Dickson invariants and a new description of $H^*(Q_0S^0, \mathbb{Z}/p\mathbb{Z})$ via $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$

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Abstract Let $\mathcal{A}$ be the mod p Steenrod algebra. A free $\mathcal{A}$ -algebra $V(SD^\#)$ is defined using Dickson subalgebras of various length and an algebra isomorphism $V(SD^\#) \cong H^*(Q_0S^0, \mathbb{Z}/p\mathbb{Z})$ is provided. Certain subalgebras A^k filter $H^*(Q_0S^0, \mathbb{Z}/p\mathbb{Z})$ and their quotient A^k/A^{k-1} is isomorphic as $\mathcal{A}$ -algebras with the universal enveloping algebra on a certain Dickson subalgebra. A consequence of this approach is a new basis for $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$.

Keywords Infinite symmetric group · Dickson algebras · Free Steenrod algebras

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1 Statement of results

Let $Q = \Omega^\infty \Sigma^\infty$ be the stabilization functor and Q_0S^0 the base point component of QS^0 . Here S^0 is the zero sphere. Let $\Sigma_\infty = \varinjlim \Sigma_n$ be the infinite symmetric group and $B\Sigma_\infty$ its classifying space. There exists a natural map between $B\Sigma_\infty$ and Q_0S^0 .

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A Theorem of Quillen [17] and, independently, Barratt and Priddy [3], shows that the (co)-homology of Q_0S^0 is isomorphic to that of $B\Sigma_\infty$. This work aims to give an alternative perspective on this result.

The role of modular invariant theory contributes important information for the cohomology of finite groups. The Dickson algebras play a basic role in describing the cohomology rings of the symmetric groups (see [2] for a full account).

As in [6] we will write the body of the paper as it would be for odd primes and put minor modifications necessary when $p = 2$ in square brackets [].

Let $D = \bigoplus_{k \geq 1} D_k^+$ be the Dickson algebra monomials of positive degree of any length and $SD = \bigoplus_{k \geq 1} SD_k^+$ a certain submodule. For $p = 2$, they coincide. SD_k is a subalgebra of the ring of invariants of the general linear group generated by certain elements:

$$\left\{ d_{k,k-r}, M_{k,j} L_k^{p-2}, M_{k;s_1,s_2} L_k^{p-2} \mid 1 \leq r \leq k, 0 \leq j < k, 0 \leq s_1 < s_2 \leq k-1 \right\},$$

$[\{ d_{k,k-r} \mid 1 \leq r \leq k \} \text{ for } p = 2]$. In this framework an explicit basis for $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ can be constructed in terms of Dickson invariants. All (co)-homologies are understood to be mod p in this work, p is any prime number.

Theorem 12 (a) *For $p = 2$, $H^*(B\Sigma_\infty)$ is isomorphic to the polynomial algebra generated by*

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} \mid (m_1, \dots, m_k) \notin (2\mathbb{N})^k \text{ for } k \geq 1 \right\}.$$

(b) *For $p > 2$, $H^*(B\Sigma_\infty)$ is isomorphic to the tensor product of exterior, polynomial, and polynomial algebras truncated at height p .*

(i) *The exterior generators are*

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} M_{k,j} L_k^{p-2} \prod_{t=1}^{\ell} M_{k;s_{2t-1},s_{2t}}^{\epsilon_t} L_k^{p-2} \mid m_r \in \mathbb{N}, \epsilon_t \in \{0, 1\}, 0 \leq j < k \right\}.$$

(ii) *The polynomial generators are*

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} \mid (m_1, \dots, m_k) \notin (p\mathbb{N})^k \text{ for } k \geq 1 \right\}.$$

(iii) *The polynomial truncated generators are*

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} \prod_{t=1}^{\ell} M_{k;s_{2t-1},s_{2t}}^{\epsilon_t} L_k^{p-2} \mid m_r \in \mathbb{N}, (\epsilon_1, \dots, \epsilon_\ell) \neq \mathbf{0} \text{ for } k \geq 1 \right\}.$$

The algebra basis above for $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ is more explicit than the one given by Nakaoka [15] for $p = 2$ and Adem–Milgram [2]. See also Hung [16].

It is known that there exists a degree preserving bijection between the algebra bases for $H_*(B\Sigma_\infty, \mathbb{Z}/2\mathbb{Z})$ and $H^*(B\Sigma_\infty, \mathbb{Z}/2\mathbb{Z})$ [2, page 204 without proof]. We prove that there does not exist a degree preserving bijection between the algebra bases for $H_*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ and $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ for p odd. However, there exists a bijection between the part of these bases not involving Bockstein elements for any prime. See Theorem 14.

Let $\mathcal{A}$ denote the mod p Steenrod algebra and R the mod p Dyer–Lashof algebra. The dual Hopf algebra R^* is related with Dickson modular invariants of the general linear group where the Steenrod algebra action is completely understood.

Another aspect of this work is to approximate $H^*(Q_0S^0)$ by certain subalgebras defined by Dickson algebras. Let $D^\#$ and $SD^\#$ be the corresponding abelian restricted Lie algebras. Here the Lie bracket is zero. There exists a monomorphism $i : SD \rightarrow H^*(Q_0S^0)$ which is not an $\mathcal{A}$ -module map. The map i provides an algebra generating set for $H^*(Q_0S^0)$. Let $V(SD^\#)$ be the universal enveloping algebra on $SD^\#$. An isomorphism between $V(SD^\#)$ and $H^*(Q_0S^0)$ is provided in Theorem 9.

The isomorphism described in the main Theorem is not an isomorphism of Steenrod modules. We wish to approximate that difference. Certain subalgebras C^k of the universal enveloping algebra $V(SD^\#)$ are defined and their images (A^k) filter $H^*(Q_0S^0)$. The interesting point is that certain quotients of those $\mathcal{A}$ -algebras are isomorphic with free Steenrod algebras generated by certain Dickson submodules.

Theorem 15 *The sequence $A^1 \subset A^2 \subset \subset H^*(Q_0S^0)$ is a complete filtration of $\mathcal{A}$ -subalgebras. There exist isomorphisms of algebras $A^{s+1} \cong A^s \otimes V(SD^\#(s+1))$ and isomorphisms of $\mathcal{A}$ -algebras $A^{s+1}/A^s \cong V(SD^\#(s+1))$.*

The action of $\mathcal{A}$ on the Dickson invariants and the cohomology classes of $B\Sigma_\infty$ is not the same but not completely different. An investigation to that direction is in progress. It would be of interest to know the action of $\mathcal{A}$ on $H^*(B\Sigma_\infty)$. Results for finite symmetric groups are given in [1] and [2].

In Sect. 2, the mod p (co)-homology of QS^0 is discussed. Our algebraic objects come with a variety of additional structures. Wherever they appear to be different, effort has been put to distinguish that difference. Explicit remarks are made on that matter both in Sects. 2 and 4. The modular invariant theory part of this work is discussed in Sect. 3. Section 4 is devoted to the definition and examination of the free object $V(SD^\#)$ and its connection with $H^*(B\Sigma_\infty)$. The proof of Theorem 1 is also given in this section. In Sect. 5, $H^*(Q_0S^0)$ is filtered by a sequence of $\mathcal{A}$ -algebras. Their associated quotients are shown to be free. Our approach is similar to that of Wellington [19]. This paper concludes with Sect. 6 where previous results are applied to prove that Q_0S^0 is H -atomic at p as studied by Campbell et al. [5]. A different proof is given for $p \geq 2$.

2 mod p (co)-homology of QS^0

The mod p cohomology of QS^0 was described by Ib Madsen and Peter May dualizing their calculation of the mod p homology of QS^0 . The description of the same object

as an algebra over the Steenrod algebra is more obscure. This is because, the Steenrod algebra action on the homology primitive elements is a hard problem.

$H^*(Q_0S^0)$ is a Hopf algebra and admits the action of the Steenrod algebra, while $H_*(Q_0S^0)$ is also a Hopf algebra and admits the action of R . The Kronecker pairing and the left $\mathcal{A}$ -module structure on $H^*(Q_0S^0)$ induces a left $\mathcal{A}^{op}$ -module structure on $H_*(Q_0S^0)$. By abuse of notation we refer to this action on $H_*(Q_0S^0)$ as an $\mathcal{A}$ -module structure.

We recall some of the results of Ib Madsen and Peter May on the $\mathcal{A}$ -coalgebra structure of $H_*(QS^0)$ [6, 10]. They described the Pontryagin algebra $H_*(Q_0S^0)$ as the homology algebra determined by local formulas for the coproduct and the action of $\mathcal{A}$.

The Pontryagin product $*$ results from the loop sum

$$\mu : QS^0 \times QS^0 \rightarrow QS^0.$$

Let $\pi_0(S^0) \cong \{x, \emptyset\}$ and $\tilde{N}\pi_0(S^0)$ be the free commutative group generated by $\pi_0(S^0)$ subject to $\emptyset = 1$, where $\emptyset$ is the component of the base point. $\pi_0(QS^0)$ is a group under product and each component is denoted by $[y]$ where y is a point in that component regarded as homology class of degree zero. For the component $[y]^{-1}$ we write y^{-1} for the corresponding point. Let $S^0 \hookrightarrow QS^0$ be the natural inclusion, the generator x of $\tilde{H}_0(S^0)$ is identified with its image in $H_*(QS^0)$.

The coproduct is denoted by

$$\psi : H_*(QS^0) \rightarrow H_*(QS^0) \otimes H_*(QS^0).$$

For $y \in H_*(QS^0)$, we have $\psi(y) = y \otimes [y] + [y] \otimes y + \text{others}$ and $[y] \in H_0(QS^0)$ denotes its connected component. The element $y * [y]^{-1}$ denotes the translation of y to the based point component. Then $H_*(Q_0S^0)$ is a Hopf-subalgebra of $H_*(QS^0)$.

There is a homotopy equivalence

$$QS^0 \simeq Q_0S^0 \times \pi_0(QS^0) \quad \text{given by } x \mapsto (xa^{-1}, [a]), x \in [a].$$

All connected components of QS^0 are homotopy equivalent and $H_0(QS^0)$ is isomorphic to the group ring $\mathbb{Z}_p \tilde{N}\pi_0(S^0) \cong \mathbb{Z}_p[\mathbb{Z}]$. Here the multiplication is given by $[i] * [j] = [i + j]$ and the coproduct by $\psi[i] = [i] \otimes [i]$.

The structure of $H_*(Q_0S^0)$ has been described in terms of Dyer–Lashof operations. For their properties see May [6]. We recall some properties needed for this work. The operation Q^0 acts trivially on $H_*(Q_0S^0)$, $Q^s y = 0$ if $2s < |y|$ while $Q^s y = y^p$ if $2s = |y|$ and $Q^s \emptyset = 0$ for $s > 0$. The element $\emptyset \in H_0(Q_0S^0)$ is the identity for the Pontryagin algebra $H_*(Q_0S^0)$.

