Bull. Math. Sci. (2012) 2:205–223
DOI 10.1007/s13373-011-0015-2

Ramanujan's modular equations and Weber–Ramanujan class invariants G_n and g_n

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Received: 20 August 2011 / Accepted: 10 November 2011 / Published online: 8 December 2011

Abstract In this paper, we use Ramanujan's modular equations and transformation formulas to find several new values of Weber–Ramanujan class invariants g_n. We also give new approach to some known values of class invariants G_n.

Keywords Weber–Ramanujan class-invariants · Modular equations · Transformation formulas

Mathematics Subject Classification (2000) Primary 33D90; Secondary 11F20

1 Introduction

The Dedekind eta-function $\eta(z)$ and Ramanujan's function $f(-q)$ are defined by

$$f(-q) := (q; q)_\infty = q^{-1/24}\eta(z), \quad q = e^{2\pi i z}, \quad \mathrm{Im}(z) > 0. \tag{1.1}$$

where$(a; q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$.
Now, Weber–Ramanujan class invariants G_n and g_n [4, p. 183, (1.3)] are defined by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q) \quad \text{and} \quad g_n = 2^{-1/4}q^{-1/24}\chi(-q), \quad q := e^{-\pi\sqrt{n}}, \tag{1.2}$$

where n is a positive rational number and

Communicated by S.K. Jain.

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$$\chi(q) = (-q; q^2)_\infty = \frac{f(q)}{f(-q^2)}. \tag{1.3}$$

In his notebooks [11] and paper [10], Ramanujan recorded a total of 116 class invariants. The table at the end of Weber's book [12, p. 721–726] contains the values of 107 class invariants. There are many applications of Weber–Ramanujan class invariants G_n and g_n. Weber primarily was motivated to calculate class invariants so that he could construct Hilbert class fields. On the other hand Ramanujan calculated class invariants to approximate π, and probably for finding explicit values of Rogers–Ramanujan continued fractions, theta-functions, etc. For further applications of class invariants see [5–9]. An account of Ramanujan's class invariants and applications can also be found in Berndt's book [4].

In 2001, Yi [13] evaluated several new class invariants G_n and g_n by finding explicit values of the parameter $r_{k,n}$ (see [13, p. 11, (2.1.1)] or [14, p. 4, (1.11)]) defined as

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}, \tag{1.4}$$

where n and k are positive real numbers. In particular, she established the result [13, p. 18, Theorem 2.2.3]

$$g_n = r_{2,n/2}. \tag{1.5}$$

Adiga et al. [1] and Baruah [2] also evaluated some new values of G_n and g_n.

In this paper, we find further new values of Ramanujan's class invariant g_n and also give new approach to some known values of G_n. We consider the parameter A_n defined by

$$A_n = \frac{f(-q)}{2^{1/4}q^{1/24}f(-q^2)}, \quad q := e^{-2\pi\sqrt{n/2}} \tag{1.6}$$

where n is a positive rational number. Clearly, the parameter A_n is the particular case, $k = 2$, of the Yi's parameter $r_{k,n}$ defined above.

In Sect. 2, we record some transformation formulas and modular equations. We also prove four eta-function identities which are also particular type of modular equations. In Sect. 3, we find several values of Weber–Ramanujan class invariants g_n. Finally in Sect. 4, we use some new values of g_n evaluated in Sect. 3 to prove some known values on G_n.

Since modular equations are key in our evaluations, we now define a modular equation. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.7}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.7). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α.

Denoting $z_r = \phi^2(q^r)$, where

$$q = \exp(-\pi K'/K), \phi(q) = f(q, q), |q| < 1;$$

the multipliers m associated with α, β is defined by $m = z_1/z_n$.

2 Transformation formulas and modular equations

The section is devoted to record some transformation formulas and modular equations which will be used in next section. The eta-function identities in Lemmas 2.7–2.9 are found by Yi [13] by using *Garvan's Maple q-series package* and then proved by verification. We give direct proofs of these identities. In our proofs, it is not necessary to know the identities in advance and employ Ramanujan's modular equations and transformation formulas. Eta-function identity in Lemma 2.10 is new.

Lemma 2.1 ([3, p. 43, Entry 27 (iii)]) *If α and β are such that the modulus of each exponential argument is less than* 1 *and $\alpha\beta = \pi^2$, then*

$$e^{-\alpha/12}\sqrt[4]{\alpha} f(-e^{-2\alpha}) = e^{-\beta/12}\sqrt[4]{\beta} f(-e^{-2\beta}). \tag{2.1}$$

Lemma 2.2 ([3, p. 124, Entry 12(ii), (iii), & (iv)])

$$f(-q) = \sqrt{z}2^{-1/6}(1-\alpha)^{1/6}\alpha^{1/24}q^{-1/24}. \tag{2.2}$$
$$f(-q^2) = \sqrt{z}2^{-1/3}(\alpha(1-\alpha))^{1/12}q^{-1/12}. \tag{2.3}$$
$$f(-q^4) = \sqrt{z}2^{-2/3}(1-\alpha)^{1/24}\alpha^{1/6}q^{-1/6}. \tag{2.4}$$

Lemma 2.3 ([3, p. 230, Entry 5(ii)]) *If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1. \tag{2.5}$$

Lemma 2.4 ([3, p. 280, Entry 13(i) & (x)]) *If β has degree 5 over α, then*

$$(\alpha\beta)^{1/2} + ((1-\alpha)(1-\beta))^{1/2} + 2\left\{16\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/6} = 1. \tag{2.6}$$

$$\{\alpha(1-\beta)\}^{1/4} + \{\beta(1-\alpha)\}^{1/4} = 4^{1/3}\left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/24}. \tag{2.7}$$

