A spectral sequence for Bredon cohomology

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Abstract For any finite group G , we construct a spectral sequence for computing the Bredon cohomology of a G -CW complex $\mathbf{X}$, starting with the cohomology of $\mathbf{X}^H / \bigcup_{K > H} \mathbf{X}^K$ with suitable local coefficients, for various $H \leq G$.

Keywords Bredon cohomology · Cohomology with local coefficients · Equivariant homotopy theory · Spectral sequence

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1 Introduction

Equivariant homotopy theory is the study of G -spaces—topological spaces equipped with a continuous action of a group G —using homotopy-theoretic methods. In [3], Bredon proposed a framework for studying a G -space $\mathbf{X}$ using the system of fixed point sets $\mathbf{X}^H$ for various subgroups $H \leq G$. In particular, he introduced an equivariant cohomology theory $\mathbf{H}_G^*(\mathbf{X}; \underline{M})$, for any coefficient system $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{A}bGp$ defined on the orbit category $\mathcal{O}_G$ (cf. Sect. 2.1).

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Bredon cohomology has become one of the major theoretical tools of equivariant homotopy theory. However, it is notoriously difficult to calculate. Our goal here is to describe a spectral sequence converging to $\mathbf{H}_G^*(\mathbf{X}; \underline{M})$, for a finite group G , starting from “local” information at the various fixed point sets $\mathbf{X}^H$. In fact, the spectral sequence takes the form

$$E_1^{k,i} = \bigoplus_{[G/H]} \tilde{H}_{W_H}^i(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M_H) \implies \mathbf{H}_G^i(\mathbf{X}; \underline{M}),$$

where $\tilde{H}_\Gamma^*(-; M)$ are reduced local cohomology groups, $W_H := N_G H/H$, and $\mathbf{X}_H^H = \mathbf{X}^H / \bigcup_{K > H} \mathbf{X}^K$. See Theorem 5.1 below.

The idea for this spectral sequence is based on a more general construction of local-to-global spectral sequences for the cohomology of a diagram $\underline{X} : I \rightarrow \mathcal{C}$ (cf. [1], and compare [14, 22]); however, the latter only works for *directed* indexing categories I , so it does not apply to Bredon cohomology. Note also that Moerdijk and Svensson have a different construction of a spectral sequence for computing Bredon cohomology (see [19]).

One might expect the spectral sequence constructed here to start from the Bredon cohomology $\mathbf{H}_{W_H}^*(\mathbf{X}^H; \widehat{M}_H)$ at the various fixed point sets. We were not able to obtain such a spectral sequence directly. However, we do have another spectral sequence of the form:

$$E_1^{m,i} = \bigoplus_{G/L} \tilde{H}_{\bar{W}_L}^i(\mathbf{E}W_L \times_{\bar{W}_L} \mathbf{X}_L^L; M_{N_G H \cap L}) \implies \mathbf{H}_{W_H}^i(\mathbf{X}^H; \widehat{M}_H),$$

thus allowing us to compute these fixed-point-set Bredon cohomology groups, too, from the reduced cohomology with local coefficients. See Theorem 5.7 below.

Remark 1.1 There is a version of Bredon cohomology for any topological group G (in particular, for a compact Lie group), based on choosing a family of (closed) subgroups of G (see [8, 12]). However, in this case the orbit category $\mathcal{O}_G$ is itself topologically (or simplicially) enriched, so the diagram systems involved are more complicated. Our methods do not work in this situation, in general, because the filtration (3.2) that we use need not be exhaustive. Therefore, throughout this paper we assume that G is a *finite* group.

We would like to thank Wolfgang Lück for pointing out that our spectral sequence is similar in spirit to the much more general p -chain spectral sequence of [5]. However, our construction is different, and in this special case we can describe the E_1 -term and differential quite explicitly.

1.1 Notation and conventions

All mapping spaces $\text{Map}(-, -)$ are simplicial sets, and the category of simplicial sets will be denoted by $\mathcal{S}$. The category of topological spaces will be denoted by $\mathcal{T}$, and its objects will be denoted by boldface letters: $\mathbf{X}, \mathbf{Y}, \dots$. We use $\mathbf{H}_G^*$ for Bredon cohomology, to distinguish it from cohomology with local coefficients, denoted simply by H_Γ^* .

1.2 Organization

In Sect. 2 we provide some background on G -spaces, the orbit category $\mathcal{O}_G$, and Bredon cohomology. In Sect. 3 we define the filtration on the orbit category which is the basis for our spectral sequences. In Sect. 4 we recall some basic facts about cohomology with local coefficients in our connection, and in Sect. 5 we construct the two spectral sequences.

2 Bredon cohomology

Bredon introduced a cohomology theory for G -spaces, using the following notions:

2.1 The orbit category

Let G be a fixed (finite) group. A *basic G -set* is the set of left cosets G/H for some subgroup $H \leq G$, with the left G -action. The *orbit category* $\mathcal{O}_G$ of G has the basic G -sets as objects, and G -equivariant maps as morphisms.

Any map $G/H \rightarrow G/K$ in $\mathcal{O}_G$ can be factored as an epimorphism $G/H \xrightarrow{i_*} G/K^{a^{-1}}$ (induced by the inclusion $i : H \hookrightarrow K^{a^{-1}}$), followed by an isomorphism $\phi_a^{K^{a^{-1}}} : G/K^{a^{-1}} \rightarrow G/K$. Here $K^{a^{-1}} = aKa^{-1}$, $a \in G$, and $\phi_a^{K^{a^{-1}}}$ is induced by the right translation $R_a : G \rightarrow G$ (with $R_a(g) = ag$), that is:

$$\phi_a^{K^{a^{-1}}} : gK^{a^{-1}} = gaKa^{-1} \mapsto gaK \quad (2.1)$$

(see [3, I, Sect. 3]). This map can also be decomposed as $i_*^a \circ \phi_a^H$, where $\phi_a^H : G/H \rightarrow G/H^a$ is induced by R_a and $i^a : H^a \hookrightarrow K$ is the conjugate of $i : H \hookrightarrow K^{a^{-1}}$ by a .

Fact 2.1 *Two maps $\phi_a^{K^{a^{-1}}} \circ i_*$ and $\phi_b^{K^{b^{-1}}} \circ j_*$ from G/H to G/K are the same in $\mathcal{O}_G$ if and only if $a^{-1}b \in K$ (so $K^{a^{-1}} = K^{b^{-1}}$ and $\phi_{ba^{-1}}^{K^{b^{-1}}}$ is the identity). Therefore, the automorphism group $W_H := \text{Aut}_{\mathcal{O}_G}(G/H)$ of $G/H \in \mathcal{O}_G$ is $N_G H/H$, where $N_G H$ is the normalizer of H in G .*

Note that if a is in $K^{a^{-1}}$, the right multiplication R_a induces the *identity* map $\phi_a^{K^{a^{-1}}} : G/K^{a^{-1}} \rightarrow G/K$, even though the conjugation isomorphisms $\rho_a^H : H \rightarrow H^a$ and $\rho_a^{K^{a^{-1}}} : K^{a^{-1}} \rightarrow K$ may be non-trivial.

2.2 G -spaces

For any (finite) group G , a G -space is a topological space $\mathbf{X} \in \mathcal{T}$ equipped with a left G -action. The category of G -spaces with G -equivariant continuous maps, simply called G -maps, will be denoted by $G\text{-}\mathcal{T}$. We write $\mathbf{X}^H$ for the *fixed point set* $\{x \in \mathbf{X} : hx = x \ \forall h \in H\}$ of X under a subgroup $H \leq G$.

The notion of G -CW complexes (for topological groups) was introduced in [12, 16]: a G -CW complex $\mathbf{X}$ is the union of sub G -spaces $\mathbf{X}^n$ such that $\mathbf{X}^0$ is a disjoint union of basic G -sets G/H , and $\mathbf{X}^{n+1}$ is obtained from $\mathbf{X}^n$ by attaching G -cells of the form $G/H \times \mathbf{D}^{n+1}$ (where $\mathbf{D}^{n+1}$ is a $(n+1)$ -disc with boundary $\mathbf{S}^n$) with attaching G -maps $G/H \times \mathbf{S}^n \rightarrow \mathbf{X}^n$. For finite G , this is equivalent to $\mathbf{X}$ being a CW-complex on which G acts cellularly (see [6]). Subcomplexes and relative G -CW complexes are defined in the obvious way. For any G -space $\mathbf{X}$, there is a G -CW complex $\widehat{\mathbf{X}}$ and a weak G -homotopy equivalence $\gamma : \widehat{\mathbf{X}} \rightarrow \mathbf{X}$.

For any collection $\mathcal{F}$ of subgroups of G closed under conjugation, there is a simplicial model category structure on $G\text{-}\mathcal{T}$, due to Dwyer and Kan, in which

- (i) A G -map $f : \mathbf{X} \rightarrow \mathbf{Y}$ is a *weak equivalence* (respectively, a *fibration*) if for each $H \in \mathcal{F}$, the restriction $f|_{\mathbf{X}^H}$ is a weak equivalence (respectively, a Serre fibration).
- (ii) A G -map $f : \mathbf{X} \rightarrow \mathbf{Y}$ is a *cofibration* if it is a retract of a (transfinite) composite of inclusions of relative G -CW pairs (see [7, Sect. 2.1(Q1)]).
- (iii) The function complex $\text{Map}_G(\mathbf{X}, \mathbf{Y})$ for the simplicial structure on $G\text{-}\mathcal{T}$ is defined by: $\text{Map}_G(\mathbf{X}, \mathbf{Y})_n := \text{Hom}_{G\text{-}\mathcal{T}}(\mathbf{X} \times \Delta[n], \mathbf{Y})$ (with a trivial G -action on $\Delta[n]$).

See [7, Sect. 2].

2.3 $\mathcal{O}_G$ -diagrams

Bredon's approach to G -equivariant homotopy theory, extended by Elmendorf in [9] to compact Lie groups, reduces the study of a G -space $\mathbf{X}$ to the system of fixed point sets under the subgroups of G :

For any category $\mathcal{C}$, an $\mathcal{O}_G^{\text{op}}$ -*diagram* in $\mathcal{C}$ is a functor $\underline{X} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{C}$, and the category of all such diagrams will be denoted by $\mathcal{C}^{\mathcal{O}_G^{\text{op}}}$. When $\mathcal{C}$ is a simplicial model category (cf. [21, II, Sect. 2]), $\mathcal{C}^{\mathcal{O}_G^{\text{op}}}$ has a projective simplicial model category structure in which a map $f : \underline{X} \rightarrow \underline{Y}$ of $\mathcal{O}_G^{\text{op}}$ -diagrams is a weak equivalence (respectively, a fibration) if for each $H \leq G$, $f(G/H) : \underline{X}(G/H) \rightarrow \underline{Y}(G/H)$ is a weak equivalence (respectively, a fibration). The mapping spaces are defined using the simplicial structure in $\mathcal{C}$ by $\text{Map}_{\mathcal{C}^{\mathcal{O}_G^{\text{op}}}}(\underline{X}, \underline{Y})_n := \text{Hom}_{\mathcal{C}^{\mathcal{O}_G^{\text{op}}}}(\underline{X} \otimes \Delta[n], \underline{Y})_n$ (cf. [7, §1.3] and compare [20]).

2.4 $\mathcal{O}_G$ -diagrams in $\mathcal{T}$

When $\mathcal{C} = \mathcal{T}$, the fixed point set functor $\Phi : G\text{-}\mathcal{T} \rightarrow \mathcal{T}^{\mathcal{O}_G^{\text{op}}}$, sending a G -space $\mathbf{X}$ to the diagram $\Phi\mathbf{X} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ defined:

$$(\Phi\mathbf{X})(G/H) := \mathbf{X}^H, \quad (2.2)$$

has a left adjoint $\Psi : \mathcal{T}^{\mathcal{O}_G^{\text{op}}} \rightarrow G\text{-}\mathcal{T}$ (see [9, Theorem 1]). We shall usually denote $\Phi\mathbf{X}$ by $\underline{X}$.

In fact, this adjoint pair constitutes a simplicial Quillen equivalence between $G\text{-}\mathcal{T}$ and $\mathcal{T}^{\mathcal{O}_G^{\text{op}}}$. See [7, Theorem 3.1] for $\mathcal{C} = \mathcal{S}$ (with $\underline{X}(G/H) := \text{Map}_G(G/H, \mathbf{X})$). Using

the “singular-realization” adjoint pair $\mathcal{S} \rightleftharpoons \mathcal{T}$, one can translate the Dwyer–Kan result to our context.