Iterates of the Dyer–Lashof operations are of the form $Q^{(I, \varepsilon)} = \beta^{\varepsilon_k} Q^{i_k} \dots \beta^{\varepsilon_1} Q^{i_1}$, where $(I, \varepsilon) = ((i_k, \dots, i_1), (\varepsilon_k, \dots, \varepsilon_1))$ with $\varepsilon_j = 0$ or 1 and i_j a non-negative integer for $j = 1, \dots, k$. For $p = 2$, $\varepsilon_j = 0$ for all j . Its degree is defined by $|I, \varepsilon| := |Q^{(I, \varepsilon)}| = 2(p - 1) \left(\sum_{i=1}^k i_i \right) - \left(\sum_{i=1}^k \varepsilon_i \right) |Q^{(I, \varepsilon)}| = \left(\sum_{i=1}^k i_i \right)$, for $p = 2$. Let $l(I, \varepsilon) = k$ denote the length of (I, ε) or $Q^{(I, \varepsilon)}$ and let the excess

of (I, ε) or $Q^{(I, \varepsilon)}$, denoted by $\text{exc}(Q^{(I, \varepsilon)}) = i_k - \epsilon_k - |Q^{(I(k-1), \varepsilon(k-1))}|$, where $(I(t), \varepsilon(t)) = ((i_t, \dots, i_1), (\epsilon_t, \dots, \epsilon_1))$.

$$\text{exc}(Q^{(I, \varepsilon)}) = i_k - \epsilon_k - 2(p-1) \sum_1^{k-1} i_t, \left[\text{exc}(Q^I) = i_k - \sum_1^{k-1} i_t \right].$$

The empty sequence corresponds to the identity operation and has length zero, degree zero and infinite excess.

We omit the sequence $(\epsilon_k, \dots, \epsilon_1)$, if all $\epsilon_i = 0$ and refer to elements Q^I as having non-negative excess, if $\text{exc}(Q^{(I(t), \varepsilon(t))}) \geq 0$ for all t . We define the weight of $Q^{(I, \varepsilon)}$ to be p^k , if $l(I, \varepsilon) = k$.

There are relations among the iterated operations called Adem relations, so an operation can be reduced to a sum of *admissible* operations after applying Adem relations. A sequence (I, ε) is called *admissible*, if $pi_{j-1} - \epsilon_{j-1} \geq i_j$ for $1 \leq j \leq k-1$ and having non-negative excess.

The Adem relations are given by:

$$Q^r Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i, \quad \text{if } r > ps;$$

and if $p > 2$ and $r \geq ps$,

$$\begin{aligned} Q^r \beta Q^s &= \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} Q^i \\ &\quad - \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \beta Q^i. \end{aligned}$$

The Hopf algebra R of Dyer–Lashof operations, known as the Dyer–Lashof algebra, was studied by Madsen and May. It is not connected.

The opposite of the Steenrod algebra $\mathcal{A}$ acts on R via Nishida relations.

Let $P_*^r = Sq_*^r$ and $\epsilon = 0$, if $p = 2$.

$$\begin{aligned} P_*^r \beta^\epsilon Q^s x &= \sum_{j \geq 0} (-1)^{r+j} \binom{(p-1)(s-r)-\epsilon}{r-pj} \beta^\epsilon Q^{s+j-r} P_*^j x \\ &\quad + \epsilon \sum_{j \geq 0} (-1)^{r+j} \binom{(p-1)(s-r)-1}{r-pj-1} Q^{s+j-r} P_*^j \beta x. \end{aligned}$$

Let us make a remark at this point concerning the sequences (I, ε) . We consider $\mathbb{N}$ as a monoid, then each (I, ε) is an element of $\mathbb{N}^n \times (\mathbb{Z}/2\mathbb{Z})^n$, $[I \in \mathbb{N}^n]$.

Definition 1 Let the admissible sequences be denoted by:

$$(1) \quad I_{k, k-i} = \underbrace{(p^{k-1} - p^{i-1}, \dots, p^{k-i} - 1)}_i, \underbrace{p^{k-i-1}, \dots, p, 1}_{k-i} \times (0, \dots, 0).$$

$$\begin{aligned}
(2) \quad J_{k;k-i} &= (\underbrace{p^{k-1} - p^{i-1}, \dots, p^{k-i} - 1}_i, \underbrace{p^{k-i-1}, \dots, p, 1}_{k-i}) \times (\underbrace{0, \dots, 0}_i, \underbrace{1, 0, \dots, 0}_{k-i-1}). \\
(3) \quad K_{k;k-m,k-s} &= (\underbrace{p^{k-1} - p^{m-1} - p^{s-1}, \dots, p^{k-s} - p^{m-s} - 1}_s, \\
&\quad \underbrace{p^{k-s-1} - p^{m-s-1}, \dots, p^{k-m} - 1}_{m-s}, \underbrace{p^{k-m-1}, \dots, p, 1}_{k-m}) \\
&\quad \times (\underbrace{0, \dots, 0}_s, \underbrace{1, 0, \dots, 0}_{m-s-1}, \underbrace{1, 0, \dots, 0}_{k-m-1}).
\end{aligned}$$

Here $0 \leq i \leq k-1$, $0 \leq s < m \leq k-1$.

The following lemma has been proved by May in [6].

Lemma 1 *Let (I, ε) be an admissible sequence, then each (I, ε) can be decomposed uniquely as follows:*

$$(I, \varepsilon) = \sum a_i I_{k,k-i} + \sum b_i J_{k;k-i} + \sum c_{s,m} K_{k;k-m,k-s}.$$

Here $a_i \in \mathbb{N}$ and all indices involved in $\sum b_i J_{k;k-i} + \sum c_{s,m} K_{k;k-m,k-s}$ are distinct and satisfy $\sum_i b_i, c_{s,m} \in \{1, 0\}$.

The Dyer–Lashof algebra can be decomposed as coalgebras over the opposite Steenrod algebra with respect to length:

$$R = \bigoplus_{k \geq 0} R[k].$$

Let R_0 be the submodule of R generated by all elements $Q^0 - Q^I$ such that $I = (0, \dots, 0)$ with $l(I) > 1$. Then R_0 is a co-ideal of R .

Definition 2 Let πR be the quotient coalgebra R/R_0 and let $(\pi R)^+$ be the submodule of πR consisting of positive degree elements.

Lemma 2 πR is a connected unstable $\mathcal{A}$ -coalgebra.

Definition 3 Let $\overline{\pi R}$ be the submodule of πR generated by those admissible sequences $Q^{(I, \varepsilon)}$ with $\varepsilon \neq \mathbf{0}$ or $I \neq pJ$, i.e. some of its entries is not divisible by p .

Proposition 1 Let $\overline{R}$ ($\overline{\pi R}$) be the submodule of R (πR) respectively generated by elements Q^I ($\overline{Q}^I$) and $Q^{(J, \varepsilon)}$ ($\overline{Q}^{(J, \varepsilon)}$) respectively such that $|I| > 0$; $I \notin p\mathbb{N}^k$ or $\varepsilon \neq \mathbf{0}$. They are $\mathcal{A}$ -modules.

Proof We show that $\overline{R}$ is an $\mathcal{A}$ -module. Let $Q^I \in \overline{R}$ and P_*^r act on the Q^I . The general case is reduced to the following two cases.

(a) Let $c Q^{pi'_k} \dots Q^{pi'_t} P_*^{r'} Q^{i_{t-1}} Q^{pi'_{t-2}} \dots Q^{pi'_1} = c' Q^{pi'_k} \dots Q^{pi'_t} Q^{pi'_{t-1}} Q^{pi'_{t-2}} \dots Q^{pi'_1}$ be a summand of $P_*^r Q^{(i_k, \dots, i_1)}$. Here $i_{t-1} = pm + a_0$ and $1 \leq a_0 \leq p-1$. The Nishida relations imply:

$$pi'_{t-1} = i_{t-1} - r' = pm + a_0 - r'$$

and

$$\binom{(p-1)(pm + a_0 - r')}{r'} = \binom{(p-1)pi'_{t-1}}{p(m - i_{t-1}) + a_0} = 0 \pmod{p}.$$

(b) Let $cQ^{pi'_k} \dots Q^{pi'_t} P_*^{r'} Q^{i_{t-1}} Q^{pi'_{t-2}} \dots Q^{pi'_1} = c'Q^{pi'_k} \dots Q^{pi'_t} Q^{pi'_{t-1}} P_*^j Q^{pi'_{t-2}} \dots Q^{pi'_1} =$

$$c''Q^{pi'_k} \dots Q^{pi'_t} Q^{pi'_{t-1}} Q^{pi'_{t-2}} P_*^{j'} Q^{pi'_{t-3}} \dots Q^{pi'_1}.$$

be a summand of $P_*^r Q^{(i_k, \dots, i_1)}$. Here $i_{t-1} = pm + a_0$ and $1 \leq a_0 \leq p-1$. We again take into account the Nishida relations:

$$i_{t-1} + j - r' = pi'_{t-1}, pj_{t-2} = pi'_{t-2} + j' - j, r' = i_{t-1} + j - pi'_{t-1}$$

and

$$\binom{(p-1)(pm + a_0 - r')}{r' - pj} = \binom{(p-1)(pi'_{t-2} - j)}{r' - pj}.$$

We may suppose that $j' = 0$, otherwise an iterated process results that case. Now,

$$j = p(i'_{t-2} - j_{t-2}), r' = p(m - j_{t-2}) + a_0$$

and

$$\binom{(p-1)(pm + a_0 - r')}{r' - pj} = \binom{(p-1)pA}{pB + a_0} = 0 \pmod{p}.$$

Here A and B are non-zero positive integers. □

Definition 4 Let (I, ε) be an admissible sequence with positive degree, then $Q^{(I, \varepsilon)}x$ corresponds to

$$x_{(I, \varepsilon)} := Q^{(I, \varepsilon)}x * [Q^{(I, \varepsilon)}x]^{-1} \in H_*(Q_0S^0).$$

Here $[Q^{(I, \varepsilon)}x]^{-1} = [-p^{l(I)}]$.

Notation. The modules M discussed in this work all come with a natural basis which is denoted by BM .

Definition 5 Let MS^0 be the $\mathbb{F}_p$ -vector space on

$$BMS^0 = \{Q^{(I, \varepsilon)}x \mid (I, \varepsilon) \text{ is admissible with positive degree}\} \cup \{\emptyset\}.$$

We denote $(BMS^0)^+ = BMS^0 - \{\emptyset\}$.

An ordering is given for $(BMS^0)^+$.

Definition 6 There is a natural order on the elements $Q^{(I, \varepsilon)}$ defined as follows:
 $(I, \varepsilon) < (I', \varepsilon')$; if

- (i) $l(I, \varepsilon) < l(I', \varepsilon')$ or
- (ii) $l(I, \varepsilon) = l(I', \varepsilon')$ and $(I, \varepsilon) < (I', \varepsilon')$ anti-lexicographically.

By the conventions above MS^0 is embedded in $H_*(Q_0S^0)$ and we regard MS^0 as a subcoalgebra.

Theorem 1 (Madsen, May) $H_*(Q_0S^0)$ is the free commutative algebra generated by $(BMS^0)^+$ modulo the ideal generated by

$$\left\{ Q^s y - y^p \mid |y| = 2s \text{ and } y \in (BMS^0)^+ \right\}.$$

Here $y^2 = 0$, if $|y| = \text{odd}$ and $p > 2$.

Another useful version of the Theorem above is the following.

Theorem 2 $H_*(Q_0S^0)$ is the free commutative algebra generated by the subset of $(BMS^0)^+$ consisting of elements of positive excess.

The following Proposition is a bookkeeping application.

