Thermal Analysis of Natural Convection in Porous Fins with Homotopy Perturbation Method (HPM)

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Abstract In this paper, porous fin has been studied and its nonlinear ordinary differential equation has been solved through homotopy perturbation method. In this method, a homotopy is introduced to be constructed for the equation. The initial approximations can be freely chosen with possible unknown constants, which can be determined by imposing the boundary and initial conditions. To study the thermal performance, one type case is considered: finite-length fin with insulated tip.

Keywords Porous fin · Homotopy perturbation method (HPM) · Nonlinear ordinary differential equation · Finite-length fin with insulated tip

الخلاصة

تمت - في هذه الورقة العلمية - دراسة النهاية الغشائية وحل معادلتها التفاضلية العادية غير الخطية من خلال طريقة اضطراب هوموتوبي (HPM). وفي هذا الأسلوب يتم إدخال homotopy لكي يتم بناؤها لأجل المعادلة. ويمكن اختبار التقريبات الأولية بحرية مع ثوابت معروفة محتملة يمكن تحديدها من خلال فرض الحدود والظروف الأولية. ولدراسة الأداء الحراري عُدت حالة من نوع واحد: نهاية الطول-المحدود الزعنفه مع رأس معزول.

Nomenclature

k	Thermal conductivity
Da	Darcy number, K/t^2
Kr	Thermal conductivity ratio, (k_{eff}/k_f)
K	Permeability of porous fin
L	Length
m	convection parameter
q	Heat transfer rate
Ra	Rayleigh number, $Gr \times Pr$
Sh	Porous parameter
$T(x)$	Temperature at any point
T_b	Temperature at fin base
t	Thickness of the fin
$v_{W(x)}$	Velocity of fluid passing through the fin anypoint
W	Width of the fin
x	Axial coordinate
X	Dimensionless axial coordinate, (x/L)
θ	Dimensionless temperature,
θ_b	Base temperature difference, $(T_b - T_\infty)$
s	Solid properties
f	Fluid properties
eff	Porous properties

1 Introduction

The homotopy perturbation method (HPM) was first proposed by He [14–24]. The HPM does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$, which is considered as a “small parameter”. The HPM was successfully applied to nonlinear oscillators with discontinuities [1–4] and bifurcation of nonlinear problems [5]. In [2], a comparison of HPM and homotopy analysis method was made. Most of the scientific

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problems and phenomena occur nonlinearly. Except a limited number of these problems, most of them do not have precise analytical solutions so that they have to be solved using other methods. Many different new methods have recently presented some techniques to eliminate the small parameter; for example, the variational iteration method (VIM) [6–9] and the Exact solutions (ADM) [10, 11], HPM [12–17], and others [18, 19]. In [25], a new perturbation algorithm combining the method of multiple scales and lindstedt–Poincare techniques is proposed for first time. The new perturbation algorithm combining the method of multiple scales (MS) and lindstedt–Poincare techniques is applied to an equation with quadratic and cubic nonlinearities [26]. In [27], forced vibrations of duffing equation with damping is considered. Paper [28] shows two new perturbation-iteration algorithms for solving differential equations of first order. A nonlinear fin equation in which thermal conductivity is an arbitrary function of temperature and heat transfer coefficient is an arbitrary function of spatial variable is considered [29], [30]. Scaling, translational and spiral group symmetries of the equations are determined. In [31], they propose an analytical solution to a nonlinear second-order boundary value problem using HPM and 4q-Boubaker polynomials Expansion scheme (BPES). In [32], they investigated magnetohydrodynamic (MHD) viscous flow due to a shrinking sheet by employing the HPM and pade approximation. A series solution of the long porous slider problem [33] where fluid is injected through the porous bottom is obtained using the HPM. In this article, HPM is used to solve nonlinear equations of finite-length fin with insulated tip with the initial conditions $\theta(0) = 1$, $\theta'(1) = 0$ for the first time.

2 Material and Methods

As shown in Fig. 1, a rectangular fin profile is considered. The dimensions of the fin are length L , width w and thickness t . The cross section area of the fin is constant. This fin is porous to allow the flow of infiltrate through it. The following assumptions are made to solve this problem.

