On Quillen's calculation of graded K -theory

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Abstract We adapt Quillen's calculation of graded K -groups of $\mathbb{Z}$ -graded rings with support in $\mathbb{N}$ to graded K -theory, allowing gradings in a product $\mathbb{Z} \times G$ with G an arbitrary group. This in turn allows us to use induction and calculate graded K -theory of $\mathbb{Z}^m$ -graded rings.

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1 Introduction

Let G be a group, written additively, and let A be a G -graded ring. We denote the category of finitely generated G -graded projective right A -modules by $\mathcal{P}gr^G(A)$. This is an exact category with the usual notion of (split) short exact sequence, and we denote

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the Quillen K -groups $K_i(\mathcal{P}gr^G(A))$ by $K_i^G(A)$. The group G acts on the category $\mathcal{P}gr^G(A)$ from the right via $(P, g) \mapsto P(g)$, where $P(g)_h = P_{g+h}$. By functoriality of K -groups this equips $K_i^G(A)$ with the structure of a right $\mathbb{Z}[G]$ -module.

If A is strongly graded, i.e., if $1 \in A_g A_{-g}$ for all $g \in G$, then by Dade's Theorem ([3, Theorem 3.1.1]) the functor $(-)_0 : \mathcal{P}gr^G(A) \rightarrow \mathcal{P}r(A_0)$, $M \mapsto M_0$, is an additive functor with an inverse $-\otimes_{A_0} A : \mathcal{P}r(A_0) \rightarrow \mathcal{P}gr^G(A)$ so that it induces an equivalence of categories. This implies there is a group isomorphism $K_i^G(A) \cong K_i(A_0)$, for $i \geq 0$.

The relation between graded K -groups and non-graded K -groups is not always apparent. For example consider the $\mathbb{Z}$ -graded matrix ring $A = \mathbb{M}_5(F)(0, 1, 2, 2, 3)$, where F is a field (see [3, §9.2] for details). Using graded Morita theory one can show that $K_0(A_0) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ whereas $K_0^{\mathbb{Z}}(A) \cong \mathbb{Z}[x, x^{-1}]$. Note also that $K_0(A) \cong \mathbb{Z}$.

For a $\mathbb{Z}$ -graded ring A with support in $\mathbb{N}$ the graded K -theory of A was determined by Quillen [4, Proposition, p. 107]: The functor $P \mapsto P \otimes_{A_0} A$ induces an isomorphism of $\mathbb{Z}[x, x^{-1}]$ -modules

$$K_i(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}] \cong K_i^{\mathbb{Z}}(A). \quad (1)$$

Contrary to other central theorems in the subject, such as fundamental theorem of K -theory (i.e., $K_i(R[x, x^{-1}]) = K_i(R) \times K_{i-1}(R)$, for R a regular ring), one cannot use an easy induction on (1) to write a similar statement for "multi-variables" rings. For example, it appears that there is no obvious inductive approach to generalise (1) to $\mathbb{Z}^m \times G$ -graded rings. However, by generalising Quillen's argument to take gradings into account on both sides of the isomorphism, such a procedure becomes feasible. We will prove the following statement:

Theorem 1 *Let G be a group, and let A be a $\mathbb{Z} \times G$ -graded ring with support in $\mathbb{N} \times G$. Then there is a $\mathbb{Z}[\mathbb{Z} \times G]$ -module isomorphism*

$$K_i^G(A_{(0, -)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\mathbb{Z} \times G] \cong K_i^{\mathbb{Z} \times G}(A),$$

where $A_{(0, -)} = \bigoplus_{g \in G} A_{(0, g)}$.

By a straightforward induction this now implies:

Corollary 2 *For a $\mathbb{Z}^m \times G$ -graded ring A with support in $\mathbb{N}^m \times G$ there is a $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_m, x_m^{-1}]$ -module isomorphism*

$$K_i^G(A_{(0, -)}) \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_m, x_m^{-1}] \cong K_i^{\mathbb{Z}^m \times G}(A).$$

For a trivial group G this is a direct generalisation of Quillen's theorem to $\mathbb{Z}^m$ -graded rings.