Proposition 2 A set of algebra generators for $H_*(Q_0S^0)$ is given by

$$\mathcal{N} = \{ Q^{(I, \varepsilon)} x \mid$$

$$\left. \begin{aligned} \varepsilon = \mathbf{0} \text{ and } I = (i_k, \dots, i_1) &= \sum_0^{k-1} a_i I_{k, k-i}, \quad a_0 \geq 1; \\ (I, \varepsilon) &= \sum a_i I_{k, k-i} + \sum_{0 \leq s < m \leq k-1} c_{s, m} K_{k; k-m, k-s}, \quad \sum_{s \geq 1} c_{s, m} \geq 1; \\ (I, \varepsilon) &= \sum a_i I_{k, k-i} + \sum b_i J_{k; k-i} + \sum_{0 \leq s < m \leq k-1} c_{s, m} K_{k; k-m, k-s}, \quad \sum b_i = 1 \end{aligned} \right\}.$$

Here $a_i \in \mathbb{N}_s$, $b_i, c_{s, m} \in \{0, 1\}$, and the s 's and m 's are all different each other. This set is called a Nakaoka basis.

Proof Let $|Q^I[y]| = 2s$ [$|Q^I[y]| = s$], then according to the given relation in $H_*(Q_0S^0)$, we have:

$$Q^s \left(Q^I[y] * \langle Q^I[y] \rangle^{-1} \right) = \left(Q^I[y] * \langle Q^I[y] \rangle^{-1} \right)^p.$$

The relation above suggests that $\text{exc}(Q^s Q^I) > 0$.

Now let $\varepsilon = \mathbf{0}$ and $I = (i_k, \dots, i_1)$ such that $i_{l+1} = (p-1) \sum_{t=1}^l i_t$ and $i_{l+1} \leq p i_l$ for $l+1 < k$. Suppose that $i_{l+2} > (p-1) \sum_{t=1}^{l+1} i_t = p i_{l+1}$, then our sequence is not admissible. It is known that any admissible sequence (I, ε) can be expressed uniquely by

$$(I, \varepsilon) = \sum_0^{k-1} a_i I_{k,k-i} + \sum_1^k b_i J_{k;k-i} + \sum_{0 \leq s < m \leq k-1} c_{s,m} K_{k;k-m,k-s}.$$

We know that $\text{exc}(I_{k,k-i}) = 0$ for $1 \leq i \leq k$. Since $\varepsilon = \mathbf{0}$, our discussion above suggests that $a_0 \geq 1$.

Let $\varepsilon \neq \mathbf{0}$, then $\text{exc}(J_{k,k-i}) = 1$ and $\text{exc}(K_{k;k-m,k-s}) = 0$ for all i, s and m such that $0 \leq s < m \leq k-1$. The other two cases follow again by the imposed relation. $\square$

A vector space basis for $H_*(Q_0 S^0)$ is given by

$$B = \left\{ y_1^{s_1} \dots y_k^{s_k} \mid \begin{array}{l} k \geq 1, y_i \in (BMS^0)^+, s_i = 1 \text{ if } |y_i| = \text{odd} \\ \text{and } 0 < s_i < p \text{ if } |y_i| = \text{even.} \end{array} \text{ Moreover, } y_1 < y_2 < \dots < y_k \right\}. \quad (2.1)$$

The opposite of the Steenrod algebra $\mathcal{A}$ acts on MS^0 via Nishida relations. The action of $\mathcal{A}$ on MS^0 is extended to $H_*(Q_0 S^0)$ by Cartan formulas which relate the operations to Pontryagin product. Thus MS^0 is an $\mathcal{A}$ -submodule of $H_*(Q_0 S^0)$.

Proposition 3 $MS^0 \cong \pi R$ as $\mathcal{A}$ -coalgebras. This isomorphism is denoted by

$$\phi : \pi R \rightarrow MS^0.$$

Proof Firstly, MS^0 and πR are both coalgebras. We consider the coproduct on basic elements. Let $a = -p^{l(I)}$.

$$\begin{aligned} \psi \left(Q^{(I, \epsilon)} x * [Q^{(I, \epsilon)} x]^{-1} \right) &= \psi \left(Q^{(I, \epsilon)} x * [a] \right) = \psi(Q^{(I, \epsilon)} x) * \psi([a]) \\ &= \left[Q^{(I, \epsilon)} x \otimes [a] + [a] \otimes Q^{(I, \epsilon)} x + \sum_{\substack{I' < I \\ \epsilon' < \epsilon}} Q^{(I', \epsilon')} x \otimes Q^{(I-I', \epsilon-\epsilon')} x \right] * [a] \otimes [a] \\ &= Q^{(I, \epsilon)} x * [a] \otimes \emptyset + \emptyset \otimes Q^{(I, \epsilon)} x * [a] + \sum_{\substack{I' < I \\ \epsilon' < \epsilon}} a_{(I', \epsilon')} Q^{(I', \epsilon')} x * [a] \otimes Q^{(I-I', \epsilon-\epsilon')} x * [a]. \end{aligned}$$

We recall that $\pi R = R/\pi R_0$ is a quotient coalgebra. Let $\varphi : \pi R \rightarrow MS^0$ be defined by $\varphi(\overline{Q^{(I, \epsilon)}}) = Q^{(I, \epsilon)} x$ and $\varphi(\overline{Q^0}) = \emptyset$, then φ is a coalgebra isomorphism. $\square$

There exists a natural epimorphism connecting $H_*(Q_0 S^0)$ and MS^0 and we extend it to πR via the isomorphism of Proposition 3. The map j plays an important role in the proof of our main Theorem. Its hom-dual shall be considered later. Let

$$j : H_*(Q_0 S^0) \rightarrow M S^0 \rightarrow \pi R \quad (2.2)$$

be given by $j(Q^{(I,\varepsilon)}x) = \overline{Q^{(I,\varepsilon)}}$ on $M S^0$ and 0 on $y_1^{s_1} \dots y_k^{s_k}$ for $\sum s_i > 1$. It is convenient to identify $\overline{Q^{(I,\varepsilon)}}$ with $Q^{(I,\varepsilon)}$.

Remark 1 Since the map j induces the isomorphism of our main Theorem, we must note that j is not an $\mathcal{A}$ -module map:

$$j(Q^2 x * \underbrace{Q^1 x * \dots * Q^1 x}_{p-1}) = 0.$$

$P_*^1(Q^2 x * Q^1 x * \dots * Q^1 x) = (Q^1 x)^p \approx Q^{(p-1,1)}x$, but $j(Q^{(p-1,1)}x) = Q^{(p-1,1)}$. Thus, πR can not be considered as the image of an $\mathcal{A}$ -module direct summand.

Let $Q(H_*(Q_0 S^0))$ be the module of the indecomposable elements, i.e. it is generated by the lifts of a basis for the quotient $\mathcal{I}(H_*(Q_0 S^0)) / (\mathcal{I}(H_*(Q_0 S^0)))^2$. Here $\mathcal{I}(-)$ stands for the associated augmentation ideal.

A mapping Φ is induced by φ (Proposition 3) connecting πR and $Q(H_*(Q_0 S^0))$:

$$\pi R \rightarrow H_*(Q_0 S^0) \xrightarrow{\text{onto}} Q(H_*(Q_0 S^0)).$$

Lemma 3 $\ker \Phi$ is an $\mathcal{A}$ -coalgebra.

Proof It suffices to prove $\ker \Phi = \{Q^{(I,\varepsilon)} | \text{exc}(I, \varepsilon) = 0, |(I, \varepsilon)| > 0\} \cup \{\overline{Q^0}\}$.

Let $\beta^\varepsilon Q^s Q^I$ be an element of excess zero, i.e. $2s - \varepsilon = |I|$. Using the Nishida relations, we shall prove that $P_*^r \beta^\varepsilon Q^s Q^I$ is a linear combination of elements of excess zero.

$$\begin{aligned} P_*^r \beta^\varepsilon Q^s Q^I &= \sum_{j \geq 0} (-1)^{r+j} \binom{(p-1)(s-r) - \varepsilon}{r-pj} \beta^\varepsilon Q^{s+j-r} P_*^j Q^I \\ &\quad + \varepsilon \sum_{j \geq 0} (-1)^{r+j} \binom{(p-1)(s-r) - 1}{r-pj-1} Q^{s+j-r} P_*^j \beta Q^I. \end{aligned}$$

Because $\text{exc}(\beta^\varepsilon Q^{s+j-r} P_*^j Q^I) = 2(s+j-r) - \varepsilon - |I| + 2j(p-1) = 2jp - 2r \geq 0$ and $r - pj \geq 0$, we have $r = pj$. Thus $\text{exc}(\beta^\varepsilon Q^{s+j-r} P_*^j Q^I) = 0$.

Let us consider $Q^{s+j-r} P_*^j \beta Q^I$ as well:

$$\text{exc}(Q^{s+j-r} P_*^j \beta Q^I) = 2(s+j-r) - (|I| - 1 - 2j(p-1)) = 2jp - 2r + 2 \geq 0$$

and $r - pj - 1 \geq 0$. Thus $\text{exc}(Q^{s+j-r} P_*^j \beta Q^I) = 0$. □

Φ is an $\mathcal{A}$ -module map and induces an $\mathcal{A}$ -module isomorphism:

$$\pi R/\ker \Phi \cong \mathcal{Q} \left(H_*(Q_0 S^0) \right). \quad (2.3)$$

We close this section by the following Lemma which refines our last statement.

Lemma 4 *The module generated by the Nakaoka basis $\mathcal{N}$ is a co-ideal and isomorphic with $\pi R/\ker \Phi$.*

3 Dickson algebras

In this section basic results concerning the Dickson algebra and its relation with the hom-dual of the Dyer–Lashof algebra are recalled.

It is known that

$$\begin{aligned} H^*(B\mathbb{Z}/p\mathbb{Z} \times \cdots \times B\mathbb{Z}/p\mathbb{Z}) &\cong E(x_1, \dots, x_k) \\ &\otimes P[y_1, \dots, y_k], [P[y_1, \dots, y_k], \text{ for } p = 2] \end{aligned}$$

as $\mathcal{A}$ -algebras. Here $|x_i| = 1$ and $\beta x_i = y_i$ ($|y_i| = 1$ for $p = 2$).

Theorem 3 [7]. *The classical Dickson algebra is described as follows*

$$P[y_1, \dots, y_k]^{GL_k} = P[d_{k,1}, \dots, d_{k,k-1}, d_{k,k}].$$

$$|d_{k,i}| = 2(p^k - p^{k-i}), [|d_{k,i}| = 2^k - 2^{k-i}].$$

Proposition 4 [9, Th. 30, p. 169].

$$P^{p^n}(d_{k,i}^{p^j}) = \begin{cases} d_{k,i+1}^{p^j}, & \text{if } n = j + k - i - 1 \text{ and } i < k \\ -d_{k,i}^{p^j} d_{k,1}^{p^j}, & \text{if } n = j + k - 1 \\ 0, & \text{otherwise} \end{cases} \quad (1) \quad (2)$$

Mui (see [13]) gave an invariant theoretic description of the cohomology algebra of the symmetric group and calculated the associated rings of invariants involving the exterior subalgebra $E(x_1, \dots, x_k)$ of $H^*(B\mathbb{Z}/p\mathbb{Z} \times \cdots \times B\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$.