Lemma 2.5 ([3, p. 314, Entry 19(i)]) *If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} = 1. \tag{2.8}$$

Lemma 2.6 ([4, p. 387, Entry 62]) *Let*

$$P = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)},$$
$$Q = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right),$$

and

$$R = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)},$$

then, if β *has degree 9 over* α*,*

$$P^6 - R(14P^3 + PQ) - 3R^2 = 0. \tag{2.9}$$

Lemma 2.7 ([4, p. 387, Entry 62]) *Let P, Q, and R be as defined in Lemma 2.6, then, if* β *has degree 13 over* α*,*

$$\sqrt{P}(P^3 + 8R) - \sqrt{R}(11P^2 + Q) = 0. \tag{2.10}$$

Lemma 2.8 ([4, p. 387, Entry 62]) *Let P, Q, and R be as defined in Lemma 2.6, then, if* β *has degree 17 over* α*,*

$$P^3 - R^{1/3}(10P^2 + Q) + 13R^{2/3}P + 12R = 0. \tag{2.11}$$

Lemma 2.9 ([4, p. 386, Entry 58]) *Let*

$$P = 1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4},$$
$$Q = 16\left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right),$$

and

$$R = 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4},$$

then, if β *has degree 19 over* α*,*

$$P^5 - 7P^2R - QR = 0. \tag{2.12}$$

Lemma 2.10 ([4, p. 386, Entry 59]) *Let P, Q, and R be as defined in Lemma 2.9, then, if* β *has degree 27 over* α*,*

$$P^9 - RP^2(29P^4 + 11P^2Q + Q^2) - 17R^2P^3 - 3R^2(PQ + R) = 0. \tag{2.13}$$

Lemma 2.11 ([4, p. 385, Entry 53]) *Let*

$$P = 1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8},$$
$$Q = 4\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right),$$

and

$$R = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8},$$

then, if β has degree 15 over α,

$$P(P^2 - Q) + R = 0. \tag{2.14}$$

Lemma 2.12 ([13, p. 21, Theorem 3.2.2]) *If* $P = \dfrac{f(-q)}{q^{1/24} f(-q^2)}$ *and* $Q = \dfrac{f(-q^2)}{q^{1/12} f(-q^4)}$

$$\textit{then}\quad (PQ)^4 + \left(\frac{4}{PQ}\right)^4 = \left(\frac{Q}{P}\right)^{12}.$$

Proof Transcribing using Lemma 2.2, we get

$$P = 2^{1/6}(1-\alpha)^{1/12}\alpha^{-1/24} \quad \text{and} \quad Q = 2^{1/3}(1-\alpha)^{1/24}\alpha^{-1/12}. \tag{2.15}$$

From (2.15), we find that

$$\alpha = \frac{16}{16 + (PQ)^8} \tag{2.16}$$

and

$$\left(\frac{Q}{P}\right)^{24} = 2^4 \left\{\alpha(1-\alpha)\right\}^{-1}. \tag{2.17}$$

Applying (2.16) in (2.17) and simplifying, we get

$$\left(16P^8 + P^{16}Q^8 - Q^{16}\right)\left(16P^8 + P^{16}Q^8 + Q^{16}\right) = 0. \tag{2.18}$$

By examining the behavior near origin, it can be shown that the second factor of the left hand side of (2.18) is non-zero in a neighborhood of the origin. Thus, the first factor vanishes in that neighborhood. Hence, by the identity theorem, this factor vanishes identically, i.e.,

$$16P^8 + P^{16}Q^8 - Q^{16} = 0. \tag{2.19}$$

Dividing (2.19) by $P^{12}Q^4$, we complete the proof. □

Lemma 2.13 ([13, p. 37, Theorem 3.5.2]) *If* $P = \dfrac{f(-q)}{q^{1/8} f(-q^4)}$ *and* $Q = \dfrac{f(-q^3)}{q^{3/8} f(-q^{12})}$

$$\textit{then}\quad PQ + \frac{4}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2.$$

Proof Transcribing P and Q by using Lemma 2.2 and simplifying, we get

$$\alpha=\frac{16}{16+P^8} \quad \text{and} \quad \beta=\frac{16}{16+Q^8}, \tag{2.20}$$

where β has degree 3 over α. It follows that

$$1-\alpha=\frac{P^8}{16+P^8} \quad \text{and} \quad 1-\beta=\frac{Q^8}{16+Q^8}. \tag{2.21}$$

Using (2.20) and (2.21) in Lemma 2.3, we arrive at

$$(4+P^2Q^2)^4=(16+P^8)(16+Q^8). \tag{2.22}$$

Factorizing (2.22) using *Mathematica*, we get

$$(P^4-4PQ-P^3Q^3+Q^4)(P^4+4PQ+P^3Q^3+Q^4)=0. \tag{2.23}$$

Since the second factor is non-zero in a neighborhood of the origin, we deduce

$$P^4-4PQ-P^3Q^3+Q^4=0. \tag{2.24}$$

Dividing above equation by P^2Q^2, we complete the proof. □

Lemma 2.14 ([13, p. 38, Theorem 3.5.3]) *If* $P=\dfrac{f(-q)}{q^{1/8}f(-q^4)}$ *and* $Q=\dfrac{f(-q^5)}{q^{5/8}f(-q^{20})}$

$$\textit{then} \quad \left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3=(PQ)^2+\left(\frac{4}{PQ}\right)^2-5\left(\frac{P}{Q}+\frac{Q}{P}\right).$$