For any topological space $\mathbf{Z}$, with trivial G -action, the associated *basic G -spaces* are those of the form $G/K \times \mathbf{Z}$ ($K \leq G$). We denote the corresponding fixed point diagrams $\Phi(G/K \times \mathbf{Z})$ by $\underline{Z}_{G/K} \in \mathcal{T}_G^{\text{op}}$, so

$$\underline{Z}_{G/K}(G/H) := \coprod_{\psi^{\text{op}}: G/K \rightarrow G/H \text{ in } \mathcal{O}_G^{\text{op}}} \mathbf{Z}_{\psi}, \quad (2.3)$$

where $\mathbf{Z}_{\psi}$ is a copy of $\mathbf{Z}$, and the structure map $\underline{Z}_{G/K}(\phi^{\text{op}})$ sends $\mathbf{Z}_{\psi}$ by the identity homeomorphism to $\mathbf{Z}_{\psi \circ \phi}$. In particular, $\mathbf{Z}_{\text{Id}}$ is the copy of $\mathbf{Z}$ indexed by $\text{Id} : G/K \rightarrow G/K$ in $\underline{Z}_{G/K}(G/K)$, and we have:

Fact 2.2 *For any $\underline{Y} \in \mathcal{T}_G^{\text{op}}$, a map of $\mathcal{O}_G^{\text{op}}$ -diagrams $\underline{f} : \underline{Z}_{G/K} \rightarrow \underline{Y}$ is uniquely determined by a map of spaces $f : \mathbf{Z}_{\text{Id}} = \mathbf{Z} \rightarrow \underline{Y}(G/K)$.*

Proof The summand $\mathbf{Z}_{\psi}$ in $\underline{Z}_{G/K}(G/H)$ is sent to $\underline{Y}(G/H)$ by $\underline{Y}(\psi) \circ f$. $\square$

2.5 Cellular $\mathcal{O}_G$ -diagrams

In particular, an n -cell in $\mathcal{T}_G^{\text{op}}$ is a diagram of the form $\underline{D}_{G/K}^n := \Phi(\mathbf{D}^n \times G/K)$, where $\mathbf{D}^n$ is an n -cell in $\mathcal{T}$, and similarly for the n -sphere $\underline{S}_{G/K}^n$. A *cellular complex* in $\mathcal{T}_G^{\text{op}}$ is a diagram $\underline{X} = \text{colim}_p \underline{X}^p$ constructed inductively by a process of “attaching cells”: i.e.,

$$\underline{X}^p = \underline{X}^{p-1} \cup_{(f_{\alpha})_{\alpha}} \coprod_{\alpha \in I_p} \underline{D}_{G/K_{\alpha}}^{n_{\alpha}} \quad (2.4)$$

for some indexing set I_p and diagram maps $f_{\alpha} : \underline{S}_{G/K_{\alpha}}^{n_{\alpha}-1} \rightarrow \underline{X}^{p-1}$. There is also a notion of a *relative cellular complex*, and the cofibrations in the model category $\mathcal{T}_G^{\text{op}}$ are retracts of inclusions of a relative cellular complex. See [7, Theorem 2.2].

The notion of a cellular diagram can be defined more generally—see [20, Sect. 3].

Fact 2.3 (cf. [9]). *For any G -space $\mathbf{X}$ and $\underline{X} \in \mathcal{T}_G^{\text{op}}$, $\Psi\Phi\mathbf{X}$ is a G -CW complex, $\Phi\Psi\underline{X}$ is a cellular diagram, the unit $\mathbf{X} \rightarrow \Psi\Phi\mathbf{X}$ is a G -weak homotopy equivalence, and the counit $\Phi\Psi\underline{X} \rightarrow \underline{X}$ is a weak equivalence of diagrams.*

Assumption 2.4 From now on we assume that all our G -CW-complexes are of the form $\mathbf{X} = \Psi\Phi\mathbf{X}'$ for some G -space $\mathbf{X}'$.

2.6 Bredon cohomology

Let G be a (finite) group, and let $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \text{Ab}Gp$ be an $\mathcal{O}_G^{\text{op}}$ -diagram in abelian groups, known as a *coefficient system* for G . Bredon showed that for each

$n \geq 1$, one can construct a natural $\mathcal{O}_G^{\text{op}}$ -diagram $K(\underline{M}, n) : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ with each $K(\underline{M}, n)(G/H) := \mathbf{K}(\underline{M}(G/H), n)$ an Eilenberg-Mac Lane space (cf. [3, Sect. 6]). Equivalently, applying the functor Ψ to $K(\underline{M}, n)$ yields a G -space $\mathbf{K}(\underline{M}, n)$ with the property that $\mathbf{K}(\underline{M}, n)^H$ is an (ordinary) Eilenberg-Mac Lane space of type $\mathbf{K}(\underline{M}(G/H), n)$. In particular, for $H = \{e\}$ we see that $\mathbf{K}(\underline{M}, n)$ is an ordinary $\mathbf{K}(\underline{M}(G/\{e\}), n)$.

By [7, Theorem 3.1], the function complex $\text{Map}_G(\mathbf{X}, \mathbf{Y})$ of G -maps between two G -CW complexes $\mathbf{X}$ and $\mathbf{Y}$ is weakly equivalent to the mapping space $\text{Map}_{\mathcal{T}_{\mathcal{O}_G^{\text{op}}}(\underline{X}, \underline{Y})}$ between the corresponding $\mathcal{O}_G^{\text{op}}$ -diagrams (cf. (2.2)), at least if $\underline{X}$ is cofibrant and $\underline{Y}$ is fibrant. Thus the study of G -maps between G -spaces (up to homotopy) is reduced to the study of mapping spaces of diagrams.

The n th Bredon cohomology group of a G -space $\mathbf{X}$ with coefficients in $\underline{M}$ may then be defined to be

$$\mathbf{H}_G^n(\mathbf{X}; \underline{M}) := \pi_0 \text{Map}_{\mathcal{T}_{\mathcal{O}_G^{\text{op}}}(\underline{X}, K(\underline{M}, n))}, \quad (2.5)$$

where we assume that $\underline{X}$ is cofibrant and $K(\underline{M}, n)$ is fibrant in the model category $\mathcal{T}_{\mathcal{O}_G^{\text{op}}}$. See [3, (6.1)]. Since Ψ induces a simplicial Quillen equivalence, we obtain a natural isomorphism

$$\mathbf{H}_G^{n-i}(\mathbf{X}; \underline{M}) \cong \pi_i \text{Map}_G(\mathbf{X}, \mathbf{K}(\underline{M}, n)) \quad (2.6)$$

for $0 \leq i \leq n$ (cf. [18, V, Sect. 4]).

See [4] for a definition of singular equivariant (co)homology (for finite groups), [24] for cellular equivariant homology, and [17] for cohomology, when G is an arbitrary topological group.

3 A filtration on the orbit category

The spectral sequence we construct here is based on the following filtration of $\mathcal{O}_G^{\text{op}}$:

3.1 Filtering $\mathcal{O}_G^{\text{op}}$

For any subgroup H of G , we define the *length* of H in G , denoted by $\text{len}_G H$, to be the maximal $0 \leq k < \infty$ such that there exists a sequence of proper inclusions of subgroups:

$$H = H_0 < H_1 < H_2 < \cdots < H_{k-1} < H_k = G. \quad (3.1)$$

This induces a filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_k \subset \cdots \subset \mathcal{O}_G^{\text{op}} \quad (3.2)$$

by full subcategories, where $\text{Obj } \mathcal{F}_k := \{G/H \in \mathcal{O}_G^{\text{op}} : \text{len}_G H \leq k\}$ (so $\text{Obj } \mathcal{F}_0 = \{G/G\}$).

Since G is finite, the filtration is exhaustive: if $\text{len}_G \{e\} = N$ —that is, the longest possible sequence (3.1) in G has N inclusions of proper subgroups—then $\mathcal{F}_N = \mathcal{O}_G^{\text{op}}$.

Let $\mathcal{E}_m^H$ denote the full subcategory of the slice category $\mathcal{O}_G^{\text{op}}/(G/H)$ whose objects are $G/K \rightarrow G/H$ in $\mathcal{O}_G^{\text{op}}$ with $G/K \in \mathcal{F}_m$, with the obvious commuting triangles as maps.

Definition 3.1 Denote by $\mathcal{S}_k$ the k th *stratum* of the filtration—that is, the full subcategory of $\mathcal{O}_G^{\text{op}}$ whose objects are in $\mathcal{F}_k \setminus \mathcal{F}_{k-1}$. Note that all the maps in $\mathcal{S}_k$ are isomorphisms (by the description in Sect. 2.1), and conversely, any isomorphism in $\mathcal{O}_G^{\text{op}}$ are contained in some $\mathcal{S}_k$. Any other map in $\mathcal{O}_G^{\text{op}}$ strictly increases filtration.

We let $\widehat{\mathcal{F}}_k$ denote the collection of subgroups $H < G$ such that $G/H \in \mathcal{F}_k$, and $\widehat{\mathcal{S}}_k := \widehat{\mathcal{F}}_k \setminus \widehat{\mathcal{F}}_{k-1}$.

3.2 The tower of mapping spaces

If $\mathcal{C}$ is any simplicially enriched category with colimits, the inclusion $\mathcal{F}_k \hookrightarrow \mathcal{O}_G^{\text{op}}$ induces a simplicial functor $\tau_k : \mathcal{C}^{\mathcal{O}_G^{\text{op}}} \rightarrow \mathcal{C}^{\mathcal{F}_k}$ defined $\tau_k \underline{X} := \underline{X}|_{\mathcal{F}_k}$. Similarly, the inclusion $J_k : \mathcal{F}_k \hookrightarrow \mathcal{F}_{k+1}$ induces a simplicial functor $J_k^* : \mathcal{C}^{\mathcal{F}_{k+1}} \rightarrow \mathcal{C}^{\mathcal{F}_k}$, so for any two diagrams $\underline{X}, \underline{Y} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{C}$ we obtain a tower of simplicial sets

$$\text{Map}(\tau_N \underline{X}, \tau_N \underline{Y}) \xrightarrow{p_N} \dots \text{Map}(\tau_{k+1} \underline{X}, \tau_{k+1} \underline{Y}) \xrightarrow{p_{k+1}} \text{Map}(\tau_k \underline{X}, \tau_k \underline{Y}) \rightarrow \dots \quad (3.3)$$

(mapping spaces in $\mathcal{C}^{\mathcal{F}_k}$), with p_k induced by J_k^* . If we assume $\underline{Y}$ is pointed, (3.3) becomes a tower of pointed simplicial sets.

Note that $J_k^* : \mathcal{C}^{\mathcal{F}_{k+1}} \rightarrow \mathcal{C}^{\mathcal{F}_k}$ has a left adjoint $\xi_k : \mathcal{C}^{\mathcal{F}_k} \rightarrow \mathcal{C}^{\mathcal{F}_{k+1}}$, with

$$(\xi_k \underline{Z})(G/H) := \text{colim}_{(\psi^{\text{op}}: G/K \rightarrow G/H) \text{ in } \mathcal{E}_k^H} \underline{Z}(G/K)_{\psi} \quad (3.4)$$

at $G/H \in \mathcal{F}_{k+1}$ for any $\underline{Z} : \mathcal{F}_k \rightarrow \mathcal{C}$. In particular, if $G/H \in \mathcal{F}_k$, the indexing slice category $\mathcal{E}_k^H$ has a terminal object $\text{Id} : G/H \rightarrow G/H$, so $(\xi_k \underline{Z})(G/H) = \underline{Z}(G/H)$. The counit of the adjunction will be denoted by $\eta_k : \xi_k J_k^* \underline{W} \rightarrow \underline{W}$ for $\underline{W} : \mathcal{F}_{k+1} \rightarrow \mathcal{C}$.

Proposition 3.2 When $\underline{X} \in \mathcal{T}^{\mathcal{O}_G^{\text{op}}}$ is a cellular diagram, then for each $0 < k \leq N$, $\tau_k \underline{X}$ is cofibrant in $\mathcal{T}^{\mathcal{F}_k}$, and the counit $\eta_{k-1} : \xi_{k-1} \tau_{k-1} \underline{X} \rightarrow \tau_k \underline{X}$ is a cofibration in $\mathcal{T}^{\mathcal{F}_k}$.

Proof We can filter $\underline{X}$ by sub-cellular diagrams in $\mathcal{T}^{\mathcal{O}_G^{\text{op}}}$:

$$\mathcal{Q}_0 \underline{X} \hookrightarrow \mathcal{Q}_1 \underline{X} \hookrightarrow \dots \mathcal{Q}_k \underline{X} \hookrightarrow \dots \mathcal{Q}_N \underline{X} = \underline{X},$$

where $\mathcal{Q}_k \underline{X}$ consists of all cells $\underline{D}_{G/H}^n$ with $G/H \in \mathcal{F}_k$ [see (2.3)]. Then $\tau_k \underline{X} = \mathcal{Q}_k \tau_k \underline{X} = \tau_k \mathcal{Q}_k \underline{X}$ is a cellular diagram in $\mathcal{T}^{\mathcal{F}_k}$, so it is cofibrant, for all $k \geq 0$.