Theorem 4 [13] *The extended Dickson algebra is described as follows*

$$D_k := (E(x_1, \dots, x_k) \otimes P[y_1, \dots, y_k])^{GL_k}.$$

It is a tensor product of the polynomial algebra $P[y_1, \dots, y_k]^{GL_k}$ and the $\mathbb{Z}/p\mathbb{Z}$ -module spanned by the set of elements consisting of the following polynomials:

$$M_{k;s_1, \dots, s_m} L_k^{p-2}; \quad 1 \leq m \leq k, \text{ and } 0 \leq s_1 < \cdots < s_m \leq k-1.$$

For the relations among the generators above see [13].

The elements $M_{k;s_1,\dots,s_m}$ above have been defined by Mui in [13] as follows:

$$M_{k;s_1,\dots,s_m} = \frac{1}{m!} \begin{vmatrix} x_1 & \cdots & x_1 \\ \vdots & & \vdots \\ x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{p^{k-1}} & \cdots & y_k^{p^{k-1}} \end{vmatrix}, \quad L_k = \begin{vmatrix} y_1 & \cdots & x_k \\ y_1^p & \cdots & y_k^p \\ \vdots & & \vdots \\ y_1^{p^{k-1}} & \cdots & y_k^{p^{k-1}} \end{vmatrix}.$$

Here there are m rows of x_i 's and the s_i -th's powers are omitted, where $0 \leq s_1 < \cdots < s_m \leq k-1$ in the first determinant.

The degrees of the elements above are $|M_{k;s_1,\dots,s_m}(L_k)^{p-2}| = m + 2((p^k - 1) - (p^{s_1} + \cdots + p^{s_m}))$.

Definition 7 (a) Let SD_k be the subalgebra of D_k generated by:

$$d_{k;s+1}, M_{k;s}(L_k)^{p-2} \text{ and } M_{k;s_1,s_2}(L_k)^{p-2}.$$

Here $0 \leq s \leq k-1$. $0 \leq s_1 < s_2 \leq k-1$.

(b) Let I_k stand for the ideal of SD_k generated by

$$\{d_{k,k}, M_{k;i}(L_k)^{p-2} \text{ and } M_{k;0,j}(L_k)^{p-2} \mid 0 \leq i < k, 0 < j < k\}.$$

Proposition 5 [8] D_k and SD_k are $\mathcal{A}$ -algebras.

Definition 8 Let $p(SD_k)$ denote the submodule of SD_k generated by monomials $d = \prod_1^k d_{k,i}^{n_i}$ such that $(n_k, \dots, n_1) \in p^{\mathbb{N}^k}$. Let $\overline{SD}_k$ denote the quotient module of $SD_k/p(SD_k)$.

The next lemma is an application of Proposition 4 for $j = 1$.

Lemma 5 $p(SD_k)$ is an $\mathcal{A}$ -module and so is $\overline{SD}_k$.

4 A free unstable $\mathcal{A}$ -algebra and $H^*(Q_0S^0)$

The first part of this section is devoted to the study of the primitive elements of $H^*(Q_0S^0)$.

The description of $R[k]^*$, the hom-dual of $R[k]$, as an algebra is due to Madsen [10] for $p = 2$ and for p odd to May [6]. Madsen expressed the connection between $R[k]^*$ and Dickson invariants around 1975.

Theorem 5 (Madsen $p = 2$, May $p > 2$) $R[k]^*$ is generated by $\{\omega_{k,i+1} = (Q^{I_{k,i+1}})^*, \tau_{k;i} = (Q^{J_{k,i}})^*, \text{ and } \sigma_{k;s,i} = (Q^{K_{k;s,i}})^* \mid 0 \leq i \leq k-1, \text{ and } 0 \leq s < i\}$, $[\{\omega_{k,i+1} \mid 0 \leq i \leq k-1\}, \text{ for } p = 2]$, modulo the following relations:

- (a) commutativity;
- (b) $\tau_{k;i} \tau_{k;i} = 0$;

- (c) $\tau_{k;s}\tau_{k;i} = \sigma_{k;s,i}\omega_{k,k}$. Here $0 \leq s < i \leq k-1$;
 (d) $\tau_{k;s}\tau_{k;i}\tau_{k;j} = \tau_{k;s}\sigma_{k;i,j}\omega_{k,k}$. Here $0 \leq s < i < j \leq k-1$;
 (e) $\tau_{k;s}\tau_{k;i}\tau_{k;j}\tau_{l;k} = \sigma_{k;s,i}\sigma_{k;j,l}\omega_{k,k}^2$. Here $0 \leq s < i < j < l \leq k-1$.

Theorem 6 (Mù $p = 2$ [14] and Kechagias $p > 2$ [9]) Let $T_k : SD_k \rightarrow R[k]^*$ be given by $T_k(d_{k,i+1}) = \omega_{k,i+1}$, $T_k(M_{k;i}L_k^{p-2}) = \tau_{k,i}$, and $T_k(M_{k;s,i}L_k^{p-2}) = \sigma_{k;s,i}$. Then T_k is a Steenrod algebra isomorphism.

Remark 2 The dual of an element of $R[k]$ is given by an upper triangular matrix. For example, let $p = 3$, then

$$(T_2(d_{2,2}))^2 (T_2(d_{2,1}))^{19} = \omega_{2,2}^2 \omega_{2,1}^{19} = (Q^{44,21})^* + (Q^{45,22})^*.$$

Let D be the submodule generated by positive degree elements of $\bigoplus_{k \geq 1} D_k$ (Theorem 4), it is an $\mathcal{A}$ -module. The weight of a non constant monomial of D_k is defined to be p^k . The pair degree and weight distinguishes uniquely each monomial of D and D is of finite type. SD and $\mathcal{I}$ are defined similarly (Definition 7). Thus, we have the following $\mathcal{A}$ -modules:

$$D = \bigoplus_{k \geq 1} D_k^+, \quad SD = \bigoplus_{k \geq 1} SD_k^+, \quad \mathcal{I} = \bigoplus_{k \geq 1} I_k. \quad (4.1)$$

Madsen and May described $R^* = \prod_{k \geq 0} R[k]^*$. The quotient map $R \twoheadrightarrow \pi R$ induces a monomorphism $(\pi R)^* \hookrightarrow R^*$. Now the Theorem 6 implies the $\mathcal{A}$ -module isomorphism

$$T : SD \rightarrow (\pi R^*)^*. \quad (4.2)$$

Next we define a free unstable $\mathcal{A}$ -algebra following Steenrod [18]. The structure of $H^*(Q_0 S^0)$ will be compared with the corresponding free object.

Definition 9 Let $D^\#$ be the Lie algebra with underlying graded module D and its Lie product zero. Let a restriction (according to Milnor–Moore) ξ be defined on the components $D_k^\#$ of $D^\#$ and extended naturally:

$$\xi : D_k^\# \rightarrow D_k^\#$$

by $\xi(d) = P^m d$, if $|d| = 2m$ and zero otherwise for $p > 2$. If $p = 2$, then $\xi(d) = Sq^{|d|}d$. Here d is a monomial. Let $V(D^\#)$ be the universal enveloping algebra on $D^\#$ given by

$$V(D^\#) = U(D^\#)/I.$$

Here I is the ideal generated by

$$\xi(d) - (d)^p \text{ for } |d| = \text{even or } p = 2.$$

Here $d \in D^\#$ and $(d)^p \in U(D^\#)$. The $\mathcal{A}$ -module structure is determined by the given action on D and the Cartan formula. We consider $SD^\#$ as a sub-restricted Lie algebra of $D^\#$ and $V(SD^\#)$ its corresponding object.

We note that there is no relation between the product in SD_k and the one imposed by the definition above.

The natural morphism $\Delta : SD \rightarrow SD \times SD$ given by $\Delta(d) = (d, d)$ induces a coproduct

$$VD : V(SD^\#) \rightarrow V(SD^\#) \otimes V(SD^\#).$$

Following May [11] the next Proposition and Theorem describe the structure of $V(SD^\#)$.

Proposition 6 *The algebra $V(SD^\#)$ is a primitively generated commutative Hopf algebra with $P(V(SD^\#)) = SD$ and $Q(V(SD^\#)) \cong SD/J$. Here J is the submodule of SD generated by the images of the Steenrod operations which are p -th power maps.*

Theorem 7 [11] *$V(SD^\#)$ is isomorphic to a tensor product of monogenic Hopf algebras.*

Although the structure of $V(SD^\#)$ is quite simple, it provides a very good estimate of $H^*(Q_0S^0)$ as we will examine in Theorem 9. However, as demonstrated by the next remark their structures as $\mathcal{A}$ -modules are quite different.

Remark 3 (1) The image of $(\pi R^+)^*$ under the map $(\varphi^*)^{-1}$ is a subspace of $H^*(Q_0S^0)$ and its isomorphic image SD under T is a subspace of $V(SD^\#)$. These images admit a different Steenrod algebra action: Let $p = 2$ and $(Q^i x)^* \in (MS^0)^* \leq H^*(Q_0S^0)$. $(Q^i x)^*$ has been identified with $(Q^i)^*$ which is the image of $d_{1,1}^i$ under the isomorphism T . The monomial $d_{1,1}^i$ is an element of $V(SD^\#)$. Then $(\varphi^*)^{-1}(Q^i x)^* = (Q^i)^*$ and $T(d_{1,1}^i) = (Q^i)^*$, where $d_{1,1}^i \in SD \leq V(SD^\#)$.

In $V(SD^\#)$: $Sq^1 d_{1,1}^3 = d_{1,1}^4$, $Sq^1(d_{1,1}^1 \cdot d_{1,1}^2) = (d_{1,1}^2)^2 \approx Sq^2 d_{1,1}^2 = d_{1,1}^4$.

Next, let us consider their hom-duals in $H_*(Q_0S^0)$,

$$Sq_*^1(Q^4 x) = Q^3 x, Sq_*^1(Q^2 x \cdot Q^2 x) \approx Sq_*^1(Q^{2,2} x) = 0.$$

But in $H^*(Q_0S^0)$ we only have $Sq^1(Q^3 x)^* = (Q^4 x)^*$.

- (2) Let πR^+ be identified with $(SD^\#)^*$ under T^* . The induced inclusion $\pi R^+ \rightarrow (V(SD^\#))^*$ as a coalgebra map is not the same as an inclusion of vector spaces: $d_{1,1}^1$ and $d_{1,1}^3$ are generators in $V(SD^\#)$ and hence $T^*(Q^1) = (d_{1,1})^*$ and $T^*(Q^3) = (d_{1,1}^3)^*$ are primitives in $(V(SD^\#))^*$. But Q^3 is not a primitive element in πR^+ .

We note that the order defined in Definition 6 can be extended to SD .

Definition 10 Let χ be the function between the module basis of SD_k and the monoid $\mathbb{N}^k \times (\mathbb{Z}/2\mathbb{Z})^k$ given by

$$\chi(d_{k,k-i}) = I_{k,k-i}, \quad \chi(M_{k,i} L_k^{(p-2)}) = J_{k,k-i}, \quad \chi(M_{k,s,m} L_k^{(p-2)}) = K_{k;k-m,k-s},$$

and the rule $\chi(uv) = \chi(u) + \chi(v)$. Here u, v are monomials.