Proof By using Lemma 2.2, we find that

$$\alpha=\frac{16}{16+P^8} \quad \text{and} \quad \beta=\frac{16}{16+Q^8}, \tag{2.25}$$

where β has degree 5 over α. It follows that

$$1-\alpha=\frac{P^8}{16+P^8} \quad \text{and} \quad 1-\beta=\frac{Q^8}{16+Q^8}. \tag{2.26}$$

Now, combining (2.6) and (2.7), we obtain

$$2\left\{(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right\}=2-\left\{\{\alpha(1-\beta)\}^{1/4}+\{\beta(1-\alpha)\}^{1/4}\right\}^4. \tag{2.27}$$

Employing (2.25) and (2.26) in (2.27) and simplifying, we obtain

$$(16+P^4Q^4)^2(16+P^8)(16+Q^8)=\left\{(16+P^8)(16+Q^8)-8(P^2+Q^2)^4\right\}^2 \tag{2.28}$$

Factorizing (2.28) using *Mathematica*, we obtain

$$\begin{aligned}&(P^2+Q^2)^2(P^6-16PQ-5P^4Q^2-5P^2Q^4-P^5Q^5+Q^6)\\&\times(P^6-16PQ-5P^4Q^2-5P^2Q^4+P^5Q^5+Q^6)=0.\end{aligned} \tag{2.29}$$

Now, proceeding as in the proof of Lemma 2.13, it can be shown that the first and last factors of (2.29) are non-zero in a neighborhood of zero. Thus, we have

$$P^6-16PQ-5P^4Q^2-5P^2Q^4-P^5Q^5+Q^6=0. \tag{2.30}$$

Dividing above equation by P^3Q^3 and rearranging the terms, we complete the proof. □

Lemma 2.15 *If* $P=\dfrac{f(-q)}{q^{1/8}f(-q^4)}$ *and* $Q=\dfrac{f(-q^7)}{q^{5/8}f(-q^{28})}$

$$\textit{then } \left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4=(PQ)^3+\left(\frac{4}{PQ}\right)^3+7\left(\left(\frac{4}{PQ}\right)^2+(PQ)^2\right)$$
$$+28\left(\frac{4}{PQ}+PQ\right)+70.$$

Proof Proceeding as in the proof of Lemma 2.13, we obtain

$$\alpha=\frac{16}{16+P^8},\quad \beta=\frac{16}{16+Q^8},\quad 1-\alpha=\frac{P^8}{16+P^8}\quad\text{and}\quad 1-\beta=\frac{Q^8}{16+Q^8}. \tag{2.31}$$

where β has degree 7 over α.

Employing (2.31) in Lemma 2.5 and simplifying, we deduce that

$$(2+PQ)^8=(P^8+16)(Q^8+16). \tag{2.32}$$

Simplifying (2.32), we get

$$\begin{aligned}&P^8-64PQ-112P^2Q^2-112P^3Q^3-70P^4Q^4-28P^5Q^5-7P^6Q^6\\&-P^7Q^7+Q^8=0.\end{aligned} \tag{2.33}$$

Dividing above equation by P^4Q^4 and rearranging the terms, we complete the proof. □

3 Evaluation of class invariants g_n

Theorem 3.1 *If A_n as defined in (1.6), then*

$$(i) A_{1/n} = 1/A_n, \quad (ii) A_1 = 1, \quad and \quad (iii) A_n = g_{2n}.$$

Proof To prove (i) we use the definition of A_n and Lemma 1.4. We set $n = 1$ in (i) to prove (ii). (iii) follows directly from the definitions of A_n and g_n from (1.6) and (1.2), respectively. □

Theorem 3.2 *One has*

$$\alpha = \frac{1}{1 + (A_n A_{4n})^8} \quad and \quad \beta = \frac{1}{1 + (A_{k^2n} A_{4k^2n})^8},$$

where β has degree k over α.

Proof For positive number k, set

$$P_1 = \frac{f(-q)}{q^{1/8} f(-q^4)} \quad \text{and} \quad Q_1 = \frac{f(-q^k)}{q^{k/8} f(-q^{4k})}. \tag{3.1}$$

Transcribing using Lemma 2.2, we obtain

$$\alpha = \frac{16}{16 + P_1^8} \quad \text{and} \quad \beta = \frac{16}{16 + Q_1^8}. \tag{3.2}$$

Setting $q := e^{-2\pi\sqrt{n/2}}$ in (3.2) and using the definition of A_n, we complete the proof. □

Next theorem is due to Yi [13, p. 42, Theorem 4.1.1].

Theorem 3.3 *We have*

$$4\left((A_n A_{4n})^4 + \frac{1}{(A_n A_{4n})^4}\right) = \left(\frac{A_{4n}}{A_n}\right)^{12}.$$

Proof follows easily from Lemma 2.12 and the definition of A_n.

