dX_0 \cdots dX_n = (\sum_{j=0}^n X_j^2) dX_0 \cdots dX_n = 0.$$

Lemma 2.5 *The $\mathcal{B}$ -module $\Omega^n(\mathcal{B})$ is free and generated by*

$$\omega_n = \sum_{i=0}^n (-1)^i X_i dX_0 \cdots \widehat{dX_i} \cdots dX_n.$$

Proof Notice that $X_j \omega_n = (-1)^j dX_0 \cdots \widehat{dX_j} \cdots dX_n$ for $j = 0, \dots, n$. Given $\Phi \in \Omega^n(\mathcal{B})$, we have

$$\Phi = \sum_{i=0}^n \alpha_i dX_0 \cdots \widehat{dX_i} \cdots dX_n$$

for some $\alpha_i \in \mathcal{B}$ with $i = 1, \dots, n$. Thus,

$$\Phi = \left(\sum_{i=0}^n (-1)^i \alpha_i X_i \right) \omega_n.$$

Finally, suppose $(a + bX_0)\omega_n = 0$ with $a, b \in A$. Then,

$$0 = bdX_1 \cdots dX_n + X_0 adX_1 \cdots dX_n + (dX_0)a \sum_{j=1}^n (-1)^j X_j dX_1 \cdots \widehat{dX_j} \cdots dX_n$$

and so $a = b = 0$. □

Write $H^k(\mathcal{A})$, (resp. $H^k(\mathcal{B})$) for the cohomology of the complexes $\Omega^k(\mathcal{A})$ (resp. $\Omega^k(\mathcal{B})$) defined above. Then, we have the following:

Theorem 2.6 1. $H^k(\mathcal{B}) = 0$ for all $k > n$;
 2. $H^0(\mathcal{B}) = \mathbb{F}$ provided that $A \cap \bar{\mathbb{F}} = \mathbb{F}$, where $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$;
 3. $H^k(\mathcal{B}) = 0$ for $0 < k < n$ provided that $H^k(\mathcal{A}) = 0$ for all $k \geq 1$;
 4. $H^n(\mathcal{B}) \cong \mathbb{F}$ provided there is an $\mathbb{F}$ -linear map $\varphi : \mathcal{A} \rightarrow \mathbb{F}$ such that $\varphi(1) = 1$ and the kernel of φ is generated over $\mathbb{F}$ by the elements of the form $X_i a - (1 - \sum_{j=1}^n X_j^2) \frac{\partial a}{\partial X_i}$ for $i = 1, \dots, n$ and $a \in A$.

Proof 1. This is clear because $\Omega^k(\mathcal{B}) = 0$ for $k > n$.

2. Let α and β elements of $\mathcal{A}$ such that $d(\alpha + X_0\beta) = 0$. Therefore, $d\alpha = 0$, $d\beta = 0$ and $\beta = 0$ and, in view of [5, Proposition 7], there exist such $\lambda_0, \dots, \lambda_m \in \mathbb{F}$ with $\lambda_m \neq 0$, $m \geq 1$, that $\lambda_0 + \lambda_1\alpha + \dots + \lambda_m\alpha^m = 0$. Whence, $\alpha \in \bar{\mathbb{F}} \cap \mathcal{A} = \mathbb{F}$.
3. As a consequence of the relations $\sum_{j=0}^n X_j^2 = 1$ and $\sum_{j=0}^n X_j dX_j = 0$, we see that for $0 \leq k \leq n$, any element of $\Phi \in \Omega^k(\mathcal{B})$ can be written uniquely as a sum

$$\Phi = \Phi_1 + X_0\Phi_2 + dX_0\Phi_3,$$

with $\Phi_1, \Phi_2 \in \Omega^k(\mathcal{A}) \subseteq \Omega^k(\mathcal{B})$ and $\Phi_3 \in \Omega^{k-1}(\mathcal{A}) \subseteq \Omega^{k-1}(\mathcal{B})$.