Sequences in $\mathbb{N}^k \times (\mathbb{Z}/2\mathbb{Z})^k$ have been ordered anti-lexicographically and via χ monomials of SD_k are also ordered. Moreover, using the weight and the order defined on the basis of SD_k , elements of SD are ordered. The correspondence

$$u \mapsto Q^{\chi(u)}$$

provides a bijection between the corresponding module bases.

Proposition 7 Let $\overline{R}$ (πR) be the submodule of R (πR) respectively generated by elements Q^I ($\overline{Q^I}$) and $Q^{(J,\epsilon)}$ ($\overline{Q^{(J,\epsilon)}}$) respectively such that $|I| > 0$; $I \notin p\mathbb{N}^k$ or $\epsilon \neq \mathbf{0}$. There exist $\mathcal{A}$ -module isomorphisms:

$$\overline{SD} \cong (\overline{R})^* \cong (\pi R)^*.$$

Proof It suffices to prove that $\overline{SD_k}$ and $(\overline{R[k]})^*$ are isomorphic as $\mathcal{A}$ -modules.

Let $\xi : R^* \rightarrow R^*$ be the p -th power map and $(\overline{R[k]})^* = \text{co ker}(\xi)$ where ξ is restricted on $(R[k])^*$. Wellington (page 47) proved that $(\overline{R[k]})$ is an $\mathcal{A}$ -submodule of $R[k]$ generated by elements Q^I ($\overline{Q^I}$) and $Q^{(J,\epsilon)}$ ($\overline{Q^{(J,\epsilon)}}$) such that $|I| > 0$; $I \notin p\mathbb{N}^k$ or $\epsilon \neq \mathbf{0}$.

We recall the Lemma 5. Now our claim follows using the next $\mathcal{A}$ -commutative diagram.

$$\begin{array}{ccccc} (R[k])^* & \xrightarrow{\xi} & (R[k])^* & \twoheadrightarrow & (\overline{R[k]})^* \\ T_k \uparrow \cong & & T_k \uparrow \cong & & \overline{T}_k \uparrow \cong \\ SD_k & \xrightarrow{\xi} & SD_k & \twoheadrightarrow & \overline{SD_k} \end{array}$$

□

An application of the map χ is the following isomorphism.

Theorem 8 [4] $I_k \cong \langle \mathcal{N}[k] \rangle^*$. Here $\mathcal{N}[k]$ contains the elements of length k of the Nakaoka basis $\mathcal{N}$ (Proposition 2).

Proof We follow May [6, page 30], and apply the isomorphism described in the Theorem 6,

$$\langle T_k(d), Q^{\chi(d)} \rangle = 1 \text{ and } \langle T_k(d'^{\chi(u)}), Q^{\chi(d')} \rangle \neq 0 \Rightarrow \chi(d') \leq \chi(d).$$

Here $d, d' \in SD_k$. Let $d \in I_k$ and Q^I such that $\text{exc}(I) = 0$, then $Q^{\chi(d)} > Q^I$ and $\langle T_k(d), Q^I \rangle = 0$. The image of $T_k(I_k)$ (Definition 7) is a subset of $(\pi R^+[k])^*$. □

Corollary 1 Let $Q^I \in \pi R^+[k]$ and $I = \chi(d)$ for some monomial $d \in SD_k$. Then $T^{-1}(Q^I)^* = d + \text{others}$ and $\chi(d') \leq I$ for d' a monomial in others.

The isomorphism in Theorem 8 provides the next $\mathcal{A}$ -module isomorphism via the isomorphism described in (2.3). We recall that $\mathcal{I}$ has been defined in (4.1).

Corollary 2 $P(H^*(Q_0 S^0)) \cong \mathcal{I}$.

For the rest of the section we shall study $V(SD^\#)$ in details.

Using the defined total ordering on SD and the imposed relations on $V(SD^\#)$, a basis is obtained as follows:

$$BV(SD^\#) = \{1\} \cup \left\{ (d_1)^{m_1} \cdots (d_k)^{m_k} \mid \begin{array}{l} k \geq 1, d_i \in B(SD), d_1 < \cdots < d_k, m_i = 1 \\ \text{if } |d_i| = \text{odd and } 0 < m_i < p \text{ if } |d_i| = \text{even} \end{array} \right\}. \quad (4.3)$$

Here $B(SD)$ stands for the basis of monomials of SD .

Lemma 6 The restriction ξ is a morphism of coalgebras and respects the given ordering.

Proof Let d and d' be basis elements such that $d < d'$ and $\xi(d), \xi(d') \neq 0$. If d is smaller than d' because of different weight, it follows that $\xi(d) < \xi(d')$. This is because the Steenrod operations respect the weight. Let d and d' be polynomial algebra generators of SD_k . By the definition and the Steenrod algebra action given in the Proposition 4, it follows that $\xi(d) < \xi(d')$. The Cartan formula gives $\xi(d) < \xi(d')$ for two monomials in SD_k . The natural inclusion $SD \hookrightarrow V(SD^\#)$ is compatible with ξ . Now the internal Cartan formula guarantees that ξ is a map of coalgebras.

$$\begin{array}{ccc} V(SD^\#) & \xrightarrow{\xi} & V(SD^\#) \\ \downarrow & & \downarrow \\ V(SD^\#) \otimes V(SD^\#) & \xrightarrow{\xi \otimes \xi} & V(SD^\#) \otimes V(SD^\#). \end{array}$$

□

The dual mapping ξ_* is the p -th root map defined both in $(V(SD^\#))^*$ and $H_*(Q_0 S^0)$ and it is an algebra mapping respecting the given order. The next Proposition follows.

Proposition 8 Let $Q^I x$ be a basis element for the generating set of $H_*(Q_0 S^0)$ such that $\xi_*(Q^I x) \neq 0$, then

$$\xi_*(Q^I x) = Q^{I/p} \xi_*(x) \text{ and } \xi_*(y_1^{s_1} \cdots y_k^{s_k}) = \xi_*(y_1)^{s_1} \cdots \xi_*(y_k)^{s_k}.$$

Here $y_1^{s_1} \cdots y_k^{s_k} \in B$ and I/p is non-zero as long as each entry is divisible by p .

Before proceeding to the next Theorem let us make the comment that the map (2.2) $j : H_*(Q_0 S^0) \rightarrow \pi R$ provides information on the primitive elements of $H_*(Q_0 S^0)$.

Proposition 9 (Wellington) *The map $j : H_*(Q_0S^0) \rightarrow \pi R$ restricted on the primitive elements is a monomorphism.*

Proof The proof depends on Milnor and Moore [11, 12, section 7]. We recall that $H_*(Q_0S^0)$ is a connected commutative Hopf algebra. The properties of $H_*(Q_0S^0)$ are studied using its associated bigraded algebra $E^0(H_*(Q_0S^0))$. The augmentation filtration $I(H_*(Q_0S^0))$ of $H_*(Q_0S^0)$ defines the bigraded module

$$E^0(H_*(Q_0S^0))_{p,q} = \left((I(H_*(Q_0S^0)))^p / (I(H_*(Q_0S^0)))^{p+1} \right)_{p+q}.$$

Now $E^0(H_*(Q_0S^0))$ is a primitively generated bigraded Hopf algebra with comultiplication induced by that of $H_*(Q_0S^0)$. Since $H_*(Q_0S^0)$ is connected with commutative multiplication, $H_*(Q_0S^0)$ and $\oplus E^0(H_*(Q_0S^0))$ are isomorphic as algebras. Here $\oplus E^0(H_*(Q_0S^0))$ is $E^0(H_*(Q_0S^0))$ regarded by total degree. Let f be the natural monomorphism $f : \pi R \rightarrow H_*(Q_0S^0)$. Then $j f = 1_{\pi R}$. The map j is filtration preserving. Let $F_s(\pi R) = \pi R \cap j(F_s(H_*(Q_0S^0)))$ and $E^0(\pi R)$ the associated bigraded module. The mapping

$$E^0(f) : E^0(\pi R) \rightarrow E^0(H_*(Q_0S^0))$$

is a monomorphism and let $\overline{E^0(f)} : V(E^0(\pi R)) \rightarrow E^0(H_*(Q_0S^0))$ be the induced mapping. Since

$$\mathcal{Q}E^0(H_*(Q_0S^0)) \cong \mathcal{Q}(H_*(Q_0S^0)) \cong \pi R / \ker \varphi,$$

the induced mapping

$$\mathcal{Q}(V(E^0(\pi R))) \rightarrow \mathcal{Q}(E^0(H_*(Q_0S^0)))$$

is an epimorphism and $\overline{E^0(f)}$ is also an epimorphism between Hopf algebras. Therefore

$$V(E^0(\pi R)) \cong V(E^0(H_*(Q_0S^0))).$$

Let $E^0(j)$ be the induced map

$$E^0(j) : E^0(H_*(Q_0S^0)) \rightarrow E^0(\pi R)$$

Now $E^0(\pi R) \cong P(E^0(H_*(Q_0S^0)))$ and $E^0(j)$ is a monomorphism on $P(E^0(H_*(Q_0S^0)))$.

Following Milnor and Moore [12, page 240], j is a monomorphism on primitive elements. $\square$

Theorem 9 The mapping $i : SD \rightarrow H^*(Q_0S^0)$ imposed by the composition of $j^* : (\pi R)^* \rightarrow H^*(Q_0S^0)$ and the isomorphism $T : SD \rightarrow (\pi R^+)^*$ provides a map

$$e : V(SD^\#) \rightarrow H^*(Q_0S^0)$$

which is an isomorphism of algebras.

Proof As vector spaces, $V(SD^\#)$ and $H^*(Q_0S^0)$ are both of finite type and in each dimension they have the same cardinality. Since $j : H_*(Q_0S^0) \rightarrow \pi R$ is a monomorphism on the primitive elements, $i(SD)$ generates $H^*(Q_0S^0)$ as an algebra. Thus the map e is also an epimorphism. $\square$

As demonstrated by the next remark their structures as $\mathcal{A}$ -modules are quite different.

Remark 4 $H^*(Q_0S^0)$ and $V(SD^\#)$ are not isomorphic as $\mathcal{A}$ -algebras: Let $p = 2$. As an example the algebra generators of dimension four will be considered. Firstly, we express all monomials of dimension four.

$$\begin{aligned} H^4(Q_0S^0) = & \left\langle (Q^4(x))^*, ((Q^2(x))^*)^2 \approx (Q^{2,2}(x))^*, ((Q^1(x))^*)^4 \approx (Q^{2,1,1}(x))^*, \right. \\ & \left. (Q^3(x))^* \cdot (Q^1(x))^*, (Q^2(x))^* \cdot ((Q^1(x))^*)^2, (Q^{2,1}(x))^* \cdot (Q^1(x))^* \right\rangle. \\ (V(SD^\#))_4 = & \left\langle (d_{1,1}^2)^2, (d_{2,1})^2, d_{2,1} \cdot d_{1,1}^2, d_{1,1}^3 \cdot d_{1,1}^1, d_{2,2} \cdot d_{1,1}, d_{3,1} \right\rangle. \end{aligned}$$

Here $d_{1,1}^2 \approx (d_{1,1})^2$, $d_{1,1}^4 \approx (d_{1,1}^2)^2 \approx (d_{1,1})^4$, $(d_{2,1})^2 \approx d_{2,1}^2$.