To prove the last statement, we shall show that for cellular $\underline{X}$, $\xi_{k-1}\tau_{k-1}\underline{X}$ is isomorphic to $\mathcal{Q}_{k-1}\underline{X}$, and the counit η_{k-1} is just the cellular inclusion $\mathcal{Q}_{k-1}\underline{X} \hookrightarrow \mathcal{Q}_k\underline{X}$. For this, by definition of the adjunction counit it suffices to show that the restriction map

$$\mathrm{Hom}_{\mathcal{T}^{\mathcal{F}_k}}(\mathcal{Q}_{k-1}\underline{X}, \underline{Y}) \xrightarrow{\rho} \mathrm{Hom}_{\mathcal{T}^{\mathcal{F}_{k-1}}}(\tau_{k-1}\underline{X}, \tau_{k-1}\underline{Y}).$$

is a natural isomorphism for any $\underline{Y} \in \mathcal{T}^{\mathcal{F}_k}$: in other words, that any map of $\mathcal{F}_{k-1}$ -diagrams $\underline{h} : \tau_{k-1}\underline{X} \rightarrow \tau_{k-1}\underline{Y}$ extends uniquely to a map of $\mathcal{F}_k$ -diagrams $\underline{h} : \mathcal{Q}_{k-1}\underline{X} \rightarrow \underline{Y}$. We show this by induction on the cellular skeleta $(\underline{X}^p)_{p=0}^\infty$:

For $p = 0$, we have

$$\underline{X}^0 = \coprod_{\alpha \in I_0} D_{G/H_\alpha}^{n_\alpha} \quad \text{and} \quad \mathcal{Q}_{k-1}\underline{X}^0 = \coprod_{\alpha \in I_0, G/H_\alpha \in \mathcal{F}_{k-1}} D_{G/H_\alpha}^{n_\alpha}.$$

Therefore, for $G/H_\alpha \in \mathcal{F}_{k-1}$,

$$\underline{h}(G/H^\alpha) : \tau_{k-1}\underline{X}(G/H^\alpha) \longrightarrow \underline{Y}(G/H^\alpha)$$

is well-defined, and determines $\widetilde{\underline{h}}^0|_{D_{G/H_\alpha}^{n_\alpha}}$, and thus $\widetilde{\underline{h}}^0 : \mathcal{Q}_{k-1}\underline{X}^0 \rightarrow \underline{Y}$ on the coproduct, by Fact 2.2.

For the induction step, assume we have the following solid diagram, and wish must define the unique $\widetilde{\underline{h}}^p : \mathcal{Q}_{k-1}\underline{X}^p \rightarrow \underline{Y}$ making the full diagram commute:

$$\begin{array}{ccccc}
 & & \widetilde{\underline{h}}^{p-1} & & \\
 & \searrow & \curvearrowright & \searrow & \\
 \mathcal{Q}_{k-1}\underline{X}^{p-1} & \xrightarrow{\quad} & \mathcal{Q}_{k-1}\underline{X}^p & \xrightarrow{\quad \widetilde{\underline{h}}^p \quad} & \underline{Y} \\
 \downarrow \text{rest} & & \downarrow \text{rest} & & \downarrow \text{rest} \\
 \tau_{k-1}\underline{X}^{p-1} & \xrightarrow{\quad} & \tau_{k-1}\underline{X}^p & \xrightarrow{\quad \underline{h}^p \quad} & \tau_{k-1}\underline{Y} \\
 & \nearrow & \curvearrowright & \nearrow & \\
 & & \widetilde{\underline{h}}^{p-1} & &
 \end{array} \tag{3.5}$$

where $\underline{h}^i := \underline{h}|_{\tau_{k-1}\underline{X}^i}$. Note that the diagram maps here have different indexing categories.

By definition of the cellular skeleta, the commuting square in the following diagram (in $\mathcal{T}^{\mathcal{F}_k}$) is cocartesian, so to define the dotted map $\widetilde{\underline{h}}^p : \mathcal{Q}_{k-1}\underline{X}^p \rightarrow \underline{Y}$, we need only produce a map $g : \coprod_{\alpha \in I_p, G/H_\alpha \in \mathcal{F}_{k-1}} D_{G/H_\alpha}^{n_\alpha} \rightarrow \underline{Y}$ making the solid diagram commute:

$$\begin{array}{ccc}
 \coprod_{\alpha \in I_p, G/H_\alpha \in \mathcal{F}_{k-1}} \underline{S}_{G/H_\alpha}^{n_\alpha} & \xrightarrow{\coprod_\alpha f_\alpha} & \mathcal{Q}_{k-1} \underline{X}^{p-1} \\
 \downarrow & \boxed{\text{PO}} & \downarrow \\
 \coprod_{\alpha \in I_p, G/H_\alpha \in \mathcal{F}_{k-1}} \underline{D}_{G/H_\alpha}^{n_\alpha} & \xrightarrow{\quad} & \mathcal{Q}_{k-1} \underline{X}^p \\
 & \searrow \widetilde{h}^p & \downarrow \widetilde{h}^{p-1} \\
 & & \underline{Y}
 \end{array}$$

g

Since all the disc-summands in the lower left corner are indexed by objects $G/H_\alpha \in \mathcal{F}_{k-1}$, as above $\widetilde{h}(G/H_\alpha)$ determines g by Fact 2.2, which also ensures that g and $\widetilde{h}^{p-1}$ agree on the sphere-summands, using commutativity of (3.5). $\square$

Corollary 3.3 *If $\mathbf{X}$ is a G -CW complex, and $\underline{X} = \Phi \mathbf{X}$, each $\tau_k \underline{X}$ is cofibrant and $\eta_{k-1} : \xi_{k-1} \tau_{k-1} \underline{X} \rightarrow \tau_k \underline{X}$ is a cofibration.*

Proof Use Assumption 2.4 and Fact 2.3. $\square$

Definition 3.4 For $\underline{X}$ as above, let $\underline{C}_k$ denote the (homotopy) cofiber of $\eta_{k-1} : \xi_{k-1} \tau_{k-1} \underline{X} \rightarrow \tau_k \underline{X}$ in $\mathcal{T}^{\mathcal{F}_k}$, with $s_k : \tau_k \underline{X} \rightarrow \underline{C}_k$ the structure map in the cofibration sequence. For any fibrant $\underline{Y} \in \mathcal{T}_G^{\text{op}}$, define:

$$F_k(\underline{X}, \underline{Y}) := \text{Map}_{\mathcal{T}^{\mathcal{F}_k}}(\underline{C}_k, \tau_k \underline{Y}).$$

Corollary 3.5 *For $\underline{X}$ as above and any fibrant pointed $\underline{Y} \in \mathcal{C}_G^{\text{op}}$, we have a fibration sequence of fibrant simplicial sets:*

$$F_k(\underline{X}, \underline{Y}) \xrightarrow{s_k^*} \text{Map}(\tau_k \underline{X}, \tau_k \underline{Y}) \xrightarrow{p_k} \text{Map}(\tau_{k-1} \underline{X}, \tau_{k-1} \underline{Y}) \quad (3.6)$$

Remark 3.6 Under the assumptions of the Corollary, (3.3) is a tower of fibrations (of Kan complexes), whose (homotopy) limit is the function complex $M = \text{Map}_{\mathcal{T}_G^{\text{op}}}(\underline{X}, \underline{Y})$ we are interested in. Its homotopy spectral sequence thus converges to the homotopy groups of M . In order to make use of it, we need to identify the homotopy groups of the successive fibers F_k .

Definition 3.7 If $\mathbf{X}$ is a G -CW complex and $H \leq G$ is any subgroup let $\mathbf{X}_H := \bigcup_{H < K} \mathbf{X}^K$ denote the union of the fixed point sets under all larger subgroups, which is a sub- W_H -complex of $\mathbf{X}^H$. The quotient W_H -space $\mathbf{X}_H^H := \mathbf{X}^H / \mathbf{X}_H$ will be called the *modified H -fixed point set* of $\mathbf{X}$, with $x_0^H := [\mathbf{X}_H]$ as its base point, and $s_H : \mathbf{X}^H \rightarrow \mathbf{X}_H^H$ the W_H -equivariant quotient map.

Fact 3.8 *The automorphism group W_H fixes x_0^H , and acts freely elsewhere in $\mathbf{X}_H^H$.*

Definition 3.9 If $H < K \leq G$, let:

$$W_H^K := (N_G K \cap N_G H) / H \leq W_H \quad \text{and} \quad \bar{W}_K^H := (N_G K \cap N_G H) / (K \cap N_G H) \leq W_K,$$

with a surjective homomorphism $\pi : W_H^K \rightarrow \bar{W}_K^H$.

Definition 3.10 Let Λ_m^H denote the opposite category of the partially-ordered set of subgroups K of G with $H \leq K \in \widehat{\mathcal{F}}_m$. It embeds in $\mathcal{O}_G^{\text{op}}/(G/H)$ by

$$K \mapsto (i^* : G/K \rightarrow G/H).$$

Recall that a *skeleton* of a category $\mathcal{C}$ is any full subcategory $\text{sk } \mathcal{C}$ whose objects consist of one representative for each isomorphism type (cf. [15, IV, Sect. 4]). Any two skeleta of $\mathcal{C}$ are isomorphic (and equivalent to $\mathcal{C}$).

Lemma 3.11 For every $H \leq G$, Λ_m^H is a skeleton of $\mathcal{E}_m^H$ (cf. Sect. 3.1).

Proof By Fact 2.1, any $\psi^{\text{op}} : G/K \rightarrow G/H$ in $\mathcal{E}_m^H$ can be factored uniquely as an isomorphism $\phi : G/K' \rightarrow G/K$, followed by $i^* : G/K' \rightarrow G/H$ (induced by the inclusion $i : H \hookrightarrow K'$). Thus $\psi^{\text{op}} : G/K \rightarrow G/H$ is isomorphic in $\mathcal{O}_G^{\text{op}}/(G/H)$ to a unique object $i^* : G/K' \rightarrow G/H$ by the unique (vertical) isomorphism:

$$\begin{array}{ccc} G/K' & \xrightarrow{i^*} & G/H \\ \cong \uparrow \phi^{\text{op}} & \nearrow \psi^{\text{op}} = i^* \circ \phi^{\text{op}} & \\ G/K & & \end{array} \quad (3.7)$$

□

3.3 Groups acting on categories

Note that the group W_H acts on the category $\mathcal{O}_G^{\text{op}}/(G/H)$ and its subcategories $\mathcal{E}_m^H$ via functors $\Theta_{\bar{a}} : \mathcal{E}_m^H \rightarrow \mathcal{E}_m^H$ (one for each $\bar{a} \in W_H$), where $\Theta_{\bar{a}}$ takes $\psi^{\text{op}} : G/K \rightarrow G/H$ to $(\phi_a^H)^{\text{op}} \circ \psi^{\text{op}}$.

Therefore, $\Theta_{\bar{a}}$ induces a W_H -action on each skeleton Λ_m^H . This takes $i^* : G/K \rightarrow G/H$ to $(i^a)^* : G/K^a \rightarrow G/H$, by (3.7), using commutativity of:

$$\begin{array}{ccc} G/K & \xrightarrow{(\phi_a^K)^{\text{op}}} & G/K^a \\ \downarrow i^* & & \downarrow (i^a)^* \\ G/H & \xrightarrow{(\phi_a^H)^{\text{op}}} & G/H, \end{array}$$

with $\Theta_{\bar{a}}$ having the same effect on $i^* : G/K \rightarrow G/H$ as $\Theta_{\bar{b}}$ if and only if $ba^{-1} \in N_G H \cap N_G K$.

In particular, if $\bar{a} \in W_H^K \subseteq W_H$, so $(i^a)^* = i^*$, then $\Theta_{\bar{a}}$ acts on i^* via $\pi : W_H^K \rightarrow \bar{W}_K^H$.

Proposition 3.12 If $\mathbf{X}$ is a G -CW complex, $\underline{Y}$ is a fibrant and pointed diagram in $\mathcal{T}_G^{\text{op}}$, and $\underline{X} = \Phi \mathbf{X}$, then $\underline{C}_k(G/H) \cong \mathbf{X}_H^H$ (as a W_H -space) for any $H \in \widehat{\mathcal{S}}_k$, and :

$$F_k(\underline{X}, \underline{Y}) \cong \prod_{G/H \in \text{sk } S_k} \text{Map}_{W_H}(\mathbf{X}_H^H, \underline{Y}(G/H)).$$

Proof By (3.4), $\xi_k \tau_k \underline{X}$ agrees with $\tau_k \underline{X}$ at all objects $G/K \in \mathcal{F}_{k-1}$, so $\underline{C}_k(G/K) = *$ for such G/K . Thus

$$F_k(\underline{X}, \underline{Y}) \cong \text{Map}_{\mathcal{T}^{\mathcal{S}_k}}(\underline{C}_k|_{\mathcal{T}^{\mathcal{S}_k}}, \underline{Y}|_{\mathcal{T}^{\mathcal{S}_k}}).$$

Since all maps in $\mathcal{S}_k$ are isomorphisms, by Sect. 3.2, it suffices to consider only its skeleton $\text{sk } \mathcal{S}_k$. Therefore:

$$F_k(\underline{X}, \underline{Y}) \cong \prod_{G/H \in \text{sk } \mathcal{S}_k} \text{Map}_{W_H}(\underline{C}_k(G/H), \underline{Y}(G/H)). \quad (3.8)$$

The space of $\mathcal{S}_k$ -diagram maps is thus equivalent to the indicated space of W_H -equivariant maps, so we have reduced the study of the fiber in (3.6) once more to the equivariant category of topological spaces, but for a different (and simpler!) finite group.