Suppose now that $d\Phi = 0$. Then,

$$0 = d\Phi_1 + X_0 d\Phi_2 + dX_0(\Phi_2 - d\Phi_3).$$

It is clear that the above relation says nothing if $k = n$. But, for $0 \leq k < n$ we have $d\Phi_1 = 0$, $d\Phi_2 = 0$ and $\Phi_2 = d\Phi_3$. Therefore,

$$\Phi = \Phi_1 + d(X_0\Phi_3).$$

Because, by hypothesis that $H_{dR}^k(\mathcal{A}) = 0$ for $k > 0$, there exists $\Psi_1 \in \Omega^{k-1}(\mathcal{A}) \subseteq \Omega^{k-1}(\mathcal{B})$ such that $\Phi_1 = d\Psi_1$, so we get

$$\Phi = d(\Psi_1 + X_0\Phi_3).$$

4. First, observe that any element of $\Omega^{n-1}(\mathcal{B})$ is a sum of elements of the form:

- (a) $adX_1 \cdots \widehat{dX_i} \cdots dX_n$, for $a \in \mathcal{A}$ and $i = 1, \dots, n$;
- (b) $aX_0 dX_1 \cdots \widehat{dX_i} \cdots dX_n$, for $a \in \mathcal{A}$ and $i = 1, \dots, n$;
- (c) $adX_0 dX_1 \cdots \widehat{dX_i} \cdots \widehat{dX_j} \cdots dX_n$, for $a \in \mathcal{A}$ and $1 \leq i < j \leq n$.

Next, in view of Lemma 2.5, the map $\tilde{\varphi} : \Omega^n(\mathcal{B}) = \mathcal{B}\omega_n \rightarrow \mathbb{F}$, given by $\tilde{\varphi}((a + X_0b)\omega_n) = \varphi(a)$, for $a, b \in \mathcal{A}$ is a well-defined $\mathbb{F}$ -linear extension of $\varphi : \mathcal{A} \rightarrow \mathbb{F}$.

We show that $\tilde{\varphi} \circ d = 0$:

- (a) $\tilde{\varphi}(d(adX_1 \cdots \widehat{dX_i} \cdots dX_n)) = \varphi(0) = 0$;
- (b) $d(aX_0 dX_1 \cdots \widehat{dX_i} \cdots dX_n) = (-1)^{i-1} X_0 \frac{\partial a}{\partial X_i} dX_1 \cdots dX_n + adX_0 \cdots \widehat{dX_i} \cdots dX_n = (-1)^i X_i a \omega_n$ and so $\tilde{\varphi}(d(aX_0 dX_1 \cdots \widehat{dX_i} \cdots dX_n)) = 0$;
- (c) $d(adX_0 dX_1 \cdots \widehat{dX_i} \cdots \widehat{dX_j} \cdots dX_n) = (\frac{\partial a}{\partial X_i} dX_i + \frac{\partial a}{\partial X_j} dX_j) dX_0 dX_1 \cdots \widehat{dX_i} \cdots \widehat{dX_j} \cdots dX_n = (-1)^{i-1} \frac{\partial a}{\partial X_i} dX_0 \cdots \widehat{dX_j} \cdots dX_n + (-1)^j \frac{\partial a}{\partial X_j} dX_0 \cdots \widehat{dX_i} \cdots dX_n = (-1)^{i+j} (-X_j \frac{\partial a}{\partial X_i} + X_i \frac{\partial a}{\partial X_j}) \omega_n$.

But, $X_i \frac{\partial a}{\partial X_j} - (1 - \sum_{k=1}^n X_k^2) \frac{\partial}{\partial X_i} (\frac{\partial a}{\partial X_j})$, $X_j \frac{\partial a}{\partial X_i} - (1 - \sum_{k=1}^n X_k^2) \frac{\partial}{\partial X_j} (\frac{\partial a}{\partial X_i}) \in \ker \varphi$.

Hence, $\tilde{\varphi}(d(adX_0 dX_1 \cdots \widehat{dX_i} \cdots \widehat{dX_j} \cdots dX_n)) = \varphi(X_i \frac{\partial a}{\partial X_j} - X_j \frac{\partial a}{\partial X_i}) = \varphi((1 - \sum_{k=1}^n X_k^2) (\frac{\partial}{\partial X_i} (\frac{\partial a}{\partial X_j}) - \frac{\partial}{\partial X_j} (\frac{\partial a}{\partial X_i}))) = 0$ and the proof is complete. $\square$

