Model structure on the category of small topological categories

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Abstract We give a short proof of the existence of a cofibrantly generated Quillen model structure on the category of small topologically enriched categories, obtained by a transfer from Bergner’s model structure on simplicially enriched categories.

Keywords Model categories · ∞ -Categories

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1 Introduction

We describe a Quillen model structure on the category of small topological categories. This model structure is obtained through a transfer from Bergner’s model structure on the category of simplicial categories [4]. Bergner’s original argument for the existence of a model structure on simplicial categories can be adapted to topological categories; this has been carried out in [1]. The proof to be presented here is much shorter, since it relies on an explicit construction of path-objects for topological categories. Recently, Berger–Moerdijk [3] gave a general existence result for a model structure on V -enriched categories including the cases $V = \mathbf{Top}$ and $V = \mathbf{sSet}$. We hope that our construction of path-objects for topological categories nevertheless presents some

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interest. We implicitly use the fact that all topological categories are fibrant which is not the case in the simplicial setting.

2 The model structure

In this article, the category of compactly generated topological Hausdorff spaces will be denoted by $\mathbf{Top}$, which is a simplicial symmetric monoidal closed model category. Before stating the main theorem we introduce some notations and definitions.

Remark 2.1 A topological category is a category enriched over $\mathbf{Top}$. The category of all (small) topological categories is denoted by $\mathbf{Cat}_{\mathbf{Top}}$. The morphisms in $\mathbf{Cat}_{\mathbf{Top}}$ are the enriched functors. It is a complete and cocomplete category. The category $\mathbf{V}$ will be either the category $\mathbf{sSet}$ or $\mathbf{Top}$.

Definition 2.2 Let $\mathcal{U} : \mathbf{V} \rightarrow \mathbf{Cat}_{\mathbf{V}}$ be the functor defined as follows:

For each object $S \in \mathbf{V}$, $\mathcal{U}(S)$ is the enriched category with two objects x and y such that $\mathrm{Map}_{\mathcal{U}(S)}(x, y) = S$.

Definition 2.3 There is an obvious monoidal functor $\pi_0 : \mathbf{Cat}_{\mathbf{Top}} \rightarrow \mathbf{Cat}$ which sends a topological category to its homotopy category obtained by replacing the hom-spaces with their sets of path-components.

Theorem 2.4 The category $\mathbf{Cat}_{\mathbf{Top}}$ admits a cofibrantly generated model structure defined as follows.

Weak equivalences (Dwyer–Kan equivalences) $F : \mathbf{C} \rightarrow \mathbf{D}$ are those functors which satisfy the following conditions.

WT1: The morphism $\mathrm{Map}_{\mathbf{C}}(a, b) \rightarrow \mathrm{Map}_{\mathbf{D}}(Fa, Fb)$ is a weak homotopy equivalence in the category $\mathbf{Top}$.

WT2: The induced morphism $\pi_0 F : \pi_0 \mathbf{C} \rightarrow \pi_0 \mathbf{D}$ is an equivalence in $\mathbf{Cat}$, i.e. fully faithful and essentially surjective functor.

Fibrations are morphisms $F : \mathbf{C} \rightarrow \mathbf{D}$ which satisfy:

FT1: The morphism $\mathrm{Map}_{\mathbf{C}}(a, b) \rightarrow \mathrm{Map}_{\mathbf{D}}(Fa, Fb)$ is a fibration in $\mathbf{Top}$.

FT2: For any objects $a \in \mathbf{C}$ and $b' \in \mathbf{D}$, and any weak equivalence $e : F(a) \rightarrow b'$ in $\mathbf{D}$ (i.e., induces an isomorphism in $\pi_0 \mathbf{D}$), there exists a weak equivalence $d : a \rightarrow b$ in $\mathbf{C}$ such that $Fd = e$.

The set I of generating cofibrations is given by:

CT1: $|\mathcal{U} \partial \Delta^n| \rightarrow |\mathcal{U} \Delta^n|$, for $n \geq 0$.