Theorem 3.4 *We have*

$$\left(\frac{A_n A_{4n}}{A_{9n} A_{36n}}\right)^2 + \left(\frac{A_{9n} A_{36n}}{A_n A_{4n}}\right)^2 = 2\left\{A_n A_{4n} A_{9n} A_{36n} + (A_n A_{4n} A_{9n} A_{36n})^{-1}\right\}.$$

Proof We set $q := e^{-2\pi\sqrt{n/2}}$ in Lemma 2.13 and use the definition of A_n. □

Theorem 3.5 *We have*

$$A_6 = g_{12} = 2^{1/6}(2+\sqrt{3})^{1/8}, \quad A_{1/6} = g_{1/3} = 2^{-1/6}(2-\sqrt{3})^{1/8},$$
$$A_{3/2} = g_3 = 2^{-1/6}(2+\sqrt{3})^{1/8}, \quad A_{2/3} = g_{4/3} = 2^{1/6}(2-\sqrt{3})^{1/8}.$$

Proof Setting $n = 1/6$ in Theorem 3.4 and simplifying using Theorem 3.1(i), we obtain

$$\left(A_{2/3}/A_6\right)^4 + \left(A_{2/3}/A_6\right)^{-4} = 4. \tag{3.3}$$

Solving (3.3) for $(A_{2/3}/A_6)$, we get

$$(A_{2/3}/A_6) = (2-\sqrt{3})^{1/4}. \tag{3.4}$$

Now setting $n = 1/6$ in Theorem 3.3 and simplifying by using Theorem 3.1(i), we obtain

$$4\left\{\left(A_{2/3}/A_6\right)^4 + \left(A_{2/3}/A_6\right)^{-4}\right\} = \left(A_6 A_{2/3}\right)^{12} \tag{3.5}$$

Using (3.3) in (3.5), we deduce that

$$A_6 A_{2/3} = 2^{1/3}. \tag{3.6}$$

From (3.4) and (3.6), we easily deduce the values of A_6 and $A_{2/3}$. The values of $A_{1/6}$ and $A_{3/2}$ immediately follow from Theorem 3.3(i). □

Theorem 3.6 *We have*

$$4\left\{A_n A_{4n} A_{25n} A_{100n}) + (A_n A_{4n} A_{25n} A_{100n})^{-1}\right\}$$
$$= \left(\frac{A_{25n}A_{100n}}{A_n A_{4n}}\right)^3 + \left(\frac{A_n A_{4n}}{A_{25n}A_{100n}}\right)^3 - 5\left(\frac{A_{25n}A_{100n}}{A_n A_{4n}} + \frac{A_n A_{4n}}{A_{25n}A_{100n}}\right).$$

Proof We set $q := e^{-2\pi\sqrt{n/2}}$ in Lemma 2.14 and use the definition of A_n. □

Theorem 3.7 *We have*

$$A_{10} = g_{20} = 2^{-1/8}(\sqrt{5}+2)^{1/24}\left(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}\right)^{1/4},$$

$$A_{5/2} = g_5 = 2^{-3/8}(\sqrt{5}-2)^{1/24}\left(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}\right)^{1/4},$$

$$A_{1/10} = g_{1/5} = 2^{1/8}(\sqrt{5}-2)^{1/24}\left(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}\right)^{-1/4},$$

$$A_{2/5} = g_{4/5} = 2^{3/8}(\sqrt{5}+2)^{1/24}\left(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}\right)^{-1/4},$$

Proof Setting $n = 1/10$ in Theorem 3.6 and simplifying using Theorem 3.1(i), we obtain

$$(A_{10}A_{5/2})^6 + (A_{10}A_{5/2})^{-6} - 5\left\{(A_{10}A_{5/2})^2 + (A_{10}A_{5/2})^{-2}\right\} = 8. \tag{3.7}$$

From (3.7), we deduce that

$$(A_{10}A_{5/2})^2 + (A_{10}A_{5/2})^{-2} = 1 + \sqrt{5}. \tag{3.8}$$

Solving (3.8) for $A_{10}A_{5/2}$, we obtain

$$A_{10}A_{5/2} = \frac{\sqrt{1 + \sqrt{5} + \sqrt{2 + 2\sqrt{5}}}}{\sqrt{2}}. \tag{3.9}$$

Now setting $n = 1/10$ in Theorem 3.3 and simplifying using Theorem 3.1(i), we deduce that

$$4\left\{(A_{10}A_{5/2})^4 + (A_{10}A_{5/2})^{-4}\right\} = \left(A_{10}/A_{5/2}\right)^{12}. \tag{3.10}$$

Squaring (3.8), we deduce that

$$(A_{10}A_{5/2})^4 + (A_{10}A_{5/2})^{-4} = 2(2 + \sqrt{5}). \tag{3.11}$$

Employing (3.11) in (3.10) and solving the resulting equation, we obtain

$$\left(A_{10}/A_{5/2}\right) = 2^{1/4}(2 + \sqrt{5})^{1/2}. \tag{3.12}$$

Combining (3.9) and (3.12), we derive the values of A_{10} and $A_{5/2}$. Then values of $A_{1/10}$ and $A_{2/5}$ follow from Theorem 3.1(i). □

Theorem 3.8 *We have*

$$\begin{aligned}
&\left(\frac{A_nA_{4n}}{A_{49n}A_{196n}}\right)^4 + \left(\frac{A_nA_{4n}}{A_{49n}A_{196n}}\right)^{-4} \\
&\quad = 8\left\{(A_nA_{4n}A_{49n}A_{196n})^3 + ((A_nA_{4n}A_{49n}A_{196n})^{-3}\right\} \\
&\quad\quad +28\left\{(A_nA_{4n}A_{49n}A_{196n})^2 + ((A_nA_{4n}A_{49n}A_{196n})^{-2}\right\} \\
&\quad\quad +56\left\{A_nA_{4n}A_{49n}A_{196n} + ((A_nA_{4n}A_{49n}A_{196n})^{-1}\right\} + 70.
\end{aligned}$$

Proof We set $q := e^{-2\pi\sqrt{n/2}}$ in Lemma 2.15 and use the definition of A_n. □