There exists only one algebra generator in $H^4(Q_0S^0)$ namely, $(Q^{(2,1,1)}x)^*$ (because $Q^{(2,1,1)}$ is the only primitive element in dimension four). The same holds in $(V(SD^\#))_4$, namely $d_{3,1}$. Applying Steenrod operations we see that

$$Sq_*^1(Q^{(2,1)}x \cdot Q^{(1,1)}x) = (Q^1x)^4$$

and dually

$$Sq^1(Q^{(2,1,1)}x)^* \neq 0.$$

On the other hand $Sq^1d_{3,1} = 0$.

Remark 5 Let k be a natural number not divisible by p , then the element $d_{1,1}^k$ is a generator in $V(SD^\#)$. Hence so there exists an element of $P(H_*(Q_0S^0))$ of degree $2k(p-1)$ and weight p . On the other hand, there exists only one primitive element of the given weight in R , namely, Q^1 .

Since $Q(H^*(Q_0S^0))$ is in 1-1 correspondence with $Q(V(SD^\#))$ and the latter is the cokernel of the map $\xi : SD \rightarrow SD$, the next Corollary provides a description of the primitive elements of $H_*(Q_0S^0)$.

Corollary 3 Let $\xi_* : \pi R \rightarrow \pi R$ be the dual of the p -th power map. Then $P(H_*(Q_0S^0))$ is isomorphic with the kernel of ξ_* . It is an opposite Steenrod module isomorphism.

Now the well known Theorem of Madsen and May follows. We recall that $x_{(I,\varepsilon)}$ has been defined in the Definition 4.

Theorem 10 (Madsen and May)

(a) For $p = 2$, $H^*(Q_0S^0)$ is the polynomial algebra generated by

$$\{j^*(x_I)^* \mid Q^I \in \overline{\pi R}\}.$$

(b) For $p > 2$, $H^*(Q_0S^0)$ is the tensor product of exterior, polynomial, and polynomial algebras truncated at height p . Their generators are in one to one correspondence with $B(\pi R)$.

(i) The exterior generators are

$$\{j^*(x_{(I,\varepsilon)})^* \mid |x_{(I,\varepsilon)}| \text{ is odd}\}.$$

(ii) The polynomial generators are

$$\{j^*(x_I)^* \mid I \text{ is not divisible by } p\}.$$

(iii) The polynomial truncated generators are

$$\{j^*(x_{(I,\varepsilon)})^* \mid |x_{(I,\varepsilon)}| \text{ is even, and } \epsilon_i = 1 \text{ for some } i\}.$$

It is known that the natural map

$$B\Sigma_\infty \rightarrow Q_0S^0$$

induces an isomorphism of Pontryagin algebras

$$H_*(B\Sigma_\infty) \xrightarrow{\cong} H_*(Q_0S^0).$$

Here Σ_∞ is the infinite symmetric group (see [3]). Nakaoka computed the cohomology of the infinite symmetric group for $p = 2$ (see [15]).

Theorem 11 (Nakaoka) The cohomology algebra $H^*(B\Sigma_\infty; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to a polynomial $\mathbb{Z}/2\mathbb{Z}$ -algebra in which the set of generators with dimension q ($q = 0, 1, 2, \dots$) is in one-to-one correspondence with the set of all sequences of (positive) integers

$$I = (i_1, \dots, i_l), l > 0$$

satisfying the conditions: (i) $i_t \leq 2i_{t+1}$; (ii) $i_1 > i_2 + \dots + i_l$ and (iii) $i_1 + i_2 + \dots + i_l = q$.

The cokernel of the map $\xi : SD \rightarrow SD$ is $\overline{SD}$ and provides the main ingredient for the proof of the main Theorem. Using the correspondence between the basis of $\overline{SD}_k$ and that of $(R[k])^*$ of Proposition 7, the next Theorem follows. We recall the mapping $i : SD \rightarrow H^*(Q_0 S^0)$ of Theorem 9.

Theorem 12 (a) For $p = 2$, $H^*(B\Sigma_\infty)$ is isomorphic to the polynomial algebra generated by

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} \mid (m_1, \dots, m_k) \notin (2\mathbb{N})^k \text{ for } k \geq 1 \right\}.$$

(b) For $p > 2$, $H^*(B\Sigma_\infty)$ is isomorphic to the tensor product of exterior, polynomial, and polynomial algebras truncated at height p .

(i) The exterior generators are

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} M_{k,j} L_k^{p-2} \prod_{t=1}^{\ell} M_{k;s_{2t-1},s_{2t}}^{\epsilon_t} L_k^{p-2} \mid m_r \in \mathbb{N}, \epsilon_t \in \{0, 1\}, 0 \leq j < k \right\}.$$

(ii) The polynomial generators are

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} \mid (m_1, \dots, m_k) \notin (p\mathbb{N})^k \text{ for } k \geq 1 \right\}.$$

(iii) The polynomial truncated generators are

$$\left\{ \prod_{1 \leq r \leq k} d_{k,k-r}^{m_r} \prod_{t=1}^{\ell} M_{k;s_{2t-1},s_{2t}}^{\epsilon_t} L_k^{p-2} \mid m_r \in \mathbb{N}, (\epsilon_1, \dots, \epsilon_\ell) \neq \mathbf{0} \text{ for } k \geq 1 \right\}.$$

In the spirit of Quillen's approach to $H^*(B\Sigma_\infty)$ via elementary abelian subgroups, an algebra basis for $H^*(B\Sigma_\infty)$ has been described by Adem and Milgram in [2]. We quote their Theorem 3.4, page 204.

Theorem 13 (Adem and Milgram) $H^*(B\Sigma_\infty, \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra on generators in one to one correspondence with the generators for $H_*(B\Sigma_\infty, \mathbb{Z}/2\mathbb{Z})$.

The bijection mentioned in the Theorem above was not provided. We shall define such a map.

Theorem 14 (a) There does not exist a degree preserving bijection between the algebra basis for $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ and $H_*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ for p an odd prime.

(b) There exists a bijection between the subsets of algebra basis for $H^*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ and $H_*(B\Sigma_\infty, \mathbb{Z}/p\mathbb{Z})$ not involving Bockstein elements.

Proof Let us start with (b).

(b) Let $\overline{R}(\beta) = \{Q^{(I, \varepsilon)} \mid \varepsilon = \mathbf{0} \text{ and } I \neq pJ \text{ with } J \text{ admissible}\}$ and $\mathcal{N}(\beta)$ be the part of Nakaoka basis without elements involving Bockstein operations (Proposition 2), i.e. $Q^{(I, \varepsilon)} \in \mathcal{N}$ and $\varepsilon = \mathbf{0}$.

Let $I = (i_k, \dots, i_1)$ be admissible and s the smallest integer such that $\text{exc}(I(s)) = 0$, then $\text{exc}(I(s+t)) = 0$ for $s \leq s+t \leq k$. Here $I(s) = (i_s, \dots, i_1)$. This is because $i_s = (p-1) \sum_{t=1}^{s-1} i_t$ and

$$(p-1) \sum_{t=1}^s i_t \leq i_{s+1} \leq pi_s.$$

Thus $i_{s+1} = pi_s$ and $\text{exc}(I(s+1)) = 0$.

Let $\gamma : \overline{R}(\beta) \rightarrow \mathcal{N}(\beta)$ defined as follows. Let $I = (i_k, \dots, i_1) \in \overline{R}(\beta)$ and s be the smallest integer such that $\text{exc}(I(s)) = 0$, we define:

$$\gamma(I) = p^{k-s+1} K.$$

Here $K = (i_{s-1}, \dots, i_1) \in \mathcal{N}(\beta)$. Now, let $I = (i_k, \dots, i_1) \in \mathcal{N}(\beta)$ and l be the maximal integer such that $i_t \equiv 0 \pmod{p^l}$ for $t = 1, \dots, k$. We define $\gamma^{-1}(I) = (p^l i_k, \dots, p^l i_1) \in \overline{R}(\beta)$.

(a) Since there is a bijection between $\overline{R}(\beta)$ and $\mathcal{N}(\beta)$, we only consider $\overline{R} - \overline{R}(\beta)$ and $\mathcal{N} - \mathcal{N}(\beta)$ for $p = 3$ and degree 30. The sequence

$$K_{3;1,2} = \left((0, p^2 - p - 1), (1, p - 1), (1, 1) \right)$$

defines an element of $\overline{R} - \overline{R}(\beta)$ but not in $\mathcal{N} - \mathcal{N}(\beta)$. On the other hand, $\mathcal{N} - \mathcal{N}(\beta) \subseteq \overline{R} - \overline{R}(\beta)$. $\square$

5 Describing $H^*(Q_0 S^0)$ via free $\mathcal{A}$ -algebras

Following Wellington [19], a description of $H^*(Q_0 S^0)$ by $\mathcal{A}$ -algebras will be defined. There exists an increasing sequence of algebras having the property that although none of these algebras is free, their associated algebras are free as unstable $\mathcal{A}$ -algebras.

An increasing filtration inspired by May on $B(SD^\#)$ and a decreasing one on the basis of $H_*(Q_0 S^0)$ is given by

$$\sigma : B(SD^\#) \rightarrow \mathbb{N} \text{ and } \tau : B \rightarrow \mathbb{N}.$$

Let $\prod_i d_{k,i}^{m_i} \in B(SD_k)$, define $\sigma(\prod_i d_{k,i}^{m_i}) = p^k$. Define $\tau(Q^{I, \varepsilon} x) = p^{l(I)}$ and extend to the rest of B by $\tau(y_1^{s_1} \dots y_k^{s_k}) = \sum s_i \tau(y_i)$.

Since the Steenrod operations respect the filtration, the next lemma follows.

Lemma 7 Let $P^m \in \mathcal{A}$ and d a monomial of SD . If $P^m d = \sum a_i d_i$ in SD and $a_i \neq 0$, then $\sigma(d_i) = \sigma(d)$. The same holds in B , if $P_*^m y = \sum a_j y_j$ and $a_j \neq 0$, then $\tau(y_j) = \tau(y)$.

The map σ above does not respect the restriction (according to Milnor–Moore) defined on $V(SD^\#)$: let $p = 2$, $Sq^1 d_{1,1} = d_{1,1}^2 \approx (d_{1,1})^2$, but $\sigma(d_{1,1}^2) = 2 \neq \sigma((d_{1,1})^2) = 2^2$.

Definition 11 (a) Let $F^q H^* := (H_*(Q_0 S^0)/F_{q+1} H_*(Q_0 S^0))^*$ where $F_{q+1} H_*(Q_0 S^0)$ is generated additively by those elements $y \in B$ with $\tau(y) \geq q + 1$.

(b) Let $SD(s)$ be the $\mathcal{A}$ -submodule of SD generated by the basis elements $d \in SD$ such that $\sigma(d) = s$. Thus $SD = \oplus SD(s)$. Let $C^k = V(\oplus_1^k SD^\#(s))$ and A^k its image in $H^*(Q_0 S^0)$ under the isomorphism e of Theorem 9.

The last lemma implies that SD and R^+ (positive degree elements of R) decompose with respect to σ and τ respectively as $\mathcal{A}$ -modules.

Please note that $SD(s) = SD_{k(s)}$ such that $s = p^{k(s)}$. Let $s = \sum^{k(s)} a_t p^t$ be its p -adic expansion, then $C^s = V(\oplus_1^{k(s)} SD_t^\#)$.