We see from (3.4) that for each $G/H \in \mathcal{S}_k$, the cofiber $\underline{C}_k(G/H)$ appearing in (3.8) is the (homotopy) colimit of the following diagram $\underline{W} : J \rightarrow \mathcal{T}$:

The forgetful functor $U : \mathcal{E}_{k-1}^H \rightarrow \mathcal{O}_G^{\text{op}}$ sends the object $\phi^{\text{op}} : G/K \rightarrow G/H$ to G/K , so G/H is a cocone for the image subcategory $U(\mathcal{E}_{k-1}^H) \subseteq \mathcal{O}_G^{\text{op}}$, and the indexing category J is:

$$\begin{array}{ccc} U(\mathcal{E}_{k-1}^H) & \xrightarrow{\quad v \quad} & G/H \\ \downarrow & & \\ \{\text{pt}\} & & \end{array}, \quad (3.9)$$

where each $G/K \in U(\mathcal{E}_{k-1}^H)$ is equipped with a map $v_K : G/K \rightarrow G/H$ (coming from $\mathcal{E}_{k-1}^H$). The category J has an W_H -action by endofunctors as above, and the functor $\underline{W}$ is induced by the given $\underline{X}$ is W_H -equivariant.

For a G -CW complex $\mathbf{X}$, all maps in $\underline{W}|_{U(\mathcal{E}_{k-1}^H)}$ are cofibrations (inclusions $\mathbf{X}^L \hookrightarrow \mathbf{X}^K$ for $H \leq K \leq L$), and in fact the (homotopy) colimit of the diagram $\underline{W}|_{U(\mathcal{E}_{k-1}^H)}$ is just $\mathbf{X}_H = \bigcup_{H < K} \mathbf{X}^K$, with the W_H -action induced by the action on $\mathbf{X}^H$.

Furthermore, each $\underline{W}(v_K) : \mathbf{X}^K \hookrightarrow \mathbf{X}^H$ is a cofibration, inducing together a cofibration $\underline{W}(v) : \mathbf{X}_H \hookrightarrow \mathbf{X}^H$. Therefore, the (homotopy) colimit $\underline{C}_k(G/H)$ of $\underline{W}$ over (3.9) is the (homotopy) cofiber of $\underline{W}(v)$, i.e., $\mathbf{X}_H^H$, again with the obvious W_H -action. $\square$

4 Cohomology with local coefficients

For a group Γ , let $\mathbf{E}\Gamma$ denote any contractible space with a free Γ -action, and $\mathbf{B}\Gamma = \mathbf{E}\Gamma/\Gamma$ the classifying space of Γ .

Definition 4.1 For any Γ -space $\mathbf{X}$, the *associated free Γ -space* is $\mathbf{E}\Gamma \times \mathbf{X}$ with diagonal Γ -action. The *Borel construction* on $\mathbf{X}$ is the (homotopy) quotient $X_{h\Gamma} := \mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}$ —that is, the orbit space of $\mathbf{E}\Gamma \times \mathbf{X}$.

When $\mathbf{X}$ has a point x_0 fixed by Γ , the *associated pointed free Γ -space* is the quotient

$$\mathbf{E}\Gamma \ltimes \mathbf{X} := \mathbf{E}\Gamma \times \mathbf{X} / \mathbf{E}\Gamma \times \{x_0\},$$

with Γ -action induced from diagonal action on $\mathbf{E}\Gamma \times \mathbf{X}$. The *pointed Borel construction* on $\mathbf{X}$ is

$$\mathbf{X}_{h\Gamma}^* := \mathbf{E}\Gamma \ltimes_{\Gamma} \mathbf{X} = \mathbf{E}\Gamma \ltimes \mathbf{X} / \Gamma.$$

Note that $\mathbf{E}\Gamma \times_{\Gamma} \{*\} \simeq \mathbf{B}\Gamma$, and the constant map $\mathbf{X} \rightarrow \{*\}$ thus induces a natural map $p : \mathbf{E}\Gamma \times_{\Gamma} \mathbf{X} \rightarrow \mathbf{B}\Gamma$. In particular, for a Γ -module M ,

$$\mathbf{K}_{\Gamma}(M, n) := \mathbf{E}\Gamma \times_{\Gamma} \mathbf{K}(M, n),$$

is called the *twisted Eilenberg-Mac Lane space* (cf. [10]). The canonical map $p : \mathbf{K}_{\Gamma}(M, n) \rightarrow \mathbf{B}\Gamma$ has homotopy fiber $\mathbf{K}(M, n)$.

Definition 4.2 Let M be a Γ -module. The *n th cohomology group with local coefficients* of any space $\theta : \mathbf{X} \rightarrow \mathbf{B}\Gamma$ over $\mathbf{B}\Gamma$ is defined by:

$$H_{\Gamma}^n(\mathbf{X}; M) := \pi_0 \operatorname{Map}_{\mathbf{B}\Gamma}(\mathbf{X}, \mathbf{K}_{\Gamma}(M, n)). \quad (4.1)$$

In particular, any homomorphism $\tau : \pi_1 \mathbf{X} \rightarrow \Gamma$ allows us think of M as a $\pi_1 \mathbf{X}$ -module; composing $\mathbf{B}\tau$ with the canonical map $\mathbf{X} \rightarrow \mathbf{B}\pi_1 \mathbf{X}$ shows that this definition generalizes the original notion of cohomology with a local coefficient system (see [23]).

The mapping space on the right side of (4.1) is defined by the pullback diagram:

$$\begin{array}{ccc} \operatorname{Map}_{\mathbf{B}\Gamma}(\mathbf{X}, \mathbf{K}_{\Gamma}(M, n)) & \longrightarrow & \operatorname{Map}(\mathbf{X}, \mathbf{K}_{\Gamma}(M, n)) \\ \downarrow & & \downarrow p_* \\ \{\theta\} & \longrightarrow & \operatorname{Map}(\mathbf{X}, \mathbf{B}\Gamma). \end{array}$$

Note further that by [11, VI, Sect. 4] we have a natural isomorphism:

$$\operatorname{Map}_{\Gamma}(\mathbf{E}\Gamma \times \mathbf{X}, \mathbf{K}(M, n)) \simeq \operatorname{Map}_{\mathbf{B}\Gamma}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}, \mathbf{K}_{\Gamma}(M, n)). \quad (4.2)$$

Definition 4.3 Let $\mathbf{X}$ be a Γ -space and M a Γ -module. The corresponding *n th reduced cohomology group with coefficients in M* is defined:

$$\tilde{H}_{\Gamma}^n(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}; M) := \pi_0 \operatorname{Map}_{(\mathcal{T}/\mathbf{B}\Gamma)_*}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}, \mathbf{K}_{\Gamma}(M, n))$$

(see [11, Lemma 4.13]). Here $(\mathcal{T}/\mathbf{B}\Gamma)_*$ is the pointed over-category, with objects $u : \mathbf{Y} \rightarrow \mathbf{B}\Gamma$ equipped with a splitting $\sigma : \mathbf{B}\Gamma \rightarrow \mathbf{Y}$, for u . The mapping spaces in this category from the object $p : \mathbf{E}\Gamma \times_{\Gamma} \mathbf{X} \rightarrow \mathbf{B}\Gamma$ with splitting $s : \mathbf{B}\Gamma \rightarrow \mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}$ is defined by the pullback diagram:

$$\begin{array}{ccc} \mathrm{Map}_{(\mathcal{T}/\mathbf{B}\Gamma)_*}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}, \mathbf{Y}) & \longrightarrow & \mathrm{Map}_{\mathbf{B}\Gamma}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}, \mathbf{Y}) \\ \downarrow \text{PB} & & \downarrow s^* \\ \{\sigma\} & \longrightarrow & \mathrm{Map}_{\mathbf{B}\Gamma}(\mathbf{B}\Gamma, \mathbf{Y}). \end{array}$$

Proposition 4.4 *If $\mathbf{X}$ is a pointed Γ -CW complex (with base point x_0 fixed by Γ), and M is a Γ -module, then for any $0 \leq i \leq n$ we have:*

$$\pi_i \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \mathbf{X}, \mathbf{K}(M, n)) \cong \tilde{H}_{\Gamma}^{n-i}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}; M). \quad (4.3)$$

Proof We have a cofibration sequence of Γ -spaces:

$$\begin{array}{ccccc} & & \varphi & & \\ & \swarrow & & \searrow & \\ \mathbf{E}\Gamma \times \{x_0\} & \xrightarrow{j} & \mathbf{E}\Gamma \times \mathbf{X} & \xrightarrow{s} & \mathbf{E}\Gamma \times \mathbf{X} \end{array} \quad (4.4)$$

with $\varphi \circ j = \mathrm{Id}$.

By (4.2) and (4.1) we have

$$\begin{aligned} \pi_i \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \mathbf{X}, \mathbf{K}(M, n)) &\cong H_{\Gamma}^{n-i}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}; M) \quad \text{and} \\ \pi_i \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \{x_0\}, \mathbf{K}(M, n)) &\cong H_{\Gamma}^{n-i}(\mathbf{B}\Gamma, M), \end{aligned} \quad (4.5)$$

for all $0 \leq i \leq n$, since $\mathbf{B}\Gamma := \mathbf{E}\Gamma \times_{\Gamma} \{x_0\}$.

Applying $\mathrm{Map}_{\Gamma}(-, \mathbf{K})$ for $\mathbf{K} = \mathbf{K}(M, n)$ to the top cofibration sequence in (4.4) yields a split fibration sequence of simplicial abelian groups:

$$\begin{array}{ccccc} & & \varphi^* & & \\ & \swarrow & & \searrow & \\ \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \mathbf{X}, \mathbf{K}) & \xrightarrow{s^*} & \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \mathbf{X}, \mathbf{K}) & \xrightarrow{j^*} & \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \{x_0\}, \mathbf{K}) \end{array}$$

and thus (using (4.5)), a split short exact sequence of homotopy groups:

$$0 \rightarrow \pi_i \mathrm{Map}_{\Gamma}(\mathbf{E}\Gamma \times \mathbf{X}, \mathbf{K}(M, n)) \xrightarrow{s^*} H_{\Gamma}^{n-i}(\mathbf{E}\Gamma \times_{\Gamma} \mathbf{X}; M) \xrightarrow{j^*} H_{\Gamma}^{n-i}(\mathbf{B}\Gamma; M) \rightarrow 0.$$

On the other hand, we know that the last two terms fit into a split short exact sequence:

$$0 \rightarrow \tilde{H}_\Gamma^{n-i}(\mathbf{E}\Gamma \times_\Gamma \mathbf{X}; M) \rightarrow H_\Gamma^{n-i}(\mathbf{E}\Gamma \times_\Gamma \mathbf{X}; M) \xrightarrow{j^*} H_\Gamma^{n-i}(\mathbf{B}\Gamma; M) \rightarrow 0 \quad (4.6)$$

(see [11, VI, Sect. 4]), so there is a natural isomorphism (4.3) for each $0 \leq i \leq n$. $\square$

Corollary 4.5 *If $\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z}$ is a cofibration sequence of pointed Γ -CW complexes and M is a Γ -module, we have a long exact sequence in reduced cohomology with local coefficients:*

$$\begin{aligned} \dots \tilde{H}_\Gamma^n(\mathbf{E}\Gamma \times_\Gamma \mathbf{Z}; M) &\xrightarrow{g^*} \tilde{H}_\Gamma^n(\mathbf{E}\Gamma \times_\Gamma \mathbf{Y}; M) \xrightarrow{f^*} \tilde{H}_\Gamma^n(\mathbf{E}\Gamma \times_\Gamma \mathbf{X}; M) \\ &\xrightarrow{\delta} \tilde{H}_\Gamma^{n+1}(\mathbf{E}\Gamma \times_\Gamma \mathbf{Z}; M) \dots \end{aligned}$$

Proof The functorial colimit $\mathbf{E}\Gamma \ltimes -$ preserves cofibration sequences, so

$$\mathbf{E}\Gamma \ltimes \mathbf{X} \xrightarrow{f_*} \mathbf{E}\Gamma \ltimes \mathbf{Y} \xrightarrow{g_*} \mathbf{E}\Gamma \ltimes \mathbf{Z} \quad (4.7)$$

is a homotopy cofibration sequence of pointed Γ spaces. Applying $\mathrm{Map}_\Gamma(-, \mathbf{K}(M, n))$ to (4.7) yields a fibration sequence in $\mathcal{S}_*$, whose long exact sequence in homotopy groups is the required one, by Proposition 4.4.