2 Swan's theorem

As in Quillen's calculation the proof of the Theorem is based on a version of Swan's Theorem, modified to the present situation: it provides a correspondence between

isomorphism classes of $\mathbb{Z} \times G$ -finitely generated graded projective A -modules and of G -finitely generated graded projective $A_{(0,-)}$ -modules.

Proposition 3 *Let Γ be a (possibly non-abelian) group. Let A be a Γ -graded ring, A_0 a graded subring of A and $\pi: A \rightarrow A_0$ a graded ring homomorphism such that $\pi|_{A_0} = 1$. (In other words, A_0 is a retract of A in the category of Γ -graded rings.) We denote the kernel of π by A_+ .*

Suppose that for any finitely generated graded right A -module M the condition $MA_+ = M$ implies $M = 0$. Then the natural functor

$$S = - \otimes_{A_0} A: \mathcal{P}r^\Gamma(A_0) \rightarrow \mathcal{P}r^\Gamma(A)$$

induces a bijective correspondence between the isomorphism classes of finitely generated graded projective A_0 -modules and of finitely generated graded projective A -modules. An inverse of the bijection is given by the functor

$$T = - \otimes_A A_0: \mathcal{P}r^\Gamma(A) \rightarrow \mathcal{P}r^\Gamma(A_0).$$

There is a natural isomorphism $T \circ S \cong \text{id}$, and for each $P \in \mathcal{P}r^\Gamma(A)$ a non-canonical isomorphism $S \circ T(P) \cong P$. The latter is given by

$$T(P) \otimes_{A_0} A \rightarrow P, \quad x \otimes a \mapsto g(x) \cdot a, \quad (2)$$

where g is an A_0 -linear section of the epimorphism $P \rightarrow T(P)$.

Proof For any finitely generated graded projective A_0 -module Q we have a natural isomorphism $TS(Q) \cong Q$ given by

$$\nu_Q: TS(Q) = Q \otimes_{A_0} A \otimes_A A_0 \rightarrow Q, \quad q \otimes a \otimes a_0 \mapsto q\pi(a)a_0. \quad (3)$$

We will show that for a graded projective A -module P there is a non-canonical graded isomorphism $ST(P) \cong_{\text{gr}} P$. The lemma then follows.

Consider the natural graded A -module epimorphism

$$f: P \rightarrow T(P) = P \otimes_A A_0, \quad p \mapsto p \otimes 1.$$

Here $T(P)$ is considered as an A -module via the map π . Since $T(P)$ is a graded projective A_0 -module, the map f has a graded A_0 -linear section $g: T(P) \rightarrow P$. This section determines an A -linear graded map

$$\psi: ST(P) = P \otimes_A A_0 \otimes_{A_0} A \rightarrow P, \quad p \otimes a_0 \otimes a \mapsto g(p \otimes a_0) \cdot a,$$

and we will show that ψ is an isomorphism. First note that the map

$$T(f): T(P) \rightarrow TT(P), \quad p \otimes a_0 \mapsto f(p) \otimes a_0 = p \otimes 1 \otimes a_0$$

is an isomorphism (consider $T(P)$ as an A -module via π here). In fact the inverse is given by the isomorphism $TT(P) = P \otimes_A A_0 \otimes_A A_0 \rightarrow P \otimes_A A_0$ which

maps $p \otimes a_0 \otimes b_0$ to $p \otimes (a_0 b_0)$. Tracing the definitions now shows that both composites

$$TST(P) \xrightarrow[\nu_{T(P)}]{T(\psi)} T(P) \xrightarrow[\cong]{T(f)} TT(P)$$

map $p \otimes a_0 \otimes a \otimes b_0 \in P \otimes_A A_0 \otimes_{A_0} A \otimes_A A_0 = TST(P)$ to the element $f(g(p \otimes a_0) \cdot a) \otimes b_0 = p \otimes (a_0 \pi(a) b_0) \otimes 1 \in TT(P)$. This implies that $T(\psi) = \nu_{T(P)}$, which is an isomorphism.