CT1: $\emptyset \rightarrow \{x\}$, where $\emptyset$ is the empty topological category and $\{x\}$ is the category with one object and one morphism.

The set J of generating acyclic cofibrations is given by:

ACT1: $|\mathcal{U} \Lambda_i^n| \rightarrow |\mathcal{U} \Delta^n|$, for $0 \leq n$ and $0 \leq i \leq n$.

ACT2: (**Interval categories**) $\{x\} \rightarrow |\mathcal{H}|$ where $\{\mathcal{H}\}$ runs through all countable interval categories in the sense of Bergner [4].

Moreover, the adjunction:

$$\text{Cat}_{\text{sSet}} \begin{array}{c} \xrightarrow{|-|} \\ \xleftarrow{\text{sing}} \end{array} \text{Cat}_{\text{Top}}$$

is a Quillen equivalence.

Proof The proof is given in Corollary 3.11. $\square$

Remark 2.5 The category Top with the standard model structure (i.e. equipped with weak homotopy equivalence and Serre fibrations) is Quillen equivalent to the standard model category of simplicial sets. The Quillen equivalence is given by the adjoint pair (geometric realization, singular functor). More precisely, a map of $f : X \rightarrow Y$ of topological spaces is a Serre fibration (resp. weak homotopy equivalence) if and only if $\text{sing}(f)$ is a Kan fibration (resp. weak homotopy equivalence) [6, Proposition 4.5.26, Corollary 4.5.31]. Therefore, our main Theorem 2.4 can be understood as asserting the existence of a transfer of Bergner’s model structure along the aforementioned adjunction, in the sense of [2, 2.6]. The only point to check is that our condition FT2 corresponds to Bergner’s condition FT2 under transfer, which is a straightforward verification.

Remark 2.6 Interval categories in the sense of Bergner [4] are V -intervals in the sense of Berger–Moerdijk [3] for $V = \text{sSet}$. Since all objects in Top are fibrant, a result of Berger–Moerdijk [3] implies that it is actually sufficient to choose a single interval category in ACT2.

3 Proof of the main Theorem 2.4

In this section, we give a proof of the existence of a cofibrantly model structure on Cat_{Top} . The proof is essentially based on a construction of a functorial path object in the category Cat_{Top} . We take our inspiration from [5] and [7]. Let $I = [0, 1]$, and X a topological space. It is well known that the concatenation of paths (when it is possible) in the standard path object $X^I = \text{map}(I, X)$ is *not* strictly associative. To solve this inconvenient problem we introduce the *Moore path space*.

Definition 3.1 Let X be a topological space, and $\mathcal{M}(X)$ be the subspace of $X^{\mathbb{R}^+} \times \mathbb{R}^+$ with the following property:

$$\mathcal{M}(X) = \{(\alpha : \mathbb{R}^+ \rightarrow X, r) \in X^{\mathbb{R}^+} \times \mathbb{R}^+ \mid \alpha(s) = \alpha(r), \forall s \geq r\}.$$

The positive real number r is called the length of the path. We denote by $\mathcal{M}^r(X)$ the space of paths of length r .

There is a natural inclusion of $X^I = \mathcal{M}^1(X) \rightarrow \mathcal{M}(X)$ which is the inclusion of paths of length 1. This inclusion is a weak equivalence. Moreover, we have two natural maps $(\delta_-, \delta_+) : \mathcal{M}(X) \rightarrow X$ defined by $\delta_-(\alpha, r) = \alpha(0)$ and $\delta_+(\alpha, r) = \alpha(r)$.