Theorem 3.9 *We have*

$$A_{14}=g_{28}=2^{1/4}(127+48\sqrt{7})^{1/16},\quad A_{7/2}=g_7=2^{-1/4}(127+48\sqrt{7})^{1/16},$$
$$A_{1/14}=g_{1/7}=2^{-1/4}(127-48\sqrt{7})^{1/16},\quad A_{2/7}=g_{4/7}=2^{1/4}(127-48\sqrt{7})^{1/16}.$$

Proof Setting $n=1/14$ in Theorem 3.8 and simplifying using Theorem 3.1(i), we deduce that

$$(A_{14}A_{7/2})^8+(A_{14}A_{7/2})^{-8}=254. \tag{3.13}$$

Solving (3.13), we obtain

$$A_{14}A_{7/2}=(127+48\sqrt{7})^{1/8}. \tag{3.14}$$

Now setting $n=1/14$ in Theorem 3.3 and simplifying using Theorem 3.1(i), we arrive at

$$4\left\{(A_{14}A_{7/2})^4+(A_{14}A_{7/2})^{-4}\right\}=\left(A_{14}/A_{7/2}\right)^{12} \tag{3.15}$$

Squaring (3.15) and then using (3.13), we deduce that

$$\left(A_{14}/A_{7/2}\right)=\sqrt{2}. \tag{3.16}$$

Combining (3.14) and (3.16), we derive the values of A_{14} and $A_{7/2}$. Then the values of $A_{1/14}$ and $A_{2/7}$ follow from Theorem 3.1(i). □

Theorem 3.10 *We have*

$$A_{18}=g_{36}=2^{1/8}\left(7+4\sqrt{3}\right)^{1/6}\left(1+2\sqrt{-24+14\sqrt{3}}\right)^{1/8},$$
$$A_{9/2}=g_9=2^{-1/8}\left(7+4\sqrt{3}\right)^{1/12}\left(1+2\sqrt{-24+14\sqrt{3}}\right)^{1/8},$$
$$A_{1/18}=g_{1/9}=2^{-1/8}\left(7-4\sqrt{3}\right)^{1/6}\left(1+2\sqrt{-24+14\sqrt{3}}\right)^{-1/8},$$
$$A_{2/9}=g_{4/9}=2^{1/8}\left(7-4\sqrt{3}\right)^{1/12}\left(1+2\sqrt{-24+14\sqrt{3}}\right)^{-1/8}.$$

Proof Setting $k=9$ in Theorem 3.2, we get

$$\alpha=\frac{1}{1+(A_nA_{4n})^8}\quad\text{and}\quad\beta=\frac{1}{1+(A_{81n}A_{324n})^8}. \tag{3.17}$$

Now setting $n=1/18$ in (3.17) and simplifying using Theorem 3.1(i), we obtain

$$\alpha=\frac{(A_{18}A_{9/2})^8}{1+(A_{18}A_{9/2})^8}\quad\text{and}\quad\beta=\frac{1}{1+(A_{18}A_{9/2})^8}. \tag{3.18}$$

From (3.18) it is readily seen that

$$\alpha = (A_{18}A_{9/2})^8\beta, \; 1-\alpha=\beta, \quad \text{and} \quad 1-\beta=\alpha. \tag{3.19}$$

Employing (2.3) in Lemma 2.6 and simplifying, we get

$$P = 1-2x, \; Q = 64x(2-x), \quad \text{and} \quad R = 32x^2, \tag{3.20}$$

where

$$x = (A_{18}A_{9/2})^4\beta. \tag{3.21}$$

Employing (3.20) in (2.9) and factorizing, we obtain

$$(1+8x-4x^2)^2(1-28x+4x^2) = 0. \tag{3.22}$$

The first factor in (3.22) is not zero and do not give real value of x such the $0 < x < 1$, so we have

$$4x^2 - 28x + 1 = 0. \tag{3.23}$$

Solving (3.23) and noting $0 < x < 1$ and $(A_{18}A_{9/2})^4 > 1$ is real, we get

$$x = (7-4\sqrt{3})/2. \tag{3.24}$$

Combining (3.18), (3.21), and (3.24), we deduce that

$$(A_{18}A_{9/2})^4 = \left(7+4\sqrt{3}\right)\left(1+2\sqrt{-24+14\sqrt{3}}\right). \tag{3.25}$$

Now, setting $n = 1/18$ in Theorem 3.3 and simplifying using Theorem 3.1(i), we arrive at

$$4\left\{(A_{18}A_{9/2})^4 + (A_{18}A_{9/2})^{-4}\right\} = \left(A_{18}/A_{9/2}\right)^{12}. \tag{3.26}$$

Using (3.25) in (3.26) and solving the resulting equation, we obtain

$$\left(A_{18}/A_{9/2}\right) = 2^{1/4}\left(7+4\sqrt{3}\right)^{1/12}. \tag{3.27}$$

From (3.25) and (3.27), we easily deduce the values of A_{18} and $A_{9/2}$. Then the values of $A_{1/18}$ and $A_{2/9}$ follow from Theorem 3.1(i). □

Theorem 3.11 *We have*

$$A_{26} = g_{52} = 2^{1/8}\left(5\sqrt{13}+18\right)^{1/6}\left(1+6\sqrt{5\sqrt{13}-18}\right)^{1/8},$$

$$A_{13/2} = g_{13} = 2^{-1/8}\left(5\sqrt{13}+18\right)^{1/12}\left(1+6\sqrt{5\sqrt{13}-18}\right)^{1/8},$$

$$A_{1/26} = g_{1/13} = 2^{-1/8}\left(5\sqrt{13}-18\right)^{1/6}\left(1+6\sqrt{5\sqrt{13}-18}\right)^{-1/8},$$
$$A_{2/13} = g_{4/13} = 2^{1/8}\left(5\sqrt{13}-18\right)^{1/12}\left(1+6\sqrt{5\sqrt{13}-18}\right)^{-1/8}.$$