The exact sequence

$$0 \rightarrow \ker \psi \rightarrow ST(P) \xrightarrow{\psi} P \rightarrow \operatorname{coker} \psi \rightarrow 0 \quad (4)$$

gives rise, upon application of the right exact functor T , to an exact sequence

$$TST(P) \xrightarrow{T(\psi)} T(P) \rightarrow T(\operatorname{coker} \psi) \rightarrow 0.$$

Since $T(\psi)$ is an isomorphism we have $T(\operatorname{coker} \psi) = \operatorname{coker} T(\psi) = 0$. Since $\operatorname{coker} \psi$ is finitely generated by (4) this implies $\operatorname{coker} \psi = 0$ (note that $T(M) = M/MA_+$ for every finitely generated module M). In other words, ψ is surjective and (4) becomes the short exact sequence $0 \rightarrow \ker \psi \rightarrow ST(P) \xrightarrow{\psi} P \rightarrow 0$. This latter sequence splits since P is projective; this immediately implies that $\ker \psi$ is finitely generated, and since $T(\psi)$ is injective we also have $T(\ker \psi) = \ker T(\psi) = 0$. The hypotheses guarantee $\ker \psi = 0$ now so that ψ is injective as well as surjective, and thus is an isomorphism as claimed. $\square$

3 A lemma on graded K -theory

Lemma 4 *Let G and Γ be groups, and let A be a G -graded ring. Then, considering A as a $\Gamma \times G$ -graded ring in a trivial way where necessary, the functorial assignment $(M, \gamma) \mapsto M(\gamma, 0)$ induces a $\mathbb{Z}[\Gamma \times G]$ -module isomorphism*

$$K_i^{\Gamma \times G}(A) \cong K_i^G(A) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\Gamma \times G].$$

Proof Let $P = \bigoplus_{(\gamma, g) \in \Gamma \times G} P_{(\gamma, g)}$ be a $\Gamma \times G$ -finitely generated graded projective A -module. Since the support of A is $G = 1 \times G$, there is a unique decomposition $P = \bigoplus_{\gamma \in \Gamma} P_\gamma$, where the $P_\gamma = \bigoplus_{g \in G} P_{(\gamma, g)}$ are finitely generated G -graded projective A -modules. This gives a natural isomorphism of categories

$$\Psi: \mathcal{P}gr^{\Gamma \times G}(A) \xrightarrow{\cong} \bigoplus_{\gamma \in \Gamma} \mathcal{P}gr^G(A).$$

The natural right action of $\Gamma \times G$ on these categories is described as follows: for a given module $P \in \mathcal{P}gr^{\Gamma \times G}(A)$ as above and elements $(\gamma, g) \in \Gamma \times G$ we have

$$P(\gamma, g)_{(\delta, h)} = P_{(\gamma+\delta, g+h)} \text{ and } \Psi(P)_\delta = P_{\gamma+\delta}(g)$$

so that $\Psi(P(\gamma, g)) = \Psi(P)(\gamma, g)$. Since K -groups respect direct sums we thus have a chain of $\mathbb{Z}[\Gamma \times G]$ -linear isomorphisms

$$\begin{aligned} K_i^{\Gamma \times G}(A) &= K_i(\mathcal{P}gr^{\Gamma \times G}(A)) \cong K_i \left(\bigoplus_{\gamma \in \Gamma} \mathcal{P}gr^G(A) \right) \\ &= \bigoplus_{\gamma \in \Gamma} K_i^G(A) \cong K_i^G(A) \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] = K_i^G(A) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma \times G]. \end{aligned}$$

□

4 Proof of Theorem 1

Let A be a $\mathbb{Z} \times G$ -graded ring with support in $\mathbb{N} \times G$. That is, A comes equipped with a decomposition

$$A = \bigoplus_{\omega \in \mathbb{N}} A_{(\omega, -)} \quad \text{where} \quad A_{(\omega, -)} = \bigoplus_{g \in G} A_{(\omega, g)}.$$