Lemma 3.2 *The map $(\delta_-, \delta_+) : \mathcal{M}(X) \rightarrow X \times X$ is a fibration in $\mathbf{Top}$.*

Proof Let $I^n \rightarrow I^{n+1}$ be a generating trivial cofibration in $\mathbf{Top}$, and consider a commutative diagram:

$$\begin{array}{ccc} I^n & \xrightarrow{f} & \mathcal{M}(X) \\ \downarrow & & \downarrow (\delta_-, \delta_+) \\ I^{n+1} & \longrightarrow & X \times X. \end{array}$$

Since I^n is compact, there exists $r \in \mathbb{R}^+$ such that the image of f lives in $\mathcal{M}^r(X) \subset \mathcal{M}(X)$. Consequently, we have to construct a lifting for the following diagram:

$$\begin{array}{ccc} I^n & \xrightarrow{f} & \mathcal{M}^r(X) \subset \mathcal{M}(X) \\ \downarrow & \nearrow & \downarrow (\delta_-, \delta_+) \\ I^{n+1} & \longrightarrow & X \times X. \end{array}$$

The existence of a lift is obvious, since $\mathcal{M}^r(X)$ is just a rescaling of X^I by r . $\square$

Remark 3.3 The diagonal map $\Delta : X \rightarrow X \times X$ is factored by $X \rightarrow \mathcal{M}(X) \rightarrow X \times X$, where the first map (weak equivalence) is the evident embedding of $X = \mathcal{M}^0(X)$ (constant paths of length 0) in $\mathcal{M}(X)$. The second map is the fibration (δ_-, δ_+) (Lemma 3.2).

Remark 3.4 Let (α, r) and (β, s) two elements of $\mathcal{M}(X)$ such that $\alpha(r) = \beta(0)$, then we can define $(\beta * \alpha, r + s)$ by concatenation $*$. The composition law $*$ is strictly associative and unital when it is defined.

Definition 3.5 Let $\mathcal{P} : \mathbf{Cat}_{\mathbf{Top}} \rightarrow \mathbf{Cat}_{\mathbf{Top}}$ be a functor defined as follows. For any topological category $\mathbf{C}$, we declare:

- (1) Objects of $\mathcal{P}\mathbf{C}$ to be morphisms $a : x \xrightarrow{\sim} y$ such that a is a weak equivalence in $\mathbf{C}$, i.e., induces an isomorphism in $\pi_0\mathbf{C}$.
- (2) The space of maps $\mathrm{Map}_{\mathcal{P}\mathbf{C}}(x \rightarrow y, x' \rightarrow y')$ between two objects $a : x \rightarrow y$ and $b : x' \rightarrow y'$ is given by the following pullback of the diagram in $\mathbf{Top}$.

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{P}\mathbf{C}}(x \rightarrow y, x' \rightarrow y') & \longrightarrow & \mathcal{M}(\mathrm{Map}_{\mathbf{C}}(x, y')) \\ \downarrow (ev_s, ev_t) & & \downarrow (\delta_-, \delta_+) \\ \mathrm{Map}_{\mathbf{C}}(x, x') \times \mathrm{Map}_{\mathbf{C}}(y, y') & \xrightarrow{a^*, b_*} & \mathrm{Map}_{\mathbf{C}}(x, y') \times \mathrm{Map}_{\mathbf{C}}(x, y') \end{array}$$

where $\mathcal{M}(\mathrm{Map}_{\mathbf{C}}(x, y'))$ is the Moore path space associated to $\mathrm{Map}_{\mathbf{C}}(x, y')$ (Definition 3.1).

Remark 3.6 The composition of morphisms in the category $\mathcal{PC}$ is explained in Sect. 3.1. In order to get an intuition about maps in the topological category $\mathcal{PC}$, we should remark that

$$\mathrm{Map}_{\mathcal{PC}}(x \rightarrow y, x' \rightarrow y')$$

is a model for the homotopy pullback in **Top** of the diagram

$$\begin{array}{ccc} & & \mathrm{Map}_{\mathbf{C}}(y, y') \\ & & \downarrow a^* \\ \mathrm{Map}_{\mathbf{C}}(x, x') & \xrightarrow{b_*} & \mathrm{Map}_{\mathbf{C}}(x, y'). \end{array}$$