Lemma 4.6 *If $\mathbf{X}$ is a Γ -CW complex with x_0 fixed by Γ , and free action on $\mathbf{X} \setminus \{x_0\}$, and M is a Γ -module, then for any $0 \leq i \leq n$:*

$$\pi_i \mathrm{Map}_\Gamma(\mathbf{X}, \mathbf{K}(M, n)) \cong \tilde{H}_\Gamma^{n-i}(\mathbf{E}\Gamma \times_\Gamma \mathbf{X}; M).$$

Proof In our case the diagram (4.4) fits into a diagram of Γ -spaces:

$$\begin{array}{ccccc} & & \varphi & & \\ & \swarrow & & \searrow & \\ \mathbf{E}\Gamma \times \{x_0\} & \xrightarrow{j} & \mathbf{E}\Gamma \times \mathbf{X} & \xrightarrow{s} & \mathbf{E}\Gamma \ltimes \mathbf{X} \\ \text{h.e.} \downarrow p & & \text{h.e.} \downarrow q & & \downarrow r \\ \{x_0\} & \xrightarrow{\quad} & \mathbf{X} & \xrightarrow{\mathrm{Id}} & \mathbf{X} \end{array} \quad (4.8)$$

where the vertical maps are projections onto the second factor; since p and q are (non-equivariant) homotopy equivalences, so is r .

Note that $r : \mathbf{E}\Gamma \ltimes \mathbf{X} \rightarrow \mathbf{X}$ induces homotopy equivalences on all fixed point sets (which consist only of the basepoint for all $\{e\} \neq H \leq G$), so by [13, Theorem (1.1)] r is in fact a Γ -homotopy equivalence. Therefore, it induces a weak equivalence:

$$\mathrm{Map}_\Gamma(\mathbf{X}, \mathbf{K}(M, n)) \xrightarrow{r^*} \mathrm{Map}_\Gamma(\mathbf{E}\Gamma \ltimes \mathbf{X}, \mathbf{K}(M, n)), \quad (4.9)$$

so the claim follows from Proposition 4.4. $\square$

From Fact 3.8 and Lemma 4.6 we deduce:

Lemma 4.7 *For any finite group G , G -CW complex $\mathbf{X}$, coefficient system $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \text{AbGp}$, and $n > 0$, let $\underline{X}$ be the fixed point set diagram for $\mathbf{X}$, and $\underline{Y} := \underline{K}(\underline{M}, n)$ the (fibrant) diagram of Eilenberg-Mac Lane spaces in $\mathcal{T}_G^{\text{op}}$ corresponding to $\underline{M}$. Then for each $k \leq \text{len}_G\{e\}$ we have*

$$\pi_i F_k(\underline{X}, \underline{Y}) \cong \bigoplus_{G/H \in \text{sk } S_k} \tilde{H}_{W_H}^{n-i}(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M_H)$$

for $0 \leq i < n$, where $F_k(\underline{X}, \underline{Y})$ is as in Definition 3.4 and $M_H := \underline{M}(G/H)$.

Definition 4.8 Given a G -CW complex $\mathbf{X}$ and fixed $G/H \in S_{k+1}$, we have a diagram of W_H -spaces:

$$\begin{array}{ccccc} \bigcup_{H < L \in \hat{\mathcal{F}}_k} \mathbf{X}_L & \xrightarrow{=} & \bigcup_{H < L \in \hat{\mathcal{F}}_k} \mathbf{X}_L & \longrightarrow & \{*\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{X}_H = \bigcup_{H < L \in \hat{\mathcal{F}}_k} \mathbf{X}_L & \xrightarrow{\quad} & \mathbf{X}^H & \twoheadrightarrow & \mathbf{X}_H^H \\ \downarrow q & & \downarrow & & \downarrow = \\ \bigvee_{H < L \in \hat{\mathcal{S}}_k} \mathbf{X}_L^L & \xrightarrow{\quad} & \mathbf{X}^H / \left(\bigcup_{H < L \in \hat{\mathcal{F}}_k} \mathbf{X}_L \right) & \twoheadrightarrow & \mathbf{X}_H^H \end{array} \quad (4.10)$$

in which all rows and columns are cofibration sequences.

Denote the connecting map for the bottom row in (4.10) (a cofibration sequence of pointed W_H -spaces) by

$$\mathbf{X}_H^H \xrightarrow{\bar{\delta}_H} \Sigma \bigvee_{H < L \in \hat{\mathcal{S}}_k} \mathbf{X}_L^L. \quad (4.11)$$

By Corollary 4.5 we have a long exact sequence in reduced cohomology with coefficients in a W_H -module M , with connecting homomorphism:

$$\tilde{H}_{W_H}^i \left(\mathbf{E}W_H \times_{W_H} \left(\bigvee_{H < L \in \hat{\mathcal{S}}_k} \mathbf{X}_L^L \right); M \right) \xrightarrow{\bar{\delta}_H^*} \tilde{H}_{W_H}^{i+1} \left(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M \right). \quad (4.12)$$

Remark 4.9 If Γ acts on a (pointed) space $\mathbf{Y}$ as a subgroup of Γ' , we can replace $\mathbf{E}\Gamma$ by $\mathbf{E}\Gamma'$ (which is still a contractible space with a free Γ action) in constructing $\mathbf{E}\Gamma \times \mathbf{Y}$, $\mathbf{E}\Gamma \times_{\Gamma} \mathbf{Y}$, $\mathbf{E}\Gamma \rtimes \mathbf{Y}$, and $\mathbf{E}\Gamma \ltimes_{\Gamma} \mathbf{Y}$.

4.1 The change of groups map

Given a G -CW complex $\mathbf{X}$ and $H \in \widehat{\mathcal{F}}_{k+1}$, we have a homeomorphism of pointed spaces:

$$\mathbf{E}W_H \ltimes \left(\bigvee_{H < L \in \widehat{\mathcal{S}}_k} \mathbf{X}_L^L \right) = \bigvee_{H < L \in \widehat{\mathcal{S}}_k} \mathbf{E}W_H \ltimes \mathbf{X}_L^L. \quad (4.13)$$

The W_H -action on the left-hand side transfers to the right-hand side as follows: for each $L > H$ in $\widehat{\mathcal{S}}_k$, any element $\bar{a} \in W_H$ takes $\mathbf{X}_L^L$ to $\mathbf{X}_{L^{\bar{a}}}^{L^{\bar{a}}}$ under left multiplication by a . In particular, $\bar{a} \in W_H^L \leq W_H$ acts on $\mathbf{X}_L^L = \mathbf{X}_{L^{\bar{a}}}^{L^{\bar{a}}}$ by automorphisms (via $\bar{W}_L^H$ —cf. Sect. 3.9).

For any G -module $\underline{M}$, we define the *change of groups map*:

$$\Phi_H^L : \text{Map}_{W_L}(\mathbf{E}W_L \ltimes \mathbf{X}_L^L, \mathbf{K}(M_L, n)) \rightarrow \text{Map}_{W_H} \left(\mathbf{E}W_H \ltimes \left(\bigvee_{H < L \in \widehat{\mathcal{S}}_k} \mathbf{X}_L^L \right), \mathbf{K}(M_H, n) \right)$$

to be the following composite:

$$\begin{aligned} \text{Map}_{W_L}(\mathbf{E}W_L \ltimes \mathbf{X}_L^L, \mathbf{K}(M_L, n)) &\xrightarrow{i} \text{Map}_{\bar{W}_L^H}(\mathbf{E}W_L \ltimes \mathbf{X}_L^L, \mathbf{K}(M_L, n)) \\ &\xrightarrow{\simeq} \text{Map}_{\bar{W}_L^H}(\mathbf{E}\bar{W}_L^H \ltimes \mathbf{X}_L^L, \mathbf{K}(M_L, n)) \xrightarrow{\pi^*} \text{Map}_{W_H^L}(\mathbf{E}W_H^L \ltimes \mathbf{X}_L^L, \mathbf{K}(M_L, n)) \\ &\xrightarrow{\mu_*} \text{Map}_{W_H^L}(\mathbf{E}W_H^L \ltimes \mathbf{X}_L^L, \mathbf{K}(M_H, n)) \xrightarrow{\simeq} \text{Map}_{W_H^L}(\mathbf{E}W_H \ltimes \mathbf{X}_L^L, \mathbf{K}(M_H, n)) \\ &\xrightarrow{\eta} \text{Map}_{W_H} \left(\mathbf{E}W_H \ltimes \left(\bigvee_{H < L' \in \widehat{\mathcal{S}}_k} \mathbf{X}_{L'}^{L'} \right), \mathbf{K}(M_H, n) \right). \end{aligned} \quad (4.14)$$

Here i is an inclusion (since $\bar{W}_L^H \leq W_L$), the homotopy equivalences exist by Remark 4.9, π_* is induced from $\pi : W_H^L \rightarrow \bar{W}_L^H$ by functoriality of $\Gamma \mapsto \mathbf{E}\Gamma$, and $\mu : M_L \rightarrow M_H$

is the $\underline{M}$ -structure map.

The map η is constructed by an “averaging” process, as follows:

First, note that by (4.13), we have:

$$\begin{aligned} &\text{Map}_{W_L} \left(\mathbf{E}W_L \ltimes \left(\bigvee_{L < H' \in \widehat{\mathcal{S}}_k} \mathbf{X}_{H'}^{H'} \right), \mathbf{K}(M_L, n) \right) \\ &\subseteq \prod_{L < H' \in \widehat{\mathcal{S}}_k} \text{Map}_{W_L^{H'}}(\mathbf{E}W_L \ltimes \mathbf{X}_{H'}^{H'}, \mathbf{K}(M_L, n)) \end{aligned} \quad (4.15)$$

so to define η we must specify a $W_L^{H'}$ -equivariant map $f' : \mathbf{E}W_L \ltimes \mathbf{X}_{H'}^{H'} \rightarrow \mathbf{K}(M_L, n)$ in each mapping space in the right hand side of (4.15) (and verify that the result is indeed W_L -equivariant).

For any $a \in N_G H$ and W_H^L -equivariant map $f \in \text{Map}_{W_H^L}(\mathbf{E}W_H \ltimes \mathbf{X}_L^L, \mathbf{K}(M_H, n))$, define:

$$f' := f_a \in \text{Map}_{W_H}(\mathbf{E}W_H \ltimes X_{L^a}^{L^a}, \mathbf{K}(M_H, n))$$

(for $L' := L^a$ to be $f_a := \mu_a \circ f \circ (\phi_a^L)^{\text{op}}$. As in Sect. 2.1,

$$(\phi_a^L)^* : \mathbf{E}W_H \ltimes X_{L^a}^{L^a} \rightarrow \mathbf{E}W_H \ltimes \mathbf{X}_L^L$$

is induced by $\phi_a^L : G/L \rightarrow G/L^a$ and $\mu_a : M_H \rightarrow M_H$ is induced by the automorphism $\phi_a^H : G/H \rightarrow G/H$. It is readily verified that f_a is $W_H^{L^a}$ -equivariant—that is, that the following diagram commutes:

$$\begin{array}{ccccccc} \mathbf{E}W_H \ltimes X_{L^a}^{L^a} & \xrightarrow{(\phi_a^L)^*} & \mathbf{E}W_H \ltimes \mathbf{X}_L^L & \xrightarrow{f} & \mathbf{K}(M_H, n) & \xrightarrow{\mu_a} & \mathbf{K}(M_H, n) \\ \bar{b}^a_* = (\phi_{b^a}^{L^a})^* \downarrow & & \bar{b}_* = (\phi_b^L)^* \downarrow & & \downarrow \mu_a & & \downarrow \mu_{b^a} \\ \mathbf{E}W_H \ltimes X_{L^a}^{L^a} & \xrightarrow{(\phi_a^L)^*} & \mathbf{E}W_H \ltimes \mathbf{X}_L^L & \xrightarrow{f} & \mathbf{K}(M_H, n) & \xrightarrow{\mu_a} & \mathbf{K}(M_H, n) \end{array}$$

for each $b \in N_G H \cap N_G L$ (so b represents an automorphism $\bar{b} \in W_H^L$ and b^a represents $\bar{b}^a \in W_H^{L^a}$). Moreover, we can see that f_a coincides with f when $a \in N_G L$.

If $L' \in \widehat{\mathcal{S}}_k$ which is not conjugate to the give L by an element $a \in N_G H$, we set $f' \in \text{Map}_{W_H^{L'}}(\mathbf{E}W_H \ltimes \mathbf{X}_{L'}^{L'}, \mathbf{K}(M_H, n))$ equal to zero. The resulting sequence $(f')_{H < L' \in \widehat{\mathcal{S}}_k}$ is in fact in the left hand side of (4.15)—that is, it defines a W_H -equivariant map on $\mathbf{E}W_H \ltimes (\bigvee_{H < L' \in \widehat{\mathcal{S}}_k} \mathbf{X}_{L'}^{L'})$, as required.