Remark 2.7 In Example 1.1(1), φ is the map appearing in [8] and in Example 1.1(2), $\varphi(a) = \int_{\mathbb{S}^n} a \omega_n$, for $a \in \mathcal{A}$, where ω_n is the volume n -form. Also, in both cases, it is well known that $H^k(\mathcal{A}) = 0$ for any $k > 0$ (see [1] for the case of polynomial algebra), and the elements of the form $X_i a - (1 - \sum_{j=1}^n X_j^2) \frac{\partial a}{\partial X_i}$ generate the kernel of φ . Instead in Example 1.1(3) is not true that $H^k(\mathcal{A}) = 0$ for $k > 0$. For instance for $\mathbb{F}$ not containing the square root of -1 , $d(\frac{dX}{X^2+1}) = 0$, but $\frac{dX}{X^2+1}$ is not df for any $f \in \mathcal{A}$.

Given a polynomial map $p = (p_0, \dots, p_n) : \mathbb{S}^n(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$, we have $p^*([\omega_n]) = B(p)[\omega_n]$ with $B(p) \in \mathbb{F}$. We say that $B(p)$ is the *Brouwer degree* of p .

Observe that if $p_j = \sum a_{\alpha_0, \dots, \alpha_n}^j X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ with $j = 0, \dots, n$, where $a_{(\alpha_0, \dots, \alpha_n)}^j \in \mathbb{F}$ and $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ for the set $\mathbb{N}$ of natural numbers then

$$B(p) \in \mathbb{Q} \left[a_{\alpha}^j \right]_{0 \leq j \leq n; \alpha \in \mathbb{N}^{n+1}},$$

i.e., the Brouwer degree $B(p)$ is a polynomial over rationals $\mathbb{Q}$ on the coefficients of the p_j for $j = 0, \dots, n$.

In fact,

$$\begin{aligned} p^*(\omega_n) &= \sum_{j=0}^n (-1)^j p_j dp_0 \cdots \widehat{dp_j} \cdots dp_n \\ &= \sum_{j=0}^n (-1)^j p_j \left(\sum_{k=0}^n \frac{\partial p_0}{\partial X_k} dX_k \right) \cdots \left(\sum_{k=0}^n \frac{\partial p_j}{\partial X_k} dX_k \right) \cdots \left(\sum_{k=0}^n \frac{\partial p_n}{\partial X_k} dX_k \right). \end{aligned}$$

But

$$dX_0 \cdots \widehat{dX_j} \cdots dX_n = (-1)^j X_j \omega_n$$

and so there is a polynomial $P(X_0, \dots, X_n)$ such that

$$p^*(\omega_n) = P(X_0, \dots, X_n) \omega_n$$

where coefficients of P belong to $\mathbb{Q}[a_{\alpha}^j]_{0 \leq j \leq n; \alpha \in \mathbb{N}^{n+1}}$.

On the other hand

$$X_0^{\alpha_0} \cdots X_n^{\alpha_n} \omega_n \sim 0$$

if at least one α_j is odd.

If $\alpha_j = 2\beta_j$ with $j = 0, \dots, n$ then, in view of [8] it holds:

$$X_0^{\alpha_0} \cdots X_n^{\alpha_n} \omega_n \sim \begin{cases} \frac{\alpha_0! \cdots \alpha_n! n! (\beta_0 + \cdots + \beta_n + s)!}{\beta_0! \cdots \beta_n! s! (\alpha_0 + \cdots + \alpha_n + n)!} \omega_n & \text{if } n = 2s, \\ \frac{\alpha_0! \cdots \alpha_n!}{\beta_0! \cdots \beta_n!} \frac{s!}{2^{\alpha_0 + \cdots + \alpha_n} (\beta_0 + \cdots + \beta_n + s)!} \omega_n & \text{if } n = 2s + 1. \end{cases}$$

Theorem 2.8 *The Brouwer degree $B(p)$ of any polynomial map $p = (p_0, \dots, p_n) : \mathbb{S}^n(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$ is an integer, where $\mathbb{F}$ is any field of characteristic zero.*

Proof Notice that we may assume $p : \mathbb{S}^n(\mathbb{F}_0) \rightarrow \mathbb{S}^n(\mathbb{F}_0)$, where $\mathbb{F}_0$ is a finite extension of $\mathbb{Q}$. In particular, $\mathbb{F}_0$ is a subfield of some finite transcendental extension $\mathbb{F}_0 \subseteq \mathbb{C}(T_1, \dots, T_r)$ of the complex numbers $\mathbb{C}$.