Proof Setting $k = 13$ in Theorem 3.2 and then setting $n = 1/26$ and simplifying using Theorem 3.1(i), we obtain

$$\alpha = \frac{(A_{26}A_{13/2})^8}{1+(A_{26}A_{13/2})^8} \quad \text{and} \quad \beta = \frac{1}{1+(A_{26}A_{13/2})^8}. \tag{3.28}$$

so that

$$\alpha = (A_{26}A_{13/2})^8\beta, \quad 1-\alpha=\beta, \quad \text{and} \quad 1-\beta=\alpha. \tag{3.29}$$

Employing (3.29) in Lemma 2.7 and simplifying, we get

$$(1+2x)(1+28x+4x^2)^2(-1+72x+4x^2)=0, \tag{3.30}$$

where

$$x = (A_{26}A_{13/2})^4\beta. \tag{3.31}$$

Since first two factors are not zero, so we have

$$4x^2+72x-1=0. \tag{3.32}$$

Solving (3.32), we obtain

$$x = (5\sqrt{13}-18)/2. \tag{3.33}$$

From (3.29), (3.31), and (3.33) and noting $A_{26}A_{13/2} > 1$, we deduce that

$$\left(A_{26}A_{13/2}\right)^4 = \left(5\sqrt{13}+18\right)\left(1+6\sqrt{5\sqrt{13}+18}\right). \tag{3.34}$$

Setting $n = 1/26$ in Theorem 3.3, simplifying using Theorem 3.1(i), employing (3.34) and then solving the resulting equation, we arrive at

$$\left(A_{26}/A_{13/2}\right) = 2^{1/4}\left(5\sqrt{13}+18\right)^{1/12}. \tag{3.35}$$

From (3.34) and (3.35), we easily calculate the values of A_{26} and $A_{13/2}$. Then the values of $A_{1/26}$ and $A_{2/13}$ follow from Theorem 3.1(i). □

Theorem 3.12 *We have*

$$A_{34}=g_{68}=2^{1/8}\left(20+5\sqrt{17}+2\sqrt{206+50\sqrt{17}}\right)^{1/6}$$
$$\times\left(1+\sqrt{1-\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}\right)^{1/8},$$
$$A_{17/2}=g_{17}=2^{-1/8}\left(20+5\sqrt{17}+2\sqrt{206+50\sqrt{17}}\right)^{1/12}$$
$$\times\left(1+\sqrt{1-\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}\right)^{1/8},$$
$$A_{1/34}=g_{1/17}=2^{-1/8}\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^{1/6}$$
$$\times\left(1+\sqrt{1-\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}\right)^{-1/8},$$
$$A_{2/17}=g_{4/17}=2^{1/8}\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^{1/12}$$
$$\times\left(1+\sqrt{1-\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)^2}\right)^{-1/8}.$$

Proof Setting $k = 17$ in Theorem 3.2 and then setting $n = 1/34$ and simplifying using Theorem 3.1(i), we obtain

$$\alpha=\frac{(A_{34}A_{17/2})^8}{1+(A_{34}A_{17/2})^8} \quad \text{and} \quad \beta=\frac{1}{1+(A_{34}A_{17/2})^8}. \tag{3.36}$$

so that

$$\alpha=(A_{34}A_{17/2})^8\beta, \quad 1-\alpha=\beta, \quad \text{and} \quad 1-\beta=\alpha. \tag{3.37}$$

Employing (3.37) in Lemma 2.8 and solving the resulting equation, we get

$$x=\left(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}}\right)/2. \tag{3.38}$$

where

$$x=(A_{34}A_{17/2})^4\beta. \tag{3.39}$$

From (3.36), (3.38), and (3.39) and noting $A_{34}A_{17/2} > 1$, we arrive that

$$\left(A_{34}A_{17/2}\right)^4 = \left(20 + 5\sqrt{17} + 2\sqrt{206 + 50\sqrt{17}}\right) \times \left(1 + \sqrt{1 - \left(20 + 5\sqrt{17} - 2\sqrt{206 + 50\sqrt{17}}\right)^2}\right). \tag{3.40}$$

Setting $n = 1/34$ in Theorem 3.3, simplifying using Theorem 3.1(i), employing (3.40) and then solving the resulting equation, we find that

$$\left(A_{34}/A_{17/2}\right) = 2^{1/4}\left(20 + 5\sqrt{17} + 2\sqrt{206 + 50\sqrt{17}}\right)^{1/12}. \tag{3.41}$$

From (3.40) and (3.41), we easily calculate the values of A_{34} and $A_{17/2}$. Then the values of $A_{1/34}$ and $A_{2/17}$ follow from Theorem 3.1(i). □

Theorem 3.13 *We have*

$$A_{38} = g_{76} = 2^{5/24}\, 3^{1/12}\, r^{-1/6}\left(576 + \sqrt{331776 - 4r^2}\right)^{1/8},$$
$$A_{19/2} = g_{19} = 2^{-11/24}\, 3^{-1/12}\, r^{-1/12}\left(576 + \sqrt{331776 - 4r^2}\right)^{1/8},$$
$$A_{1/38} = g_{1/19} = 2^{-5/24}\, 3^{-1/12}\, r^{1/6}\left(576 + \sqrt{331776 - 4r^2}\right)^{-1/8},$$
$$A_{2/19} = g_{4/19} = 2^{11/24}\, 3^{1/12}\, r^{1/12}\left(576 + \sqrt{331776 - 4r^2}\right)^{-1/8},$$

where $r = 528 - 40(2944 - 384\sqrt{57})^{1/3} + (2944 - 384\sqrt{57})^{2/3} - 160(46 + 6\sqrt{57})^{1/3} + 16(46 + 6\sqrt{57})^{2/3}$.