Lemma 8 With respect to the Pontryagin product in $H_*(Q_0 S^0)$ we have $F_q \otimes F_r \subseteq F_{q+r}$.

Proof Let $Q^{(I,\varepsilon)} x$ be a monomial, then

$$\tau\left((Q^{(I,\varepsilon)} x)^{p-k} \cdot (Q^{I,\varepsilon} x)^k\right) = \tau(Q^{((a,I),\varepsilon')} x) = p^{l(I)+1},$$

where $(Q^{I,\varepsilon} x)^p \approx Q^{((a,I),\varepsilon')} x$.

Let $b = y_1^{m_1} \dots y_k^{m_k} \in F_q$ and $c = c_1^{m'_1} \dots c_l^{m'_l} \in F_r$ be basis elements. Then bc can be rewritten again as an element of B taking in to account the ordering and the p -th power relation. The first line of the proof implies $\tau(dc) = \tau(d) + \tau(c)$ and dc is an element of F_{q+r} . $\square$

Since $V(SD^\#) = \varinjlim C^k$, the filtration $C^1 \subset C^2 \subset \dots \subset V(SD^\#)$ is complete.

Proposition 10 The sequence of $\mathcal{A}$ -algebras $\{C^k\}$ is a complete filtration of $V(SD^\#)$ such that $C^k \cong C^{k-1} \otimes V(SD^\#(k))$, and $C^k/C^{k-1} \cong V(SD^\#(k))$ as $\mathcal{A}$ -algebras.

Proof Since $V(\oplus_1^k SD^\#(s)) \cong \otimes_1^k V(SD^\#(s))$ as Hopf algebras and the Steenrod operations preserve $SD(s)$, the proposition follows by Proposition 6.5 in [12]. $\square$

The Theorem 9 provides a basis for $H^*(Q_0 S^0)$: $BH^*(Q_0 S^0) = \{1\} \cup$

$$\{(y_1^*)^{s_1} \dots (y_k^*)^{s_k} \mid y_i^* \in j^* T(SD), y_i < y_{i+1}, 0 < s_i < p \text{ if } |y_i| = \text{even}, \\ \text{and } y_i = 1 \text{ if } |y_i| = \text{odd}\}.$$

The relation between $BH^*(Q_0 S^0)$ and B^* (the dual of B) shall be studied along with the action of the Steenrod operations for the remaining of the section in order to prove the main Theorems of the section.

Lemma 9 Let $b = y_1^{s_1} \dots y_k^{s_k} \in B$ and $c^* = (y_1^*)^{s_1} \dots (y_k^*)^{s_k} \in BH^*(Q_0S^0)$. If $y_k^* \in A^t$, then $c^* \in A^t$. Let $\tau(b) = s$ and $\sum s_i > 1$, then $t \leq s - 1$. If $\sum s_i = 1$, then $t = s$.

Proof Since $\tau(b) = \sum s_i \tau(y_i) = s$ and $\sum s_i > 1$, then $\tau(y_i) \leq s - 1$. Let $y_i = Q^{(I_i, \varepsilon_i)}x$ and $\chi^{-1}(I_i, \varepsilon_i) = d_i \in SD_{n(I_i)}$ (see Corollary 1). Here $p^{n(I_i)} \leq s - 1$. Now,

$$(y_i^*) = j^*T(d_i) + \text{others}, (y_i^*)^{s_i} = (j^*T(d_i))^{s_i} + \text{others};$$

$$c^* = \left(\prod_i (j^*T(d_i)) \right)^{s_i} + \text{others}.$$

Here $\prod_i (d_i)^{s_i} \in \otimes (SD^\#)_{n(I_i)} \subset C^{s-1}$. It follows that $c^* \in A^{s-1}$. $\square$

Example 1 Let $p = 2$. Let $b = Q^1x \cdot Q^2x$ and $c^* = (Q^1x)^* \cdot (Q^2x)^*$, then $\tau(b) = 2 + 2 = 4$. $T^{-1}(Q^1) \cdot T^{-1}(Q^2) \in V(SD^\#)(2)$ and $c^* \in A^2$.

Proposition 11 Let $b = y_1^{s_1} \dots y_k^{s_k} \in B$ and $c^* = (y_1^*)^{s_1} \dots (y_k^*)^{s_k} \in BH^*(Q_0S^0)$. Let $\tau(b) = s$, then $c^* \in F^s H^*$ and $c^* = b^*$ modulo $F^{s-1} H^*$.

Proof We shall prove that if $b' \in B$ and the Kronecker pairing $\langle c^*, b' \rangle$ is non-zero, then $b' = b$ or $\tau(b') < \tau(b)$. Induction shall be applied on $\sum s_i$.

The assertion is obvious for $\sum s_i = 1$. Let $b' = Q^{(I, \varepsilon)}x \cdot Q^{(J, \nu)}x$ such that $\tau(Q^I) \leq \tau(Q^J)$. We know that

$$\psi(Q^{(I, \varepsilon)}x) = Q^{(I, \varepsilon)}x \otimes \emptyset + \emptyset \otimes Q^{(I, \varepsilon)}x + \sum_{I_1+I_2=I} Q^{(I_1, \varepsilon_1)}x \otimes Q^{(I_2, \varepsilon_2)}x.$$

Applying Adem relations on $Q^{(I_1, \varepsilon_1)}$ and $Q^{(I_2, \varepsilon_2)}$, we get

$$\sum_{I_1+I_2=I} Q^{(I_1, \varepsilon_1)}x \otimes Q^{(I_2, \varepsilon_2)}x = \sum_{I'_1+I'_2=I} a_{I'_1, I'_2} Q^{(I'_1, \varepsilon'_1)}x \otimes Q^{(I'_2, \varepsilon'_2)}x$$

and $l(I'_1) = l(I'_2) = l(I)$.

The coproduct is a map of algebras in $H_*(Q_0S^0)$.

$$\psi(b') = \psi(Q^{(I, \varepsilon)}x) \cdot \psi(Q^{(J, \nu)}x),$$

$$\psi(b') = b' \otimes \emptyset + \emptyset \otimes b' + \sum_{I'_1+I'_2=I, J'_1+J'_2=J} a_{I'_1, I'_2} a'_{J'_1, J'_2} Q^{(I'_1, \varepsilon'_1)}x \cdot Q^{(J'_1, \nu'_1)}x$$

$$x \otimes Q^{(I'_2, \varepsilon'_2)}x \cdot Q^{(J'_2, \nu'_2)}x.$$

From Lemma 8 we get that $Q^{(I'_t, \varepsilon'_t)}_x \cdot Q^{(J'_t, \nu'_t)}_x \in F^{\tau(Q^{I'_t}) + \tau(Q^{J'_t})}$, and the equation above implies that

$$\tau(b') < \tau(Q^{(I'_1, \varepsilon'_1)}_x \cdot Q^{(J'_1, \nu'_1)}_x) + \tau(Q^{(I'_2, \varepsilon'_2)}_x \cdot Q^{(J'_2, \nu'_2)}_x) \text{ or} \\ Q^{(I'_1, \varepsilon'_1)}_x \cdot Q^{(J'_1, \nu'_1)}_x = Q^{(I, \varepsilon)}_x \text{ and } Q^{(I'_2, \varepsilon'_2)}_x \cdot Q^{(J'_2, \nu'_2)}_x = Q^{(J, \nu)}_x \quad (5.1)$$

for some elements.

Now let us consider the general case, let $b' = x_1^{t_1} \dots x_m^{t_m}$. Then

$$\langle c^*, b' \rangle = \langle y_1^* \otimes (y_1^*)^{s_1-1} \dots (y_k^*)^{s_k}, \psi(b') \rangle.$$

Let $\psi(b') = b' \otimes \emptyset + \emptyset \otimes b' + \sum ab'' \otimes b'''$, with $b'', b''' \in B$. Using (5.1) for more than two factors we have

$$\tau(b') < \tau(b'') + \tau(b''') \text{ or there exist } 0 < j_i \leq t_i \text{ such that} \\ b'' = x_1^{j_1} \dots x_m^{j_m} \text{ and } b''' = x_1^{t_1-j_1} \dots x_m^{t_m-j_m}.$$

Thus

$$\langle c^*, b' \rangle = \sum a \langle y_1^*, b' \rangle \langle (y_1^*)^{s_1-1} \dots (y_k^*)^{s_k}, b''' \rangle.$$

Here a is a constant and the bracket $\langle c^*, b' \rangle$ is zero, unless $b'' = y_1$. Then the induction hypothesis implies either

$$b''' = (y_1)^{s_1-1} \dots (y_k)^{s_k} \text{ or } \tau(b''') < \phi(b) - \tau(y_1).$$

Finally, $b' = y_1^{s_1} \dots y_k^{s_k}$ or $\tau(b') < \tau(b)$. $\square$

The relation between $F^s H^*$ and C^s is revealed in the next Proposition.

Proposition 12 $F^s H^* \subset e(C^s) = A^s$ (for the isomorphism e see Theorem 9).

Proof Let $b \in H_*(Q_0 S^0)/F_{q+1} H_*(Q_0 S^0)$ and $b^* \in F^s H^*$ with $\phi(b^*) \leq s$. We shall find $u \in C^s$ such that $b^* = e(u)$. Induction is applied on s . Let $b = Q^{(I, \varepsilon)}_x$ and $\tau(b) = s$, then $(Q^{(I, \varepsilon)})^* \in (\pi R)^*$. Thus there exists a $u \in SD_s$ such that $T(u) = (Q^{(I, \varepsilon)})^*$. Hence $e(u) = j^* T(u) = (Q^{(I, \varepsilon)}_x)^* = b^* \in A^s$.

Next, products of elements of the form $Q^{(I, \varepsilon)}_x$ shall be considered. Let $b = y_1^{s_1} \dots y_k^{s_k} \in B$ with $\tau(b) \leq s$. Thus $y_i^* \in A^s$ for each i and e is a map of algebras. The previous Proposition implies

$$b^* = (y_1^*)^{s_1} \dots (y_k^*)^{s_k} + \text{others}$$

and $\text{others} \in F^{s-1} H^* \subset A^s$. Hence $b^* \in A^s$. $\square$

Lemma 10 Let $u \in SD_k$, $P^m \in \mathcal{A}$ and $b = y_1^{s_1} \dots y_k^{s_k} \in B$. If $\langle P^m(i(u)), b \rangle \neq 0$, then $\tau(b) = \sigma(u)$.