Therefore, it is enough to show that for any $p = (p_0, \dots, p_n) : \mathbb{S}^n(\mathbb{C}(T_1, \dots, T_r)) \rightarrow \mathbb{S}^n(\mathbb{C}(T_1, \dots, T_r))$ the Brouwer degree $B(p) \in \mathbb{Z}$.

But, there is an open subset $U \subseteq \mathbb{C}^r$ such that for any $t = (t_1, \dots, t_r) \in U$ we may evaluate all coefficients of the p_j at t for $j = 0, \dots, n$ and we get $p(t) : \mathbb{S}^n(\mathbb{C}) \rightarrow$

$\mathbb{S}^n(\mathbb{C})$, i.e., p can be thought as a map $U \rightarrow \mathcal{P}_n(\mathbb{C})$, where $\mathcal{P}_n(\mathbb{C})$ is the set of polynomial maps of $\mathbb{S}^n(\mathbb{C})$ to itself.

Thus, $B(p)$ is now a rational function of t and we have $B(p)(t) = B(p(t))$.

Finally, since $B(p) : U \rightarrow \mathbb{C}$ is continuous and takes values on $\mathbb{Z}$, it is constant and so $B(p)$ is an integer. $\square$

Example 2.9 1. For a field $\mathbb{F}$, consider the ring $\mathbb{F}[i]$ with $i^2 = -1$. Next, let $p : \mathbb{S}^n(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$ be given by

$$p(x_0, \dots, x_n) = (p_k(x_0), q_k(x_0)x_1, \dots, q_k(x_0)x_n)$$

for $(x_0, \dots, x_n) \in \mathbb{S}^n(\mathbb{F})$, where k is a natural number and polynomials p_k, q_k are given by the formula

$$(x_0 + ix_1)^k = p_k(x_0) + ix_1q_k(x_0)$$

for $x_0^2 + x_1^2 = 1$. Then, we get

$$p^*(\omega_n) = q_k(x_0)^{n-1}\omega_{n-1} \sim \begin{cases} k\omega_n & \text{if } n \text{ is odd,} \\ \omega_n & \text{if } n \text{ is even and } k \text{ is odd,} \\ 0 & \text{if } n \text{ is even and } k \text{ is even.} \end{cases}$$

Therefore $B(p) = k$ if n is odd, $B(p) = 1$ if n is even and k odd, and $B(p) = 0$ if both n and k are even.

2. This example is taken from [3, p. 64]. First, notice that $i \notin \mathbb{F}$ implies $\mathbb{F}[i] = \mathbb{F}(i)$, the field extension of $\mathbb{F}$. For $\mathbb{F}$ of characteristic different from two and $n = 2m + 1$, define polynomial maps $p : \mathbb{F}(i)^{2m+2} \rightarrow \mathbb{F}(i)^{m+1}$ and $\bar{p} : \mathbb{F}(i)^{2m+2} \rightarrow \mathbb{F}(i)^{m+1}$ by

$$\begin{aligned} p(r_0, \dots, r_{2m+1}) &= (p_k(r_0) + iq_k(r_0)r_1, q_k(r_0)(r_2 + ir_3), \dots, q_k(r_0)(r_{2m} + ir_{2m+1})), \\ \bar{p}(r_0, \dots, r_{2m+1}) &= (p_k(r_0) + iq_k(r_0)r_1, q_k(r_0)(r_2 - ir_3), \dots, q_k(r_0)(r_{2m} - ir_{2m+1})) \end{aligned}$$

respectively, for $(r_0, \dots, r_{2m+1}) \in \mathbb{F}^{2m+2}$.

Next, consider a polynomial map $\tilde{p} : \mathbb{S}^{n+1}(\mathbb{F}(i)) \rightarrow \mathbb{S}^{n+1}(\mathbb{F}(i))$ given by

$$\begin{aligned} \tilde{p}(r_0, \dots, r_{2m+1}, r_{2m+2}) &= \left(\frac{1}{2} (A_k(r_{2m+2})\bar{p}(r) + A_k(-r_{2m+2})p(r)), \right. \\ &\quad \left. -\frac{i}{2} (A_k(-r_{2m+2})p(r) - A_k(r_{2m+2})\bar{p}(r)), r_{2m+2}R_k(r_{2m+2}^2) \right) \end{aligned}$$