Proof Setting $k = 19$ in Theorem 3.2 and then setting $n = 1/38$ and simplifying using Theorem 3.1(i), we arrive at

$$\alpha = \frac{(A_{38}A_{17/2})^8}{1 + (A_{34}A_{19/2})^8} \quad \text{and} \quad \beta = \frac{1}{1 + (A_{38}A_{19/2})^8}. \tag{3.42}$$

so that

$$\alpha = (A_{38}A_{19/2})^8\beta, \quad 1 - \alpha = \beta, \quad \text{and} \quad 1 - \beta = \alpha. \tag{3.43}$$

Employing (3.43) in Lemma 2.9 and factorizing, we get

$$(1 + 2y)^2(8y^3 + 20y^2 + 14y - 1) = 0, \tag{3.44}$$

where

$$y=(A_{38}A_{19/2})^2\sqrt{\beta}. \tag{3.45}$$

Since the first factor in (3.44) is non-zero, so we have

$$8y^3+20y^2+14y-1=0. \tag{3.46}$$

Solving (3.46), we obtain

$$y=\left(-20+(2944-384\sqrt{57})^{1/3}+4(46+6\sqrt{57})^{1/3}\right)/24. \tag{3.47}$$

From (3.42), (3.45), and (3.47) and noting $A_{38}A_{19/2}>1$, we deduce that

$$\left(A_{38}A_{19/2}\right)^4=\left(576+\sqrt{331776-4r^2}\right)\Big/2r, \tag{3.48}$$

where $r=528-40(2944-384\sqrt{57})^{1/3}+(2944-384\sqrt{57})^{2/3}-160(46+6\sqrt{57})^{1/3}+16(46+6\sqrt{57})^{2/3}$.

Setting $n=1/38$ in Theorem 3.3, simplifying using Theorem 3.1(i), employing (3.48) and then solving the resulting equation, we find that

$$\left(A_{38}/A_{19/2}\right)=2^{2/3}3^{1/6}r^{-1/12}. \tag{3.49}$$

From (3.48) and (3.49), we easily find the values of A_{38} and $A_{19/2}$. Then the values of $A_{1/38}$ and $A_{2/19}$ follow from Theorem 3.1(i). □

Theorem 3.14 *We have*

$$\begin{aligned}
A_{54}&=g_{108}=2^{1/6}\left(2+\sqrt{3-100\cdot 2^{1/3}+80\cdot 2^{2/3}}\right)^{1/8}\left(6\cdot 2^{2/3}-2^{4/3}-7\right)^{-1/6},\\
A_{27/2}&=g_{27}=2^{-1/6}\left(2+\sqrt{3-100\cdot 2^{1/3}+80\cdot 2^{2/3}}\right)^{1/8}\left(6\cdot 2^{2/3}-2^{4/3}-7\right)^{-1/12},\\
A_{1/54}&=g_{1/27}=2^{-1/6}\left(2+\sqrt{3-100\cdot 2^{1/3}+80\cdot 2^{2/3}}\right)^{-1/8}\left(6\cdot 2^{2/3}-2^{4/3}-7\right)^{1/6},\\
A_{2/27}&=g_{4/27}=2^{1/6}\left(2+\sqrt{3-100\cdot 2^{1/3}+80\cdot 2^{2/3}}\right)^{-1/8}\left(6\cdot 2^{2/3}-2^{4/3}-7\right)^{1/12}.
\end{aligned}$$

Proof Setting $k=27$ in Theorem 3.2 and then setting $n=1/38$ and simplifying using Theorem 3.1(i), we arrive at

$$\alpha=\frac{(A_{54}A_{27/2})^8}{1+(A_{54}A_{27/2})^8}\quad\text{and}\quad\beta=\frac{1}{1+(A_{54}A_{27/2})^8}. \tag{3.50}$$

so that

$$\alpha=(A_{54}A_{27/2})^8\beta,\quad 1-\alpha=\beta,\quad\text{and}\quad 1-\beta=\alpha. \tag{3.51}$$

Employing (3.51) in Lemma 2.10 and factorizing *Mathematica*, we get

$$(1+6x-4x^2+8x^3)^2(-1+30y-12y^2+8y^3)=0, \tag{3.52}$$

where

$$y=(A_{54}A_{27/2})^2\sqrt{\beta}. \tag{3.53}$$

Since the first factor in (3.52) is non-zero, so we have

$$-1+30y-12y^2+8y^3=0. \tag{3.54}$$

Solving (3.54), we obtain

$$y=(-1+2^{1/3})^2/2. \tag{3.55}$$

From (3.50), (3.53), and (3.55) and noting $A_{54}A_{27/2}>1$, we deduce that

$$\left(A_{54}A_{27/2}\right)^4=\frac{2+\sqrt{3-100\cdot 2^{1/3}+80\cdot 2^{2/3}}}{6\cdot 2^{2/3}-2^{4/3}-7}. \tag{3.56}$$

Setting $n=1/54$ in Theorem 3.3, simplifying using Theorem 3.1(i), employing (3.56) and then solving the resulting equation, we find that

$$\left(A_{54}/A_{27/2}\right)=2^{1/3}\left(6\cdot 2^{2/3}-2^{4/3}-7\right)^{-1/12}. \tag{3.57}$$