From Proposition 4.4 we deduce:

Corollary 4.10 *Given a G -CW complex X , a G -module $\underline{M}$, and $H \in \widehat{\mathcal{F}}_{k+1}$, we have a change of groups homomorphism*

$$(\Phi_H^L)_* : \tilde{H}_{W_L}^*(\mathbf{E}W_L \times_{W_L} \mathbf{X}_L^L; M_L) \rightarrow \tilde{H}_{W_H}^* \left(\mathbf{E}W_H \times_{W_H} \left(\bigvee_{H < L' \in \widehat{\mathcal{S}}_k} \mathbf{X}_{L'}^{L'} \right); M_H \right).$$

5 The spectral sequences

Our goal is to understand how the Bredon cohomology of a G -CW complex $\mathbf{X}$ can be computed in terms of *local* information—that is, the fixed point sets X^H for various subgroups $H \leq G$.

Conceptually, this local data should depend only on the Bredon cohomology of $\mathbf{X}^H$ with respect to the action of the automorphism group W_H . In practice, the local data is more delicate, since it is expressed in terms of the reduced cohomology groups of $\mathbf{X}^H$ with local coefficients. In fact, we have two different spectral sequences, which allow

us to compute both Bredon cohomology groups (of $\mathbf{X}$ itself, and of $\mathbf{X}^H$) in terms of reduced cohomology with local coefficients.

First, we state our main result:

Theorem 5.1 *For any finite group G , G -CW complex $\mathbf{X}$, and coefficient system $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \text{AbGp}$, there is a first quadrant spectral sequence with:*

$$E_1^{k,i} = \bigoplus_{G/L \in \text{sk } \mathcal{S}_{k+i}} \tilde{H}_{W_L}^i(\mathbf{E}W_L \times_{W_L} \mathbf{X}_L^L; M_L) \implies \mathbf{H}_G^i(\mathbf{X}; \underline{M}).$$

The differential $d_1 : E_1^{k,i} \rightarrow E_1^{k,i+1} = \bigoplus_{G/H \in \text{sk } \mathcal{S}_{k+i+1}} \tilde{H}_{W_H}^{i+1}(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M_H)$ on each summand is non-zero only if $H < L$, in which case its component

$$\tilde{H}_{W_L}^i(\mathbf{E}W_L \times_{W_L} \mathbf{X}_L^L; M_L) \rightarrow \tilde{H}_{W_H}^{i+1}(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M_H)$$

is the composite of the connecting homomorphism δ^i of (4.12) with the change of groups homomorphism of Corollary (4.10).

Proof Fix $n > 0$ and let $\underline{Y} := \underline{K}(\underline{M}, n)$ be the (fibrant) diagram of Eilenberg-Mac Lane spaces in $\mathcal{T}_G^{\text{op}}$ corresponding to $\underline{M}$, so that $\pi_t \text{Map}_{\mathcal{T}_G^{\text{op}}}(X, \underline{Y}) \cong \mathbf{H}_G^{n-t}(\mathbf{X}; \underline{M})$ for any $0 \leq i \leq n$. If $N = \text{len}_G\{e\}$, then the tower (3.3) is a tower of fibrations (by Corollary 3.5), with $\text{Map}(\underline{X}, \underline{Y}) = \text{Map}(\tau_N \underline{X}, \tau_N \underline{Y})$ as its (homotopy) limit.

The usual homotopy spectral sequence for this tower of fibrations (cf. [2, IX, Sect. 4.2]) has:

$$E_1^{k,t} := \pi_t F_{k-t}(X, \underline{Y}) \Rightarrow \pi_t \text{Map}(\underline{X}, \underline{Y}) \cong \mathbf{H}_G^{n-t}(\mathbf{X}; \underline{M}).$$

By Lemma 4.7 we have:

$$\pi_t F_{k-t} \cong \bigoplus_{G/L \in \text{sk } \mathcal{S}_{k-t}} \tilde{H}_{W_L}^{n-t}(\mathbf{E}W_L \times_{W_L} \mathbf{X}_L^L; M_L).$$

Finally, set $t := n - i$, and note that if we replace n by $n - 1$, we simply apply the loop functor Ω to the diagram $\underline{Y}$, and thus to the tower of fibrations (3.3). Therefore, the spectral sequences for different n all agree.

The differential $d_1^{k-i,i} : E_1^{k-i,i} \rightarrow E_1^{k-i+1,i+1}$ is induced from the long exact sequences in π_* of the fibration sequences (3.6). for k and $k + 1$ by the composite:

$$\begin{aligned} \pi_t \text{Map}_{\mathcal{T}^{\mathcal{F}_k}}(\underline{C}_k, \tau_k \underline{Y}) &\xrightarrow{s_k^*} \pi_t \text{Map}_{\mathcal{T}^{\mathcal{F}_k}}(\tau_k \underline{X}, \tau_k \underline{Y}) \cong \pi_t \text{Map}_{\mathcal{T}^{\mathcal{F}_{k+1}}}(\xi_k \tau_k \underline{X}, \tau_{k+1} \underline{Y}) \\ &\xrightarrow{\partial_t} \pi_{t-1} \text{Map}_{\mathcal{T}^{\mathcal{F}_{k+1}}}(\underline{C}_{k+1}, \tau_{k+1} \underline{Y}), \end{aligned} \quad (5.1)$$

where the first group is:

$$\bigoplus_{G/L \in \text{sk } \mathcal{S}_k} \tilde{H}_{W_L}^i(\mathbf{E}W_L \times_{W_L} \mathbf{X}_L^L; M_L) \quad (5.2)$$

and the last group is:

$$\bigoplus_{G/H \in \text{sk } \mathcal{S}_{k+1}} \tilde{H}_{W_H}^{i+1}(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M_H) \quad (5.3)$$

both of which are *finite* direct sums of abelian groups.

To define a homomorphism into the direct sum (product), it suffices to describe its component in each summand of (5.3) indexed by $G/H \in \text{sk } \mathcal{S}_{k+1}$:

First, note that the connecting homomorphism ∂_i in the fibration sequence (3.6) for $k+1$ is induced by the connecting map δ_{k+1} in the cofibration sequence:

$$\xi_k \tau_k \underline{X} \xrightarrow{\eta_k} \tau_{k+1} \underline{X} \xrightarrow{s_{k+1}} \underline{C}_{k+1} \xrightarrow{\delta_{k+1}} \Sigma \xi_k \tau_k \underline{X}$$

in $\mathcal{T}^{\mathcal{F}_{k+1}}$. Thus given a map of diagrams $\underline{f} : \Sigma \xi_k \tau_k \underline{X} \rightarrow \tau_{k+1} \underline{Y}$, we need to calculate the composite $\underline{f} \circ \delta_{k+1} : \underline{C}_{k+1} \rightarrow \tau_{k+1} \underline{Y}$. However, the diagram $\underline{C}_{k+1}$ is trivial except on $\mathcal{S}_{k+1}$, so this map is completely determined by $(\underline{f} \circ \delta_{k+1})(G/H) = \underline{f}(G/H) \circ \delta_{k+1}(G/H)$ for $G/H \in \mathcal{S}_{k+1}$, where $\delta_{k+1}(G/H)$ is the connecting map δ_H in the middle horizontal cofibration sequence in (4.10).

By (3.4) we have:

$$\Sigma \xi_k \tau_k \underline{X}(G/H) = \Sigma \text{colim } \underline{X}(G/L) = \text{colim } \Sigma \underline{X}(G/L),$$

where the colimit, taken over all $G/L \rightarrow G/H$ in $\mathcal{O}_G^{\text{op}}$ for $G/L \in \mathcal{F}_k$, is $\bigcup_{H < L \in \widehat{\mathcal{F}}_k} \Sigma \underline{X}^L$. Therefore:

$$\underline{f}(G/H) = \bigcup_{H < L \in \widehat{\mathcal{F}}_k} \underline{Y}(i_*^{\text{op}}) \circ \underline{f}(G/L) \quad (5.4)$$

for $\underline{f}(G/L) : \Sigma \underline{X}^L \rightarrow \mathbf{K}(M_L, n)$ and $\underline{Y}(\psi)$ induced by the map $i_*^{\text{op}} : G/L \rightarrow G/H$ in $\mathcal{O}_G^{\text{op}}$ corresponding to the inclusion $i : H \hookrightarrow L$ (inducing the map denoted by μ_* in (4.14)).

The decomposition (5.4) is clearly unaffected by the isomorphism in (5.1), and the map s_k^* is induced by the quotient map s_k in the cofibration sequence:

$$\xi_{k-1} \tau_{k-1} \underline{X} \xrightarrow{\eta_{k-1}} \tau_k \underline{X} \xrightarrow{s_k} \underline{C}_k$$

in $\mathcal{T}^{\mathcal{F}_k}$.

It might seem that we must evaluate s_k at all $G/L \in \mathcal{F}_k$ with $H < L$. However, in order to define the homomorphism out of the direct sum (coproduct) (5.2), it suffices to describe its component on each summand, and since we are mapping into the summand for $G/H \in \text{sk } \mathcal{S}_{k+1}$, we need only consider subgroups $L > H$ with $G/L \in \text{sk } \mathcal{S}_k$. and for these we have: $s_k(G/L) = q$ (the lower left vertical quotient map in (4.10)).

We thus see that the composite (5.1) is induced by the composite:

$$\mathbf{X}_H^H \xrightarrow{\delta_H} \Sigma \mathbf{X}_H = \bigcup_{H < L \in \widehat{\mathcal{F}}_k} \Sigma \mathbf{X}_L^L \xrightarrow{\Sigma q} \bigvee_{H < L \in \widehat{\mathcal{S}}_k} \Sigma \mathbf{X}_L^{L'}$$

which is just $\bar{\delta}_H$ of (4.11) by naturality of the connecting maps in the diagram of cofibration sequences (4.10).

At this point we have shown that, given a W_H -map

$$g : \bigvee_{H < L \in \widehat{\mathcal{S}}_k} \Sigma \mathbf{X}_L^L \longrightarrow \mathbf{K}(M_H, n),$$

or its adjoint

$$\tilde{g} : \bigvee_{H < L \in \widehat{\mathcal{S}}_k} \mathbf{X}_L^L \longrightarrow \Omega \mathbf{K}(M_H, n) = \mathbf{K}(M_H, n-1),$$

we obtain a W_H -map $g \circ (\Sigma q \circ \delta_H) : \mathbf{X}_H^H \rightarrow \mathbf{K}(M_H, n)$. By Lemma 4.7 this yields a homomorphism:

$$\tilde{H}_{W_H}^i \left(\mathbf{E}W_H \times_{W_H} \left(\bigvee_{H < L \in \widehat{\mathcal{S}}_k} \Sigma \mathbf{X}_L^L \right); M_H \right) \xrightarrow{(\Sigma q \circ \delta_H)^*} \tilde{H}_{W_H}^{i+1} \left(\mathbf{E}W_H \times_{W_H} \mathbf{X}_H^H; M_H \right).$$

Now fix $L_0 > H$ in $\widehat{\mathcal{S}}_k$ —that is, our representative in the skeleton of $\mathcal{E}_k^H$ (see Lemma 3.11)—and consider the inclusion of the summand

$$\begin{aligned} \tilde{H}_{W_{L_0}}^i \left(\mathbf{E}W_{L_0} \times_{W_{L_0}} \mathbf{X}_{L_0}^{L_0}; M_{L_0} \right) &\xrightarrow{\iota_{L_0}} \bigoplus_{G/L \in \text{sk } \mathcal{S}_k} \tilde{H}_{W_L}^i \left(\mathbf{E}W_L \times_{W_L} \mathbf{X}_L^L; M_L \right) \\ &\cong \pi_t \text{Map}_{\mathcal{T}\mathcal{F}_k} (\underline{C}_k, \tau_k \underline{Y}) \end{aligned}$$

in the direct sum (5.2).

The composite $s_k^* \circ \iota_{L_0}$ into $\pi_t \text{Map}_{\mathcal{T}\mathcal{F}_k} (\tau_k \underline{X}, \tau_k \underline{Y})$ is induced by the equivalence of categories $\tau : \mathcal{F}_k/(G/H) \rightarrow \text{sk } \mathcal{F}_k/(G/H)$, which extends a W_{L_0} -map $g : \mathbf{X}_{L_0}^{L_0} \rightarrow \underline{Y}(G/L_0) = \mathbf{K}(M_{L_0}, i)$ (representing $[g] \in \tilde{H}_{W_{L_0}}^i \left(\mathbf{E}W_{L_0} \times_{W_{L_0}} \mathbf{X}_{L_0}^{L_0}; M_{L_0} \right)$) to all of $\mathcal{T}\mathcal{F}_k$ precisely by the averaging map η in (4.14). $\square$

5.1 The categories $\mathcal{E}_m^H$ with group action

As noted in Sect. 3.3, for a fixed $G/H \in \mathcal{S}_k$, the group $W_H = \text{Aut}(G/H)$ acts on the categories $\mathcal{E}_m^H$; we can incorporate this action into each $\mathcal{E}_m^H$ to obtain a new category $\tilde{\mathcal{E}}_m^H$ defined as follows:

First, we define a free category $\mathcal{C}$ with objects $\text{Obj}(\mathcal{E}_m^H)$: that is, $\psi^{\text{op}} : G/K \rightarrow G/H$ in $\mathcal{O}_G^{\text{op}}$ with $G/K \in \mathcal{F}_m$. The maps of $\mathcal{C}$ will be generated by those of $\mathcal{E}_m^H$, together with a new map $(\tilde{\phi}_a^K)^{\text{op}} : i^* \rightarrow (i^a)^*$ for any $a \in N_G H$ and $i^* : G/K \rightarrow G/H$ in $\Lambda_m^H \subseteq \text{Obj}(\mathcal{E}_m^H)$ (cf. §3.3).