3.1 Composition law in $\mathcal{PC}$

Now, we define the composition law in the categorical path object introduced in Definition 3.5. Let $\mathbf{C}$ be a topological category. The categorical path object $\mathcal{P}(\mathbf{C})$ has as objects $a : x \rightarrow y$ where a is a weak equivalence in $\mathbf{C}$. A morphism in $\mathcal{P}(\mathbf{C})$ between $a : x \rightarrow y$ and $b : x' \rightarrow y'$ is a diagram of the form

$$\begin{array}{ccc} x & \xrightarrow{a} & y \\ f \downarrow & \swarrow \alpha & \downarrow g \\ x' & \xrightarrow{b} & y' \end{array}$$

where f, g are elements of $\mathrm{Map}_{\mathbf{C}}(x, x')$ and $\mathrm{Map}_{\mathbf{C}}(y, y')$, and α_r is an element of $\mathcal{M}(\mathrm{Map}_{\mathbf{C}}(x, y'))$ such that α_r is a homotopy (path) of length r between $g \cdot a$ and $b \cdot f$. Let $a : x \rightarrow y, b : x' \rightarrow y'$ and $c : x'' \rightarrow y''$ three objects of $\mathcal{P}(\mathbf{C})$. Consider the diagram

$$\begin{array}{ccc} x & \xrightarrow{a} & y \\ f \downarrow & \swarrow \alpha_r & \downarrow g \\ x' & \xrightarrow{b} & y' \\ f' \downarrow & \swarrow \beta_s & \downarrow g' \\ x'' & \xrightarrow{c} & y'' \end{array}$$

by composition we obtain

where $(\beta \circ \alpha)_{r+s} \in \mathcal{M}(\text{Map}_{\mathbf{C}}(x, y''))$ is defined as the concatenation $\beta.f * g'.\alpha$. The identity map id_a of $a : x \rightarrow y$ is given by $id_x : x \rightarrow x$, $id_y : y \rightarrow y$ and a constant Moore path of length 0.

Lemma 3.7 *The previous composition law is associative.*

Proof Let take the following diagram:

The composition of morphisms and Moore paths is strict and we get:

$$(\gamma_t \circ \beta_s) \circ \alpha_r = (\gamma_t f' f * g'' . \beta_s) \circ \alpha_r \quad (3.1)$$

$$= (\gamma_t . f' * g'' . \beta_s) . f * g'' g' . \alpha_r \quad (3.2)$$

$$= \gamma_t . f' f * g'' . \beta_s . f * g'' g' . \alpha_r \quad (3.3)$$

$$\gamma_t \circ (\beta_s \circ \alpha_r) = \gamma_t \circ (\beta_s . f * g' . \alpha_r) \quad (3.4)$$

$$= \gamma_t . f' f * g'' . (\beta_s . f * g' . \alpha_r) \quad (3.5)$$

$$= \gamma_t . f' f * g'' . \beta_s . f * g'' g' . \alpha_r \quad (3.6)$$

□

Remark 3.8 The diagonal map $\Delta : \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$ is factored as

where

- $i : \mathbf{C} \rightarrow \mathcal{PC}$ is defined as $i(x) = id_x : x \rightarrow x$.
- $ev_s(x \rightarrow y) = x$ and $ev_t(x \rightarrow y) = y$.

Lemma 3.9 *The map $i : \mathbf{C} \rightarrow \mathcal{PC}$ is an equivalence of topological categories (cf. Theorem 2.4).*

Proof First, we check that i is homotopically fully faithful i.e.,

$$\mathrm{Map}_{\mathbf{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{PC}}(i(x), i(y))$$

is a weak equivalence of topological space, since by definition

$$\mathrm{Map}_{\mathcal{PC}}(i(x), i(y)) = \mathcal{M}(\mathrm{Map}_{\mathbf{C}}(x, y)).$$