The ring A has a $\mathbb{Z} \times G$ -graded subring $A_{(0, -)}$ (with trivial grading in $\mathbb{Z}$ -direction). The projection map $A \rightarrow A_{(0, -)}$ is a $\mathbb{Z} \times G$ -graded ring homomorphism; its kernel is denoted A_+ . Explicitly, A_+ is the two-sided ideal

$$A_+ = \bigoplus_{\omega > 0} A_{(\omega, -)}.$$

We identify the quotient ring A/A_+ with the subring $A_{(0, -)}$ via the projection.

4.1 Functors

If P is a finitely generated graded projective A -module, then $P \otimes_A A_{(0, -)}$ is a finitely generated $\mathbb{Z} \times G$ -graded projective $A_{(0, -)}$ -module. Similarly, if Q is a finitely generated graded projective $A_{(0, -)}$ -module then $Q \otimes_{A_{(0, -)}} A$ is a $\mathbb{Z} \times G$ -finitely generated graded projective A -module. We can thus define functors

$$\begin{aligned} T = - \otimes_A A_{(0, -)}: \mathcal{P}gr^{\mathbb{Z} \times G}(A) &\longrightarrow \mathcal{P}gr^{\mathbb{Z} \times G}(A_{(0, -)}) \\ \text{and } S = - \otimes_{A_{(0, -)}} A: \mathcal{P}gr^{\mathbb{Z} \times G}(A_{(0, -)}) &\longrightarrow \mathcal{P}gr^{\mathbb{Z} \times G}(A). \end{aligned}$$

Since $T(P) = P/P A_+$ we see that *the support of $T(P)$ is contained in the support of P .*

Observe now that if M is a finitely generated $\mathbb{Z} \times G$ -graded A -module and $M A_+ = M$ then $M = 0$; for if $M \neq 0$ there is a minimal $\omega \in \mathbb{Z}$ such that $M_{(\omega, -)} \neq 0$, but $(M A_+)_{(\omega, -)} = 0$. It follows from Proposition 3 that for each graded finitely generated

projective A -module P there is a non-canonical isomorphism $P \cong T(P) \otimes_{A(0,-)} A$ as in (2) which respects the $\mathbb{Z} \times G$ -grading. Explicitly, for a given $(\omega, g) \in \mathbb{Z} \times G$ we have an isomorphism of abelian groups

$$P_{(\omega,g)} \cong \bigoplus_{(\kappa,h)} T(P)_{(\kappa,h)} \otimes A_{(-\kappa+\omega, -h+g)}; \quad (5)$$

the tensor product $T(P)_{(\kappa,h)} \otimes A_{(-\kappa+\omega, -h+g)}$ denotes, by convention, the abelian subgroup of $T(P) \otimes_{A(0,-)} A$ generated by primitive tensors of the form $x \otimes y$ with homogeneous elements $x \in T(P)$ of degree (κ, h) and $y \in A$ of degree $(-\kappa + \omega, -h + g)$.

4.2 Filtration

For a $\mathbb{Z} \times G$ -graded A -module P write $P = \bigoplus_{\omega \in \mathbb{Z}} P_{(\omega,-)}$, where $P_{(\omega,-)} = \bigoplus_{g \in G} P_{(\omega,g)}$. For $\lambda \in \mathbb{Z}$ let $F^\lambda P$ denote the A -submodule of P generated by the elements of $\bigcup_{\omega \leq \lambda} P_{(\omega,-)}$; this is $\mathbb{Z} \times G$ -graded again. As an explicit example, we have

$$F^\lambda A(\omega, g) = \begin{cases} A(\omega, g) & \text{if } \lambda \geq -\omega, \\ 0 & \text{else.} \end{cases}$$

Suppose that P is a finitely generated graded projective A -module. Since the support of A is contained in $\mathbb{N} \times G$ there exists $n \in \mathbb{Z}$ such that $F^{-n} P = 0$ and $F^n P = P$. Write $\mathcal{P}gr_n^{\mathbb{Z} \times G}(A)$ for the full subcategory of $\mathcal{P}gr^{\mathbb{Z} \times G}(A)$ spanned by those modules P which satisfy $F^{-n} P = 0$ and $F^n P = P$. Then $\mathcal{P}gr_n^{\mathbb{Z} \times G}(A)$ is the filtered union of the $\mathcal{P}gr_n^{\mathbb{Z} \times G}(A)$.