The forgetful functor $U_m : \mathcal{E}_m^H \rightarrow \mathcal{O}_G^{\text{op}}$ (sending $\psi^{\text{op}} : G/K \rightarrow G/H$ to G/K) extends to $\tilde{U}_m : \mathcal{C} \rightarrow \mathcal{O}_G^{\text{op}}$ in the obvious way, with

$$\tilde{U}_m((\tilde{\phi}_a^K)^{\text{op}}) = (\phi_a^K)^{\text{op}} : G/K \rightarrow G/K^a.$$

We define $\tilde{\mathcal{E}}_m^H$ to be the quotient category of $\mathcal{C}$, with the same object set, in which two morphisms f and g are identified if and only if $\tilde{U}_m f = \tilde{U}_m g$. Thus $\mathcal{E}_m^H$ embeds faithfully (but not fully) in $\tilde{\mathcal{E}}_m^H$.

This implies that if $a \in H \leq K$, then $(\phi_a^K)^{\text{op}}$ is the identity, so $\tilde{\mathcal{E}}_m^H$ encodes the action of $W_H = N_G H/H$ on $\mathcal{E}_m^H$ (see Sect. 3.3). In particular, $(\tilde{\phi}_a^K)^{\text{op}} : i^* \rightarrow (i^a)^*$ will be identified with $(\tilde{\phi}_b^K)^{\text{op}} : i^* \rightarrow (i^b)^*$ if

$$(\phi_a^K)^{\text{op}} = (\phi_b^K)^{\text{op}} : G/K \rightarrow G/K^a = G/K^b,$$

that is, if $b^{-1}a \in K$. In particular, $N_G H \cap N_G K$ acts by automorphisms on $i^* : G/K \rightarrow G/H$, and

$$\text{Aut}_{\tilde{\mathcal{E}}_m^H}(i^*) \cong (N_G H \cap N_G K)/(N_G H \cap K) = \bar{W}_K^H.$$

The filtration (3.2) induces a filtration:

$$\tilde{\mathcal{E}}_0^H \subset \tilde{\mathcal{E}}_1^H \subset \cdots \tilde{\mathcal{E}}_m^H \subset \cdots \subset \tilde{\mathcal{E}}_k^H \quad (5.5)$$

by full subcategories. We let $\tilde{\mathcal{S}}_m^H$ be the full subcategory of $\tilde{\mathcal{E}}_m^H$ with object set $\tilde{\mathcal{E}}_m^H \setminus \tilde{\mathcal{E}}_{m-1}^H$.

Definition 5.2 Given a G -space $\mathbf{X}$, $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{A}bGp$, $n \geq 0$, and a fixed subgroup $H \leq G$, let $\mathbf{Y} := \mathbf{K}(\underline{M}, n)^H$ (an W_H -Eilenberg-Mac Lane space with $\pi_n \mathbf{Y} = M_H := \underline{M}(G/H)$).

We define two diagrams $\hat{X}, \hat{Y} : \tilde{\mathcal{E}}_k^H \rightarrow \mathcal{T}$ by:

- (a) $\hat{X} := \underline{X} \circ \tilde{U}_k$, so $\hat{X}(G/K) := \mathbf{X}^K$ with $\hat{X}(j^*)$ the inclusion $\mathbf{X}^L \hookrightarrow \mathbf{X}^K$ for $j : L \hookrightarrow K$ and $\hat{X}((\phi_a^K)^{\text{op}}) : \mathbf{X}^K \rightarrow \mathbf{X}^{K^a}$ the action of $N_G H$.
- (b) $\hat{Y}(G/K) := \mathbf{Y}^{N_G H \cap K^a}$, with $\hat{Y}(j^*)$ the inclusion, and

$$\widehat{Y} \left((\phi_a^H)^{\text{op}} \right) : \mathbf{Y}^{N_G H \cap K} \longrightarrow \mathbf{Y}^{N_G H \cap K^a}$$

again the given $N_G H$ -action.

Lemma 5.3 *If $\mathbf{X}$ and $\mathbf{Y}$ as above, there is a natural isomorphism*

$$\text{Map}_{\mathcal{T}^{\widetilde{\mathcal{E}}_k^H}}(\widehat{X}, \widehat{Y}) \cong \text{Map}_{W_H}(\mathbf{X}^H, \mathbf{Y}).$$

Proof Note that for any diagram $\widehat{Z} : \widetilde{\mathcal{E}}_k^H \rightarrow \mathcal{T}$, $\widehat{Z}(G/H)$ has a W_H -action, and the restriction $\underline{Z} : \mathcal{E}_k^H \rightarrow \mathcal{T}$ of $\widehat{Z}$ is W_H -equivariant (with respect to the W_H -action of Sect. 3.3).

Since $\widehat{X}(G/H) = \mathbf{X}^H$ and $\widehat{Y}(G/H) = \mathbf{Y}$, we have a projection

$$\text{Map}_{\mathcal{T}^{\widetilde{\mathcal{E}}_k^H}}(\widehat{X}, \widehat{Y}) \rightarrow \text{Map}(\mathbf{X}^H, \mathbf{Y})$$

which lands in $\text{Map}_{W_H}(\mathbf{X}^H, \mathbf{Y})$ (because of the equivariance).

On the other hand, given a W_H -map $f : \mathbf{X}^H \rightarrow \mathbf{Y}$ and $i : H \hookrightarrow K$, because $N_G H \cap K \leq K$, we have an inclusion $j : \mathbf{X}^K \hookrightarrow \mathbf{X}^{N_G H \cap K}$. Moreover, $f|_{\mathbf{X}^{N_G H \cap K}}$ lands in $\mathbf{Y}^{N_G H \cap K} = \widehat{Y}(G/K)$ because any W_H -map is an $N_G H$ -map. Therefore, we can define a map of $\widetilde{\mathcal{E}}_m^H$ -diagrams $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$ by setting

$$\widehat{f}(G/K) : \widehat{X}(G/K) \rightarrow \widehat{Y}(G/K)$$

to be

$$f|_{\mathbf{X}^K} = (f|_{\mathbf{X}^{N_G H \cap K}}) \circ j : \mathbf{X}^K \rightarrow \mathbf{Y}^{N_G H \cap K},$$

and noting that all the groups acting in $\widetilde{\mathcal{E}}_m^H$ do so via the $N_G H$ -action. $\square$

Definition 5.4 Given an inclusion $\ell : H \hookrightarrow L$, let $\langle \ell^* \rangle$ denote the collection of all distinct conjugates $(\ell^a)^* : G/L^a \rightarrow G/H^a = G/H$ of $\ell^* : G/L \rightarrow G/H$ by elements $a \in N_G H$. This class contains $[N_G H : N_G H \cap N_G L]$ distinct elements of $\mathcal{O}_G^{\text{op}}/(G/H)$. When $L \in \widehat{\mathcal{F}}_m$, $\langle \ell^* \rangle$ is an isomorphism class of elements in $\widetilde{\mathcal{E}}_m^H$, so by choosing one representative $\ell^* : G/L \rightarrow G/H$ in each such class $\langle \ell^* \rangle$, we obtain a skeleton $\text{sk } \widetilde{\mathcal{E}}_m^H$ for $\widetilde{\mathcal{E}}_m^H$.

Note that any object $\ell^* : G/L \rightarrow G/H$ in the skeleton (with $\ell : H \hookrightarrow L$ an inclusion) is determined by the object G/L in $\mathcal{O}_G^{\text{op}}$, so by abuse of notation we denote it simply by G/L .

Proposition 5.5 *If $\underline{X} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ is cellular, then for each $H \in \widehat{\mathcal{S}}_k$ and $m \leq k$ the restriction $\underline{X} \circ \widetilde{U}_m : \widetilde{\mathcal{E}}_m^H \rightarrow \mathcal{T}$ is cellular, too.*

Proof For any $L \in \widehat{\mathcal{F}}_m$ and inclusion $i : H \hookrightarrow K$, let

$$\mathcal{V}_K^L := \{(j \circ i)^* : G/L^a \rightarrow G/H : a \in G \text{ and } j : K \hookrightarrow L^a \text{ is an inclusion}\} \subseteq \widetilde{\mathcal{E}}_m^H.$$

Note that each object in $\mathcal{V}_K^L$ is determined by the conjugate L^a (containing K).

In particular, if $K = H$, we let $\mathcal{P}_H^L := \mathcal{V}_H^L \cap \text{sk } \tilde{\mathcal{E}}_m^H$. Moreover, given $\mathcal{P}_H^L$, we can generate all of $\mathcal{V}_H^L$ by conjugating each $j = j \circ \text{Id}_H$ by all possible elements $a \in N_G H$, so $\mathcal{V}_H^L = \coprod_{j^* \in \mathcal{P}_H^L} \langle j^* \rangle$. Since $\mathcal{V}_K^L \subseteq \mathcal{V}_H^L$, this yields a partition:

$$\mathcal{V}_K^L = \coprod_{(j \circ i)^*: G/L^a \rightarrow G/H \in \mathcal{P}_H^L} \langle (j \circ i)^* \rangle \cap \mathcal{V}_K^L \quad (5.6)$$

for any $H \leq K \leq G$.

Now consider a cell $\underline{D}_{G/L}^n$ in $\mathcal{T}_G^{\mathcal{O}_G^{\text{op}}}$ (Sect. 2.5). By definition, for each $G/K \in \mathcal{O}_G^{\text{op}}$,

$$\underline{D}_{G/L}^n(G/K) = \coprod_{\psi^{\text{op}}: G/L \rightarrow G/K \text{ in } \mathcal{O}_G^{\text{op}}} \mathbf{D}_{\psi}^n$$

(see (2.3)). However, every map $\psi^{\text{op}}: G/L \rightarrow G/K$ is determined by choosing a conjugate L^a of L containing K , with inclusion $j: K \hookrightarrow L^a$, and then $\psi^{\text{op}} = j^* \circ (\phi_a^L)^{\text{op}}$ (see Sect. 2.1). Since $(\phi_a^L)^{\text{op}}$ is an isomorphism, the possible choices of such isomorphisms $G/L \rightarrow G/L^a$ (for fixed $L^a \in \mathcal{V}_K^L$) is in one-to-one correspondence with $\text{Aut}(G/L) = W_L$. Thus we have a set bijection between $\text{Hom}_{\mathcal{O}_G^{\text{op}}}(G/L, G/K)$ and $\mathcal{V}_K^L \times W_L$, so:

$$\underline{D}_{G/L}^n(G/K) = \coprod_{\mathcal{V}_K^L} \coprod_{W_L} \mathbf{D}^n = \coprod_{(j \circ i)^* \in \mathcal{P}_H^L} \coprod_{\langle (j \circ i)^* \rangle \cap \mathcal{V}_K^L} \coprod_{W_L / \tilde{W}_L^H} \coprod_{\tilde{W}_L^H} \mathbf{D}^n,$$

where the second equality follows from (5.6) and Definition 3.9.