For the homotopical essential surjectivity, let's take an object $a : x \rightarrow y$. By definition, there is a weak equivalence in $\mathcal{PC}$ between $i(a)$ and $a : x \rightarrow y$ given by

$$\begin{array}{ccc} x & \xrightarrow{id} & x \\ \downarrow id & & \downarrow a \\ x & \xrightarrow[\sim]{a} & y. \end{array}$$

so i is an equivalence of topological categories. $\square$

Lemma 3.10 *Let $\mathbf{C}$ be a topological category, then the map $(ev_s, ev_t) : \mathcal{PC} \rightarrow \mathbf{C}$ is a fibration in $\mathbf{Cat}_{\mathrm{Top}}$ (cf. Theorem 2.4).*

Proof First, we have to prove that

$$(ev_s, ev_t) : \mathrm{Map}_{\mathcal{PC}}(x \rightarrow y, x' \rightarrow y') \rightarrow \mathrm{Map}_{\mathbf{C}}(x, x') \times \mathrm{Map}_{\mathbf{M}}(y, y')$$

is a fibration of topological spaces. By definition (ev_s, ev_t) is a pullback in $\mathbf{Top}$ of the fibration

$$(\delta_-, \delta_+) : \mathcal{M}(\mathrm{Map}_{\mathbf{C}}(x, y')) \rightarrow \mathrm{Map}_{\mathbf{C}}(x, y') \times \mathrm{Map}_{\mathbf{C}}(x, y').$$

Second, let $w_1, w_2 : (x, y) \rightarrow (x', y')$ be a weak equivalence in $\mathbf{C} \times \mathbf{C}$ such that there exists an object $a : x' \rightarrow y'$ in $\mathcal{PC}$. Choose a homotopical inverse to w_2 denoted by w_2^{-1} i.e., we have a path between id and $w_2 w_2^{-1}$, and define the map $w_2^{-1} a w_1 : x \rightarrow y$ which is also a weak homotopy equivalence. We remark, by definition, that we have a path

$$\gamma : [0, 1] \rightarrow \mathrm{Map}_{\mathbf{C}}(x, y')$$

between $w_2 w_2^{-1} a w_1$ and $a w_1$. Therefore, we obtain a following morphism in $\mathcal{PC}$, namely:

$$\begin{array}{ccc} x & \xrightarrow{w_2^{-1} a w_1} & y \\ w_1 \downarrow \sim & \nearrow \gamma & \downarrow \sim w_2 \\ x' & \xrightarrow{a} & y'. \end{array}$$

So we can lift the weak equivalence $(w_1, w_2) \in \mathbf{C} \times \mathbf{C}$ to a weak equivalence in $\mathcal{PC}$ given by (w_1, w_2, γ) . $\square$

Corollary 3.11 (*Proof of Theorem 2.4*) *The category $\mathbf{Cat}_{\mathbf{Top}}$ is a cofibrantly generated model category and Quillen equivalent to the model category $\mathbf{Cat}_{\mathbf{sSet}}$.*

Proof The standard Quillen adjunction (equivalence) between $\mathbf{sSet}$ and compactly generated topological spaces $\mathbf{Top}$ induces an adjunction between $\mathbf{Cat}_{\mathbf{sSet}}$ and $\mathbf{Cat}_{\mathbf{Top}}$. In order to apply the transfer principle [2, 2.6] we must first show that Quillen's small object argument is available. Since the domains of the generating (trivial) cofibrations in $\mathbf{Top}$ are compact, and hence $\otimes$ -finite in the sense of [3, Lemma 2.15 in loc. cit], together with an adjunction argument establishes the required smallness of the generating (trivial) cofibrations of $\mathbf{Cat}_{\mathbf{Top}}$. The existence of a transferred model structure on $\mathbf{Cat}_{\mathbf{Top}}$ is then a direct consequence of the Bergner's model structure on $\mathbf{Cat}_{\mathbf{sSet}}$ [4] and the existence of path object (Lemmas 3.9, 3.10). The Quillen equivalence between $\mathbf{Cat}_{\mathbf{Top}}$ and $\mathbf{Cat}_{\mathbf{sSet}}$ is obvious. $\square$

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