for $(r_0, \dots, r_{2m+1}, r_{2m+2}) \in \mathbb{S}^{n+1}(\mathbb{F}(i))$, where $r = (r_0, \dots, r_n)$, $A_k \in \mathbb{Q}[X]$ is the polynomial of algebraic degree $k - 1$ given by

$$A_k(X)(1 + X)^k + A_k(-X)(1 - X)^k = 2$$

and $R_k \in \mathbb{Q}[X]$ is given by

$$(A_k(X) - 1)(1 + X)^k = XR_k(X^2).$$

Then one gets $\tilde{p}^*(\omega_{n+1}) \sim k\omega_{n+1}$ and so $B(\tilde{p}) = k$.

Remark 2.10 (a) $B : \mathcal{P}_n(\mathbb{F}) \rightarrow \mathbb{Z}$ is surjective for any field $\mathbb{F}$ and odd n , see [4] or [10] for explicit constructions.

(b) The image of $B : \mathcal{P}_n(\mathbb{F}) \rightarrow \mathbb{Z}$ contains all odd integers if $\mathbb{F}$ contains the field of real algebraic numbers $\mathbb{R}_{alg}$ and even n , see [4].

3 Hopf degree of polynomial maps

Let $q : \mathbb{S}^{2n-1}(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$ be a polynomial map. It is clear that $d(q^*(\omega_n)) = q^*(d\omega_n) = 0$ and then we use Theorem 2.6 to choose $\gamma_{n-1} \in \Omega^{n-1}(A_{2n-1}(\mathbb{F}))$ such that $d\gamma_{n-1} = q^*(\omega_n)$.

Define then $H(q) \in \mathbb{F}$ by the formula

$$[\gamma_{n-1}q^*(\omega_n)] = H(q)[\omega_{2n-1}].$$

By using again Theorem 2.6 it is clear the definition of $H(q)$ is correct.

If n is even and we use standard de Rham cohomology, the Hopf invariant $h(g)$ of a smooth map $g : \mathbb{S}^{2n-1} \rightarrow \mathbb{S}^n$ (see e.g., [6, p. 229]) is given by

$$\left[\frac{\gamma_{n-1}}{\text{vol}(\mathbb{S}^n)} g^* \left(\frac{\omega_n}{\text{vol}(\mathbb{S}^n)} \right) \right] = h(g) \left[\frac{\omega_{2n-1}}{\text{vol}(\mathbb{S}^{2n-1})} \right]$$

because we have to replace the volume form ω_n by the orientation form $\alpha_n = \frac{\omega_n}{\text{vol}(\mathbb{S}^n)}$ whose integral over the sphere is one. Therefore,

$$H(g) = \frac{\text{vol}^2(\mathbb{S}^n)}{\text{vol}(\mathbb{S}^{2n-1})} h(g) = \frac{4^n}{m \binom{n}{m}} h(g)$$

for $n = 2m$.

We use now the previous formula for defining the Hopf invariant $h(q)$ of any polynomial map $q : \mathbb{S}^{2n-1}(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$ for n even. This could be done because $\frac{\text{vol}^2(\mathbb{S}^n)}{\text{vol}(\mathbb{S}^{2n-1})}$ is then a rational number.

Of course, we know that for $\mathbb{F}$ being the real or complex field, $H(q) = 0$ for odd n and the Hopf invariant $h(q)$ is an integer for even n . Then, by using the same method of proof as that of Theorem 2.8, we have the following:

Theorem 3.1 *For a field $\mathbb{F}$ of characteristic zero and any even n , the Hopf degree $h(q)$ of any polynomial map $q : \mathbb{S}^{2n-1}(\mathbb{F}) \rightarrow \mathbb{S}^n(\mathbb{F})$ is an integer.*

Example 3.2 We mimic the Hopf maps $\mathbb{S}^3 \rightarrow \mathbb{S}^2$, $\mathbb{S}^7 \rightarrow \mathbb{S}^4$ and $\mathbb{S}^{15} \rightarrow \mathbb{S}^8$ to analyse the following polynomial maps.