Let $P \in \mathcal{P}gr_n^{\mathbb{Z} \times G}(A)$; we want to identify $F^\lambda P$. By definition, the A -module $F^\lambda P$ is generated by the elements of $P_{(\omega,g)}$ for $\omega \leq \lambda$, with $P_{(\omega,g)}$ having been identified in (5). We remark that the direct summands in (5) indexed by $\kappa > \omega$ are trivial as A has support in $\mathbb{N} \times G$. On the other hand, for $\omega \geq \kappa$ a given primitive tensor $x \otimes y \in P_{(\omega,g)}$ with $x \in T(P)_{(\kappa,h)}$ and $y \in A_{(-\kappa+\omega, -h+g)}$ can always be re-written, using the right A -module structure of $T(P) \otimes_{A(0,-)} A$, as

$$x \otimes y = (x \otimes 1) \cdot y \quad \text{where } x \otimes 1 \in T(P)_{(\kappa,h)} \otimes A_{(0,0)} \subseteq P_{(\kappa,h)}.$$

That is, the A -module $F^\lambda P$ is generated by those summands of (5) with $\kappa = \omega \leq \lambda$. We claim now that $F^\lambda P$ is isomorphic to

$$M^{(\lambda)} = \bigoplus_{\kappa \leq \lambda} T(P)_{(\kappa,-)} \otimes_{A(0,-)} A_{(-\kappa, 0)}, \quad (6)$$

considering $T(P)_{(\kappa, -)}$ as a $\mathbb{Z} \times G$ -graded $A_{(0, -)}$ -module with support in $\{0\} \times G$. The homogeneous components of $M^{(\lambda)}$ are given by

$$M_{(\omega, g)}^{(\lambda)} = \bigoplus_{\kappa \leq \lambda} \bigoplus_{h \in G} T(P)_{(\kappa, h)} \otimes A(-\kappa, 0)_{(\omega, -h+g)}.$$

Now elements of the form

$$x \otimes 1 \in T(P)_{(\kappa, h)} \otimes A(-\kappa, 0)_{(\kappa, -h+h)} \subseteq M_{(\kappa, h)}^{(\lambda)}$$

clearly form a set of A -module generators for $M^{(\lambda)}$ so that, by the argument given above, $F^\lambda P$ and $M^{(\lambda)}$ have the same generators in the same degrees. The claim follows. The module $F^\lambda P$ is finitely generated (*viz.*, by those generators of P that have $\mathbb{Z}$ -degree at most λ). Since $T(P)$ is a finitely generated projective $A_{(0, -)}$ -module so is its summand $T(P)_{(k, -)}$; consequently, $P \mapsto F^k P$ is an endofunctor of $\mathcal{P}gr_n^{\mathbb{Z} \times G}(A)$. It is exact as can be deduced from the (non-canonical) isomorphism in (6), using exactness of tensor products.

4.3 Filtration quotients

From the isomorphism $F^k P \cong M^{(k)}$, cf. (6), we obtain an isomorphism

$$F^{k+1} P / F^k P \cong T(P)_{(k, -)} \otimes_{A_{(0, -)}} A(-k, 0); \quad (7)$$

in particular, $F^{k+1} P / F^k P \in \mathcal{P}gr_n^{\mathbb{Z} \times G}(A)$.