On the other hand, let $\widehat{D}_{\vartheta^{\text{op}}}^n$ be an n -cell in $\mathcal{T}_{\tilde{\mathcal{E}}_m^H}^{\tilde{\mathcal{E}}_m^H}$, for some $\vartheta^{\text{op}}: G/L \rightarrow G/H$ in $\tilde{\mathcal{E}}_m^H$. Given any $\zeta^{\text{op}}: G/K \rightarrow G/H$ in $\tilde{\mathcal{E}}_m^H$, we have:

$$\widehat{D}_{\vartheta^{\text{op}}}^n(\zeta^{\text{op}}) = \coprod_{\tilde{\psi}^{\text{op}}: \vartheta^{\text{op}} \rightarrow \zeta^{\text{op}} \text{ in } \tilde{\mathcal{E}}_m^H} \mathbf{D}_{\tilde{\psi}}^n.$$

We may assume for simplicity that ϑ^{op} and ζ^{op} are in the skeleton—that is, they are induced by inclusions: $\vartheta^{\text{op}} = \ell^*: G/L \rightarrow G/H$ and $\zeta^{\text{op}} = i^*: G/K \rightarrow G/H$. Because $\tilde{U}_m: \tilde{\mathcal{E}}_m^H \rightarrow \mathcal{O}_G^{\text{op}}$ is faithful, the possible maps $\tilde{\psi}^{\text{op}}$ are determined by $\psi^{\text{op}} := \tilde{U}_m(\tilde{\psi}^{\text{op}}): G/L \rightarrow G/K$ in $\mathcal{O}_G^{\text{op}}$, that is, by the choice of $a \in N_G H$ and $(j \circ i)^* \in \langle \ell^* \rangle \cap \mathcal{V}_K^L$ (for $j: K \hookrightarrow L^a$), and an isomorphism $G/L \rightarrow G/L^a$ in $\tilde{\mathcal{E}}_m^H$. Therefore:

$$\widehat{D}_{\ell^*}^n(i^*) = \coprod_{\langle \ell^* \rangle \cap \mathcal{V}_K^L} \coprod_{\tilde{W}_L^H} \mathbf{D}^n.$$

We conclude that for any cell $\underline{D}_{G/L}^n$ in $\mathcal{T}_G^{\text{op}}$, its restriction $\underline{D}_{G/L}^n \circ \tilde{U}_m$ to $\mathcal{T}_{\tilde{\mathcal{E}}_m^H}$ is itself a coproduct of cells, viz.:

$$\underline{D}_{G/L}^n \circ \tilde{U}_m = \coprod_{(j \circ i)^* \in \mathcal{P}_H^L} \coprod_{W_L/\tilde{W}_L^H} \hat{D}_{(j \circ i)^*}^n.$$

The cellularity of $\underline{X} \circ \tilde{U}_m$ now follows by induction from (2.4) and Fact 2.2. $\square$

Remark 5.6 Note that for $L > H$ we have $N_G H \cap N_G L \subseteq N_G(N_G H \cap L)$, so we have an inclusion

$$(N_G H \cap N_G L)/(N_G H \cap L) = \tilde{W}_L^H \hookrightarrow W_{N_G H \cap L}.$$

Therefore, $\tilde{W}_L^H$ acts on $\underline{M}(G/(N_G H \cap L))$ for any $\mathcal{O}_G^{\text{op}}$ -diagram $\underline{M}$.

Furthermore, there is a one-to-one correspondence between the lattice of groups L with $H \leq L \leq N_G H$ and the lattice of subgroups $\tilde{L} := L/H$ of $W_H = N_G H/H$, which induces an inclusion of categories $I : \mathcal{O}_{W_H}^{\text{op}} \hookrightarrow \mathcal{O}_G^{\text{op}}$. Therefore, given a diagram $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \text{AbGp}$, we can define a coefficient system $\hat{M}_H = \underline{M} \circ I : \mathcal{O}_{W_H}^{\text{op}} \rightarrow \text{AbGp}$, with $\hat{M}_H(W_H/\tilde{L}) := \underline{M}(G/L)$.

Moreover, if $\mathbf{Y} := \mathbf{K}(\underline{M}, n)^H$ as above, then $\mathbf{Y}^L \simeq \mathbf{K}(\underline{M}(G/L), n)$ for any $h \leq L \leq N_G H$, so $\mathbf{Y}$ is a W_H -space of type $\mathbf{K}(\hat{M}_H, n)$ (see Sect. 2.6), and thus

$$\mathbf{H}_{W_H}^i(\mathbf{X}^H; \hat{M}_H) \cong \pi_{n-i} \text{Map}_{W_H}(\mathbf{X}^H, \mathbf{Y})$$

for any G -space $\mathbf{X}$ by [18, II, Sect. 1].

Theorem 5.7 For any finite group G , G -CW complex $\mathbf{X}$, and coefficient system $\underline{M} : \mathcal{O}_G^{\text{op}} \rightarrow \text{AbGp}$, for each subgroup $H \leq G$ there is a first quadrant spectral sequence with:

$$E_1^{m,i} = \bigoplus_{G/L \in \text{sk } \tilde{\mathcal{S}}_{m+i}^H} \tilde{H}_{\tilde{W}_L^H}^i(\mathbf{E}W_L \times_{\tilde{W}_L^H} \mathbf{X}_L^L; M_{N_G H \cap L}) \implies \mathbf{H}_{W_H}^i(\mathbf{X}^H; \hat{M}_H).$$

Proof For $\hat{X}, \hat{Y} : \tilde{\mathcal{E}}_k^H \rightarrow \mathcal{T}$ of Definition 5.2, as in Sect. 3.2, we can restrict $\hat{X}$ and $\hat{Y}$ along (5.5) to obtain $\tau_m \hat{X}, \tau_m \hat{Y} : \tilde{\mathcal{E}}_m^H \rightarrow \mathcal{T}$. By Corollary 3.3, when $\mathbf{X}$ is a G -CW complex, $\underline{X}$ and thus $\tau_m \underline{X}$ are cellular, and thus cofibrant in $\mathcal{T}_G^{\text{op}}$ and $\mathcal{T}_{\mathcal{F}_m}$, respectively. Applying Proposition 5.5 we deduce that $\hat{X} := \underline{X} \circ \tilde{U}_k$ and each $\tau_m \hat{X} := \underline{X} \circ \tilde{U}_m$ ($m \leq k$) is cofibrant, too.

As in Sect. 3.2 the truncation functor $\mathcal{T}_{\tilde{\mathcal{E}}_{m+1}^H} \rightarrow \mathcal{T}_{\tilde{\mathcal{E}}_m^H}$ has a left adjoint $\zeta_m : \mathcal{T}_{\tilde{\mathcal{E}}_m^H} \rightarrow \mathcal{T}_{\tilde{\mathcal{E}}_{m+1}^H}$. As in the proof of Proposition 3.2, one can show that the counit $\varepsilon_{m-1} : \zeta_{m-1} \tau_{m-1} \hat{X} \rightarrow \tau_m \hat{X}$ is a cellular inclusion, and thus a cofibration, in $\mathcal{T}_{\tilde{\mathcal{E}}_m^H}$. Therefore, applying $\text{Map}_{\mathcal{T}_{\tilde{\mathcal{E}}_m^H}}(-, \tau_m \hat{Y})$ to ε_{m-1} yields a fibration

$$\varepsilon_{m-1}^* : \text{Map}_{\mathcal{T}_{\tilde{\mathcal{E}}_m^H}}(\tau_m \hat{X}, \tau_m \hat{Y}) \rightarrow \text{Map}_{\mathcal{T}_{\tilde{\mathcal{E}}_m^H}}(\zeta_{m-1} \tau_{m-1} \hat{X}, \tau_m \hat{Y}) = \text{Map}_{\mathcal{T}_{\tilde{\mathcal{E}}_{m-1}^H}}(\tau_{m-1} \hat{X}, \tau_{m-1} \hat{Y})$$

for each $m \leq k$. We therefore obtain a (finite) tower of fibrations

$$\mathrm{Map}(\tau_k \widehat{X}, \tau_k \widehat{Y}) \xrightarrow{p_k} \dots \mathrm{Map}(\tau_m \widehat{X}, \tau_m \widehat{Y}) \xrightarrow{p_m} \mathrm{Map}(\tau_{m-1} \widehat{X}, \tau_{m-1} \widehat{Y}) \rightarrow \dots$$

(mapping spaces in $\mathcal{T}^{\widetilde{\mathcal{E}}_m^H}$), as in (3.3), with successive fibers $\widehat{F}_m(\widehat{X}, \widehat{Y})$.

The homotopy spectral sequence for this tower converges to the homotopy groups of $\mathrm{Map}(\tau_k \widehat{X}, \tau_k \widehat{Y}) = \mathrm{Map}(\widehat{X}, \widehat{Y})$, which are the required Bredon cohomology groups by Lemma 5.3 and (2.6). The E_1 -term of the spectral sequence can be described in terms of the homotopy groups of the fibers $\widehat{F}_m(\widehat{X}, \widehat{Y})$, which we identify as follows:

By definition,

$$\widehat{F}_m(\widehat{X}, \widehat{Y}) = \left\{ \widehat{f} \in \mathrm{Map}_{\widetilde{\mathcal{E}}_m^H}(\widehat{X}, \widehat{Y}) : \tau_{m-1} \widehat{f} = 0 \right\}.$$

and since $\mathrm{Map}_{\widetilde{\mathcal{E}}_m^H}(\widehat{X}, \widehat{Y})$ is a subspace of $\prod_{\vartheta^{\mathrm{op}} \in \widetilde{\mathcal{E}}_m^H} \mathrm{Map}_{\mathcal{T}}(\widehat{X}(\vartheta^{\mathrm{op}}), \widehat{Y}(\vartheta^{\mathrm{op}}))$, we see that $\widehat{F}_m(\widehat{X}, \widehat{Y})$ is a subspace of $\prod_{\vartheta^{\mathrm{op}} \in \widetilde{\mathcal{S}}_m^H} \mathrm{Map}_{\mathcal{T}}(\widehat{X}(\vartheta^{\mathrm{op}}), \widehat{Y}(\vartheta^{\mathrm{op}}))$, and in fact, it suffices to consider the factors indexed by objects in a skeleton of $\widetilde{\mathcal{S}}_m^H$. Note that any map of diagrams must be equivariant with respect to the action of the automorphisms of such objects, so:

$$\widehat{F}_m(\widehat{X}, \widehat{Y}) \subseteq \prod_{G/L \in \mathrm{sk} \widetilde{\mathcal{S}}_m^H} \mathrm{Map}_{\mathrm{Aut}_{\widetilde{\mathcal{S}}_m^H}(G/L)}(\widehat{X}(G/L), \widehat{Y}(G/L)).$$

In our case $\widehat{X}(G/L) = \mathbf{X}^L$, $\widehat{Y}(G/L) = \mathbf{K}(M_{N_G H \cap L}, n)$, and $\mathrm{Aut}_{\widetilde{\mathcal{S}}_m^H}(G/L) = \bar{W}_L^H \subseteq W_L$, so

$$\widehat{F}_m(\widehat{X}, \widehat{Y}) \subseteq \prod_{G/L \in \mathrm{sk} \widetilde{\mathcal{S}}_m^H} \mathrm{Map}_{\bar{W}_L^H}(\mathbf{X}^L, \mathbf{K}(M_{N_G H \cap L}, n)).$$

However, in order for a collection of $\bar{W}_L^H$ -maps $f_L : \mathbf{X}^L \rightarrow \mathbf{K}(M_{N_G H \cap L}, n)$ ($G/L \in \mathrm{sk} \widetilde{\mathcal{S}}_m^H$) to yield a map of diagrams $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$ in $\widetilde{\mathcal{E}}_m^H$, we need the restrictions to $\widehat{X}(G/K) = \mathbf{X}^K$ to vanish for any $K \geq L$. Since $\mathbf{X}_L = \bigcup_{K \geq L} \mathbf{X}^K$ is a $\bar{W}_L^H$ -invariant subspace of $\mathbf{X}^L$, this is equivalent to each map f_L inducing a (pointed) $\bar{W}_L^H$ -map $\mathbf{X}_L^L = \mathbf{X}^L / \mathbf{X}_L \rightarrow \mathbf{K}(M_{N_G H \cap L}, n)$

Therefore,

$$\widehat{F}_m(\widehat{X}, \widehat{Y}) \cong \prod_{G/L \in \mathrm{sk} \widetilde{\mathcal{S}}_m^H} \mathrm{Map}_{\bar{W}_H^L}(\mathbf{X}_L^L, \mathbf{K}(M_H, n)).$$

where $\bar{W}_H^L$ acts on $\mathbf{K}(M_{N_G H \cap L}, n)$ via the quotient map $\pi : W_H^L \rightarrow \bar{W}_L^H$ as a subgroup of $W_{N_G H \cap L}$, and on $\mathbf{X}_L^L$ via π .

From Fact 3.8 and Lemma 4.6 we deduce:

$$\pi_i \hat{F}_m(\hat{X}, \hat{Y}) \cong \bigoplus_{G/L \in \text{sk } \tilde{\mathcal{S}}_m^H} \tilde{H}_{W_H^L}^{n-i}(\mathbf{E}W_L \times_{\bar{W}_L^H} \mathbf{X}_L^L; M_{N_G H \cap L}).$$

□

Remark 5.8 Given a G -CW complex $\mathbf{X}$ and a coefficient system $\underline{M}$, we have two spectral sequences for computing the Bredon cohomology groups $\mathbf{H}_{W_H}^*(\mathbf{X}^H; \hat{M}_H)$, given by Theorems 5.1 and 5.7, respectively, both starting with reduced cohomology groups with local coefficients. However, they are not identical, since the modified fixed point sets appearing in the E_1 -term of the first spectral sequence are restricted to $(\mathbf{X}^H)_{\bar{L}}^{\bar{L}}$ for $\bar{L} \leq W_H$, while in the second spectral sequence we allow all $\mathbf{X}_L^L$ for $H \leq L \leq G$. Note, however, that if $N_G H \leq L$, then $\bar{W}_L^H$ is trivial, so by (4.6) the reduced cohomology is just ordinary (non-equivariant) cohomology.

Remark 5.9 For finite G and any G -CW complex $\mathbf{X}$, Bredon also defined its cellular equivariant homology groups, and this definition was extended to arbitrary G -spaces by Bröcker in [4], and to arbitrary topological groups by Willson in [24]. However, these constructions are in terms of appropriate chain complexes, rather than mapping spaces of diagrams as in (2.5), so our methods do not yield analogous spectral sequences in Bredon homology.

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