From (3.56) and (3.57), we easily find the values of A_{54} and $A_{27/2}$. Then the values of $A_{1/54}$ and $A_{2/27}$ follow from Theorem 3.1(i). □

Theorem 3.15 *We have*

$$A_{30}=g_{60}=2^{7/24}\left(16+\sqrt{162+42\sqrt{5}}\right)^{1/8}\left(7-3\sqrt{5}\right)^{-1/6},$$

$$A_{15/2}=g_{15}=2^{-7/24}\left(16+\sqrt{162+42\sqrt{5}}\right)^{1/8}\left(7-3\sqrt{5}\right)^{-1/12},$$

$$A_{1/30}=g_{1/15}=2^{-7/24}\left(16+\sqrt{162+42\sqrt{5}}\right)^{-1/8}\left(7-3\sqrt{5}\right)^{1/6},$$

$$A_{2/15}=g_{4/15}=2^{7/24}\left(16+\sqrt{162+42\sqrt{5}}\right)^{-1/8}\left(7-3\sqrt{5}\right)^{1/12}.$$

Proof Setting $k=15$ in Theorem 3.2 and then setting $n=1/30$ and simplifying using Theorem 3.1(i), we arrive at

$$\alpha=\frac{(A_{30}A_{15/2})^8}{1+(A_{30}A_{15/2})^8}\quad\text{and}\quad\beta=\frac{1}{1+(A_{30}A_{15/2})^8}. \tag{3.58}$$

so that

$$\alpha = (A_{30}A_{15/2})^8\beta, \quad 1-\alpha=\beta, \quad \text{and} \quad 1-\beta=\alpha. \tag{3.59}$$

Employing (3.59) in Lemma 2.11, we get

$$4z^2+2z-1=0, \tag{3.60}$$

where

$$z=(A_{30}A_{15/2})\beta^{1/4}. \tag{3.61}$$

Solving (3.60), we obtain

$$z=(-1+\sqrt{5})/4. \tag{3.62}$$

From (3.58), (3.61), and (3.62) and noting $A_{30}A_{15/2}>1$, we deduce that

$$\left(A_{30}A_{15/2}\right)^4=\left(16+\sqrt{162+42\sqrt{5}}\right)\Big/\left(7-3\sqrt{5}\right). \tag{3.63}$$

Setting $n=1/30$ in Theorem 3.3, simplifying using Theorem 3.1(i), employing (3.63) and then solving the resulting equation, we find that

$$\left(A_{30}/A_{15/2}\right)=2^{7/12}\left(7-3\sqrt{5}\right)^{-1/12}. \tag{3.64}$$

From (3.63) and (3.64), we easily deduce the values of A_{30} and $A_{15/2}$. Then the values of $A_{1/30}$ and $A_{2/15}$ follow from Theorem 3.1(i). □

4 Evaluation of class invariants G_n

In his paper [10] and also in page 294 of his second notebook [11, Vol. II], Ramanujan recorded two simple formulas relating the class invariants g_n and G_n, namely, for $n>0$

$$g_{4n}=2^{1/4}g_nG_n \tag{4.1}$$

and

$$(g_nG_n)^8(G_n^8-g_n^8)=\frac{1}{4}. \tag{4.2}$$

Thus, if we know g_n and g_{4n} or only g_n then the corresponding G_n can be calculated by the above formulas. We now find some values of G_n by using (4.1) and the new values of g_n and g_{4n} evaluated in above section.

Theorem 4.1 *We have*

$$
\begin{aligned}
&\text{(i)}\quad G_3 = 2^{1/12}, \quad \text{(ii)}\quad G_5 = (\sqrt{5}+2)^{1/12}, \quad \text{(iii)}\quad G_7 = 2^{1/4},\\
&\text{(iv)}\quad G_9 = (7+4\sqrt{3})^{1/12}, \quad \text{(v)}\quad G_{13} = (5\sqrt{13}+18)^{1/12},\\
&\text{(vi)}\quad G_{17} = \left(20+5\sqrt{17}+2\sqrt{206+50\sqrt{17}}\right)^{1/12},\\
&\text{(vii)}\quad G_{19} = 2^{5/12}3^{1/6}r^{-1/12}, \textit{ where } r \textit{ is given in Theorem } 3.13.\\
&\text{(viii)}\quad G_{27} = 2^{1/12}(6\cdot 2^{2/3}-2^{4/3}-7)^{-1/12}, \quad \text{(ix)}\quad G_{15} = 2^{1/3}(7-3\sqrt{5})^{-1/12}.
\end{aligned}
$$

Proof For (i), we use the values of g_{12} and g_3 from Theorem 3.5 in (4.1). To prove (ii), we employ the values of g_{20} and g_5 from Theorem 3.7 in (4.1). To prove (iii), employ the values of g_{28} and g_7 from Theorem 3.9 in (4.1). To prove (iv), we use the values of g_{36} and g_9 from Theorem 3.10 in (4.1). Employing the values of g_{52} and g_{13} from Theorem 3.11 in (4.1) we arrive at (v). Employing the values of g_{17} and g_{68} from Theorem 3.12 in (4.1), we prove (vi). For (vii), we use the values of g_{19} and g_{76} from Theorem 3.13 in (4.1). To prove (viii), we use the values of g_{27} and g_{108} from Theorem 3.14 in (4.1). To prove (ix), we employ the values of g_{15} and g_{60} from Theorem 3.15 in (4.1). □

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