1. Let $q : \mathbb{S}^3(\mathbb{F}) \rightarrow \mathbb{S}^2(\mathbb{F})$ be given by

$$q(x_0, x_1, x_2, x_3) = \left(2(x_0x_2 + x_1x_3), 2(-x_0x_3 + x_1x_2), x_0^2 + x_1^2 - x_2^2 - x_3^2 \right).$$

A straightforward calculation gives

$$q^*(\omega_2) = -4(dX_0dX_1 + dX_2dX_3) = d\gamma_1$$

with $\gamma_1 = -4(X_0dX_1 + X_2dX_3)$.

But,

$$\begin{aligned} dX_1dX_2dX_3 &= X_0\omega_3, & dX_0dX_2dX_3 &= -X_1\omega_3, & dX_0dX_1dX_3 &= X_2\omega_3, \\ dX_0dX_1dX_2 &= -X_3\omega_3 \end{aligned}$$

and

$$X_0^2\omega_3 \sim X_1^2\omega_3 \sim X_2^2\omega_3 \sim X_3^2\omega_3.$$

Therefore, we have

$$\gamma_1 q^*(\omega_2) = 16 \left(X_0^2 + X_2^2 \right) \omega_3 \sim 8 \left(X_0^2 + X_1^2 + X_2^2 + X_3^2 \right) \omega_3 = 8\omega_3.$$

Thus $H(q) = 8$ and so

$$h(q) = 1$$

as expected.

2. Consider the non-commutative and unitary ring $\mathbb{F}\{i, j, k\}$ with $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$. Given $s = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{F}\{i, j, k\}$, we write $|s|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$ and $\bar{s} = x_0 - ix_1 - jx_2 - kx_3$. Then, $s\bar{s} = |s|^2$, $|s_1s_2|^2 = |s_1|^2|s_2|^2$ for $s, s_1, s_2 \in \mathbb{F}\{i, j, k\}$ and

$$\mathbb{S}^7(\mathbb{F}) \cong \left\{ (s_0, s_1) \in \mathbb{F}\{i, j, k\} \times \mathbb{F}\{i, j, k\}; |s_0|^2 + |s_1|^2 = 1 \right\}.$$

Now, let $q : \mathbb{S}^7(\mathbb{F}) \rightarrow \mathbb{S}^4(\mathbb{F})$ be given by

$$q(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \left(2s_0\bar{s}_1, |s_0|^2 - |s_1|^2 \right)$$

for $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{S}^7(\mathbb{F})$, where $s_0 = x_0 + ix_1 + jx_2 + kx_3$ and $s_1 = x_4 + ix_5 + jx_6 + kx_7$.

Then, we get $H(q) = \frac{64}{3}$ and so $h(q) = 1$.

3. Next, consider the non-associative and unitary ring $\mathbb{F}\{e_1, \dots, e_7\}$, where products e_se_t are defined by the Cayley algebra rules for $s, t = 1, \dots, 7$. Given $c = x_0 + e_1x_1 + \dots + e_7x_7 \in \mathbb{F}\{e_1, \dots, e_7\}$, write $|c|^2 = x_0^2 + x_1^2 + \dots + x_7^2$ and $\bar{c} = x_0 - e_1x_1 - \dots - e_7x_7$. Then, $c\bar{c} = |c|^2$, $|c_1c_2|^2 = |c_1|^2|c_2|^2$ for $c_1, c_2 \in \mathbb{F}\{e_1, \dots, e_7\}$ and

$$\mathbb{S}^{15}(\mathbb{F}) \cong \left\{ (c_0, c_1) \in \mathbb{F}\{e_1, \dots, e_7\} \times \mathbb{F}\{e_1, \dots, e_7\}; |c_0|^2 + |c_1|^2 = 1 \right\}.$$

Now, let $q : \mathbb{S}^{15}(\mathbb{F}) \rightarrow \mathbb{S}^8(\mathbb{F})$ be given by

$$q(x_0, \dots, x_{15}) = \left(2c_0\bar{c}_1, |c_0|^2 - |c_1|^2 \right)$$

for $(x_0, \dots, x_{15}) \in \mathbb{S}^{15}(\mathbb{F})$, where $c_0 = x_0 + e_1x_1 + \dots + e_7x_7$ and $c_1 = x_8 + e_1x_9 + \dots + e_7x_{15}$.

Then, we get $H(q) = \frac{4^7}{70}$ and so $h(q) = 1$.

We close the paper with:

Remark 3.3 Observe that proofs of Theorems 2.8 and 3.1 use topology. It would be interesting having algebraic ones.

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