Proof Our assertion $\langle P^m(i(u)), b \rangle \neq 0$ implies $\langle u, T^*j(P_*^m b) \rangle \neq 0$. On the other hand,

$$P_*^m b = \sum P_*^{m_1} y_1^{s_1} \dots P_*^{m_k} y_k^{s_k} = \sum a_{(t_1, \dots, t_j)} x_1^{t_1} \dots x_j^{t_j}.$$

Here the $a_{(t_1, \dots, t_j)}$ are constants and the $x_1^{t_1}, \dots, x_j^{t_j}$ are basis elements such that $\sum t_i = \sum s_i$. Because the Steenrod operations respect the length, we have

$$\sum t_i \tau(x_i) = \sum s_i \tau(y_i) = \tau(b).$$

Let b' be one of the terms such that $\langle u, T^*j(b') \rangle \neq 0$. From the definition of the map j (see 2.2), we have $b' = Q^{(W, \varepsilon)} x$ for some sequence (W, ε) such that $\tau(b) = \tau(b') = p^{l(W)}$. Since $u \in SD_k$ and $\langle u, T^*j(b') \rangle \neq 0$, we conclude that $\tau(b) = \tau(b') = \phi(u)$. $\square$

Example 2 Let $b = Q^1 x \cdot Q^2 x$ and $c^* = (Q^1 x)^* \cdot (Q^2 x)^*$, then

$$Sq_*^1 b = Q^1 x \cdot Q^1 x \approx Q^{1,1} x$$

and

$$\langle Sq^1 i(d_{2,1}), b \rangle = \langle i(d_{2,1}), Q^{(1,1)} x \rangle \neq 0.$$

But $\langle iSq^1(d_{2,1}), b \rangle = \langle d_{2,2}, T^*j(Q^1 x \cdot Q^2 x) \rangle = \langle d_{2,2}, 0 \rangle = 0$.

Proposition 13 Let $u \in SD_s$ and $P^m \in \mathcal{A}$, then $P^m(i(u)) \equiv i(P^m(u))$ modulo A^{s-1} .

Proof The assertion shall be proved by evaluating the Kronecker bracket. Let $b = Q^{(l, \varepsilon)} x$, then

$$\begin{aligned} \langle P^m(i(u)), b \rangle &= \langle i(u), P_*^m b \rangle = \langle j^* T(u), P_*^m b \rangle \\ &= \langle T(u), j(P_*^m b) \rangle = \langle T(u), P_*^m j(b) \rangle = \langle P^m T(u), j(b) \rangle \\ &= \langle T P^m(u), j(b) \rangle = \langle j^* T(P^m u), b \rangle = \langle i(P^m u), b \rangle. \end{aligned}$$

This is because j is an $\mathcal{A}$ -map on $MS^0 \cong \pi R$.

Let $b = y_1^{s_1} \dots y_k^{s_k} \in B$ such that $\sum s_i > 1$, then $j(b) = 0$. In this case $\langle j^* T(P^m u), b \rangle = 0$.

Let $\langle P^m(i(u)), b \rangle \neq 0$, then $\tau(b) = \phi(u) = s$. The Proposition 11 implies $b^* = c^*$ modulo $F^{s-1} H^*$ and $F^{s-1} H^* \subset A^{s-1}$ and $c^* \in A^{s-1}$ (Lemma 9). $\square$

Theorem 15 The sequence $A^1 \subset A^2 \subset \dots \subset H^*(Q_0 S^0)$ is a complete filtration of $\mathcal{A}$ -subalgebras. There exist isomorphisms of algebras $A^{s+1} \cong A^s \otimes V(SD^\#(s+1))$ and isomorphisms of $\mathcal{A}$ -algebras $A^{s+1}/A^s \cong V(SD^\#(s+1))$.

Proof Firstly, $A^{s+1} = \text{Im}(C^{s+1} = C^s \otimes V(SD^\#(s+1)) \rightarrow H^*(Q_0S^0))$. Thus,

$$A^{s+1} \cong A^s \otimes V(SD^\#(s+1))$$

as algebras.

Secondly, the last Proposition implies that A^{s+1} is an $\mathcal{A}$ -subalgebra, because each element of A^{s+1} is a product of images of elements of $SD(t)$ with $t \leq s+1$ and the Steenrod operations are applied by the Cartan formula. For the same reasons (applying the last Proposition) $A^{s+1}/A^s \cong V(SD^\#(s+1))$ as $\mathcal{A}$ -algebras. $\square$

The elements being images under the p -th power map shall be distinguished.

Definition 12 Let $B(\overline{SD})$ be the natural basis of $\overline{SD}$. For each $d + p(SD) \in B(\overline{SD})$ we define $\sigma(d + p(SD)) := \sigma(d)$.

Lemma 11 $\overline{SD}$ is the cokernel of the p -th power map in SD and $\overline{SD} = \bigoplus_{s>0} \overline{SD}(s)$ decomposes into $\mathcal{A}$ -submodules with respect to σ (Lemma 7).

The next Lemma is a consequence of the definitions.

Lemma 12 $QV(SD^\#) \cong \overline{SD}$ as $\mathcal{A}$ -modules.

It has been noted that $PH_*(Q_0S^0) \cong \overline{\pi R}$ or $QH^*(Q_0S^0) \cong \overline{SD}$ as vector spaces but not as $\mathcal{A}$ -modules. Using the same filtration defined before, an $\mathcal{A}$ -module isomorphism will be derived. The filtration F^q can be applied to $QH^*(Q_0S^0)$ as well: let $F^q QH^*(Q_0S^0)$ be the image from $F^q H^*(Q_0S^0)$ to $QH^*(Q_0S^0)$. Now we define

$$\begin{aligned} E^q QH^*(Q_0S^0) &:= F^q QH^*(Q_0S^0) / F^{q-1} QH^*(Q_0S^0) \text{ and} \\ E_0 QH^*(Q_0S^0) &:= \bigoplus_{q>0} E^q QH^*(Q_0S^0). \end{aligned}$$

Theorem 16 $E^q QH^*(Q_0S^0)$ and $\overline{SD}(q)$ are isomorphic as $\mathcal{A}$ -modules. Moreover,

$$E_0 QH^*(Q_0S^0) \cong \overline{SD}.$$

Proof For the $\mathcal{A}$ -module isomorphism the Proposition 13 shall be applied.

Let $F^q QH^*(Q_0S^0) = \text{Im}(F^q H^*(Q_0S^0) \rightarrow QH^*(Q_0S^0))$ and $F^q H^*(Q_0S^0) \subset e(C^q) = A^q$ (Proposition 12). Let $QA^q = \text{Im}(A^q \rightarrow QH^*(Q_0S^0))$. We recall that $jT(SD(q)) \subset F^q H^*(Q_0S^0)$ and $C^q = V(\bigoplus_1^q SD(k))$. Now,

$$QC^q \cong \bigoplus_1^q \overline{SD}(k) \text{ implies } QA^q \cong \bigoplus_1^q \overline{SD}(k).$$

Thus,

$$Q(A^q) \subset F^q QH^*(Q_0S^0).$$

The inclusions above imply $Q(A^q) \subset F^q QH^*(Q_0S^0)$ and $F^q QH^*(Q_0S^0) \subset Q(A^q)$.

Finally, since $E^q QH^*(Q_0S^0) = Q(A^q)/Q(A^{q-1})$ and $A^q/A^{q-1} \cong V(SD^\#(q))$, we have

$$E^q QH^*(Q_0S^0) \cong QV(SD^\#(q)) = \overline{SD}(q).$$

This isomorphism above is an $\mathcal{A}$ -module isomorphism. Now we are reached the last assertion of our statement:

$$E_0 QH^* \cong \oplus_{q>0} \overline{SD}(q).$$

□

6 Q_0S^0 is H -atomic at p

Campbell et al. proved that Q_0S^0 is H -atomic in [4]. This section is devoted to the proof of the same statement for $p \geq 2$ using the complete filtration $\{A^i\}$ of $H^*(Q_0S^0)$.

The term atomic was defined by Cohen, Moore and Neisendorfer. Let us recall from [4] that a CW complex X is atomic at p , if given any map $f : X \rightarrow X$ such that f induces an isomorphism on $H_r(X, \mathbb{Z}/p\mathbb{Z})$, then $f_{(p)}$ is a homotopy equivalence. Here r is the lowest degree such that $\overline{H}_r(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$.

For the next Theorem Campbell et al. assume that the map $f : Q_0S^0 \rightarrow Q_0S^0$ is a loop map which induces a homology isomorphism on degree $2p - 3$.

Theorem 17 *Let $f^* : H^*(Q_0S^0) \rightarrow H^*(Q_0S^0)$ be a Steenrod coalgebra map which induces an isomorphism on degrees $2p - 3$ and $2(p^2 - 1)$. Then f^* is an isomorphism.*

Proof The map f^* commutes with Steenrod operations and ξ , since the restriction (according to Milnor–Moore) is a Steenrod operation. Let $\overline{f}$ be the restriction of f^* on SD under the isomorphism e . We recall that C^q is primitively generated (Definition 11). Thus, if $\overline{f}$ is an isomorphism on $SD(s)$ for $s \leq q$, then it is an isomorphism on C^q . Induction is applied on the filtration $\{A^q\}$.

It is given that $f^*(\beta Q^1 x)^* = u(\beta Q^1 x)^*$ and

$$f^*(Q^{(p,1)} x)^* = u(Q^{(p,1)} x)^* + \text{others}.$$

Thus $\overline{f}$ restricted on C^p is an isomorphism. This is because

$$i(M_{1;0} L_1^{p-2}) = (\beta Q^1 x)^*.$$

Hence $f^*_{|A^p}$ is an $\mathcal{A}$ -module isomorphism.

It is also given that $f^*(Q^{(p,1)} x)^* = u(Q^{(p,1)} x)^* + \text{others}$ or

$$f_*(Q^{(p,1)} x) = uQ^{(p,1)} x + \text{others}.$$

Here “others” is a sum of products. We recall that $Q^{(p,1)}x$ is a primitive and a generator. Again, $i(d_{2,2}) = (Q^{(p,1)}x)^*$ and $\bar{f}$ restricted on $V(SD^\#(p^2)) = V(SD_2^\#)$ is an isomorphism. The Theorem 15 implies that f^* is an isomorphism on A^{p^2}/A^{p^2-1} .

$A^{p^2-1} = e(V(\bigoplus_1^{p^2-1} SD^\#(s)))$ and $SD(p) = SD_1$. Now $SD(s)$ is generated by suitable products in SD_1 for $s \leq p^2 - 1$. Since f^* is a coproduct map and an isomorphism on A^p , f^* is also an isomorphism on A^{p^2-1} and on A^{p^2} .

Let $q = p^n$ and the assertion is true for $q < p^n$. Let $b_0 = Q^{(p^{n-1}, \dots, 1)}x$, $b_1 = Q^{(p^{n-1}-1, p^{n-2}, \dots, 1)}x$ and $c = Q^{p^{n-2}, \dots, 1}x$. So $b_1 = (c)^p$. Since b_1 is a primitive, $(b_1)^* \in A^{p^n}$ is a generator but not a primitive. Now $i(d_{n,n-1}) = (b_1)^*$. Proposition 13 implies

$$P^1(b_1)^* = i(P^1 d_{n,n-1}) = i(d_{n,n}) = (b_0)^*.$$

Since $(b_1)^*$ is not a primitive, it decomposes with respect to the coproduct (dual to the Pontryagin product)

$$\psi(b_1)^* = (b_1)^* \otimes \emptyset + \emptyset \otimes (b_1)^* + \sum (w_i)^* \otimes (w'_i)^*.$$

Here $(w_i)^*, (w'_i)^* \in A^{p^n-1}$ where f^* is an isomorphism by induction hypothesis. Thus

$$f^*(b_1)^* = u(b_1)^* + \text{others}; \quad P^1 f^*(b_1)^* = u'(b_0)^* + \text{others}';$$

$$f^*(b_0)^* = u'(b_0)^* + \text{others}'.$$

By the same argument as above, f^* is an isomorphism on A^{p^n} . □

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