The isomorphism (7) depends on the isomorphism (6), and thus ultimately on (2). The latter depends on a choice of a section g of $P \rightarrow T(P)$. Given another section g_0 the difference $g - g_0$ has image in $\ker(P \rightarrow T(P)) = PA_+$. Since A_+ consists of elements of positive $\mathbb{Z}$ -degree only, this implies that the isomorphism $F^{k+1} P \cong M^{(k+1)}$ does not depend on g up to elements in $F^k P$; in other words, the quotient $F^{k+1} P / F^k P$ is independent of the choice of g . Thus the isomorphism (7) is, in fact, a natural isomorphism of functors.

4.4 K -theory

We are now in a position to perform the K -theoretical calculations. First define the exact functor

$$\begin{aligned} \Theta_q : \mathcal{P}gr_q^{\mathbb{Z} \times G}(A_{(0, -)}) &\longrightarrow \mathcal{P}gr_q^{\mathbb{Z} \times G}(A) \\ P &= \bigoplus_{\omega} P_{(\omega, -)} \mapsto \bigoplus_{\omega} P_{(\omega, -)} \otimes_{A_{(0, -)}} A(-\omega, 0); \end{aligned}$$

here $\mathcal{P}gr_q^{\mathbb{Z} \times G}(A_{(0, -)})$ denotes the full subcategory of $\mathcal{P}gr^{\mathbb{Z} \times G}(A_{(0, -)})$ spanned by modules with support in $[-q, q] \times G$, and $P_{(\omega, -)}$ on the right is considered as a $\mathbb{Z} \times G$ -graded $A_{(0, -)}$ -module with support in $\{0\} \times G$.

Next define the exact functor

$$\begin{aligned}\Psi_q : \mathcal{P}gr_q^{\mathbb{Z} \times G}(A) &\longrightarrow \mathcal{P}gr_q^{\mathbb{Z} \times G}(A_{(0,-)}) \\ P &\mapsto \bigoplus_{\omega \in \mathbb{Z}} T(P)_{(\omega,-)} ;\end{aligned}$$

here $T(P)_{(\omega,-)}$ is considered as an $A_{(0,-)}$ -module with support in $\{\omega\} \times G$.

Now $\Psi_q \circ \Theta_q \cong \text{id}$; indeed, the composition sends the summand $P_{(\omega,-)}$ of P to the κ -indexed direct sum of

$$T(P_{(\omega,-)} \otimes_{A_{(0,-)}} A(-\omega, 0))_{(\kappa,-)} \cong \begin{cases} P_{(\omega,-)} & \text{if } \kappa = \omega, \\ 0 & \text{else.} \end{cases}$$

In particular, $\Psi_q \circ \Theta_q$ induces the identity on K -groups. — As for the other composition, we have

$$\Theta_q \circ \Psi_q(P) = \bigoplus_{\omega} T(P)_{(\omega,-)} \otimes_{A_{(0,-)}} A(-\omega, 0) \stackrel{(7)}{=} \bigoplus_{j=-q}^{q-1} F^{j+1}P/F^jP.$$

Since $F^q = \text{id}$, additivity for characteristic filtrations [4, p. 107, Corollary 2] implies that $\Theta_q \circ \Psi_q$ induces the identity on K -groups.

For any $P \in \mathcal{P}gr^{\mathbb{Z} \times G}(A_{(0,-)})$ we have

$$(P \otimes_{A_{(0,-)}} A(-\omega, 0))(0, g) = P(0, g) \otimes_{A_{(0,-)}} A(-\omega, 0),$$

by direct calculation. Hence the functor Θ_q induces a $\mathbb{Z}[G]$ -linear isomorphism on K -groups. Since K -groups are compatible with direct limits, letting $q \rightarrow \infty$ yields a $\mathbb{Z}[G]$ -linear isomorphism $K_i^{\mathbb{Z} \times G}(A_{(0,-)}) \cong K_i^{\mathbb{Z} \times G}(A)$ and thus, by Lemma 4, a $\mathbb{Z}[\mathbb{Z} \times G]$ -module isomorphism

$$K_i^G(A_{(0,-)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\mathbb{Z} \times G] \cong K_i^{\mathbb{Z} \times G}(A).$$

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