From the Darcy's model, we have:

$$v_w = \frac{gk\beta(T - T_\infty)}{v}$$

Now applying energy balance equation at steady state condition, we have:

$$\frac{d^2\theta}{dx^2} - S_h\theta(x)^2 - m^2\theta(x) = 0 \quad (1)$$

Where, $\theta = \frac{T(x) - T_\infty}{T_b - T_\infty}$ and $X = \frac{x}{L}$

Porous parameter, $S_h = \frac{DaxRa}{k_r} \left(\frac{L}{t}\right)^2$ and Convection parameter, $m = \left(\frac{hp}{k_s A}\right)^{\frac{1}{2}}$

Fig. 1 Schematic diagram of fin profile under consideration

Here, s_h is a porous parameter that indicates the effect of the permeability of the porous medium as well as buoyancy effect, so higher value of s_h indicates higher permeability of the porous medium or higher buoyancy forces. m is a convection parameter that indicates the effect of surface convecting of the fin.

Here is the summary of case to be considered for this research.

Case: finite-length fin with insulated tip ($\theta(0) = 1$, $d\theta/dx|_{x=1} = 0$)

3 Basic Concept of He's Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following equation:

$$A(F) - f(r) = 0 \quad r \in \Omega \quad (2)$$

With the boundary condition of:

$$B\left(u, \frac{\partial F}{\partial n}\right) = 0 \quad (3)$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω . A can be divided into two parts, L and N , where L is linear and N is nonlinear. Equation 2 can, therefore, be rewritten as follows:

$$L(F) + N(F) - f(r) = 0 \quad r \in \Omega \quad (4)$$

Homotopy perturbation structure is shown as follows:

$$H(V, P) = (1 - P)[L(V) - L(u_0)] + P[A(V) - f(r)] = 0 \quad (5)$$

Where,

$$V(r, p) : \Omega \times [0, 1] \rightarrow R, \quad (6)$$

In Eq. (6), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (1) can be written as a power series in p , as following:

$$V = V_0 + pV_1 + p^2V_2 + \dots = \sum_{i=0}^n V_i p^i \quad (7)$$

And the best approximation for solution is:

$$F = \lim_{p \rightarrow 1} V = V_0 + V_1 + V_2 + \dots \quad (8)$$

4 Application of HPM to Porous Fin

In this section, we will apply the HPM to nonlinear ordinary differential Eq. (1). According to the HPM, we can construct homotopy of Eq. (1) as follows:

$$(1-p) \left[\frac{d^2}{dx^2} \theta(x) - \theta(x) \right] + p \left[\frac{d^2}{dx^2} \theta(x) - \theta(x)^2 - \theta(x) \right] = 0 \quad (9)$$

We consider θ as follows:

$$\theta(x) = \theta_0(x) + \theta_1(x) + \theta_2(x) + \dots = \sum_{i=0}^n \theta_i(x) \quad (10)$$

Substituting θ from Eq. (10) into Eq. (9) and some simplification and rearranging based on powers of p -terms, we have:

$$P^0 : -\theta_0(x) + \frac{d^2}{dx^2} \theta_0(x) = 0, \quad \theta_0(0) = 1, \quad \theta_0'(1) = 0 \quad (11)$$

$$P^1 : -\theta_0(x)^2 - \theta_1(x) + \frac{d^2}{dx^2} \theta_1(x) = 0, \quad \theta_1(0) = 0, \quad \theta_1'(1) = 0 \quad (12)$$

$$P^2 : -2\theta_0(x)\theta_1(x) = 0, \quad \theta_2(0) = 0, \quad \theta_2'(1) = 0 \quad (13)$$

$$P^3 : -\theta_1(x)^2 = 0, \quad \theta_3(0) = 0, \quad \theta_3'(1) = 0 \quad (14)$$

Solving Eqs. (11)–(14) with boundary conditions, we have:

$$\theta_0(x) = \frac{ee^{-x}}{e^{-1} + e} + \frac{e^{-1}e^x}{e^{-1} + e} \quad (15)$$

$$\begin{aligned} \theta_1(x) = & -\frac{1}{3} \frac{e^{-x}e(e^2 - 6 + e^{-2})e^2}{e^4e^{-1} + 2e^2e^{-1} + e^{-1} + e + 2ee^2 + ee^4} \\ & -\frac{1}{3} \frac{e^xe^{-1}(e^2 - 6 + e^{-2})e^2}{e^4e^{-1} + 2e^2e^{-1} + e^{-1} + e + 2ee^2 + ee^4} \\ & + \frac{(e^2 - 6e^{2x} + e^{4x-2})e^{-2x+2}}{3 + 6e^2 + 3e^4} \end{aligned} \quad (16)$$

Fig. 2 The distribution of axial non-dimensional temperature along the finite-length fin for different values of S_h

Fig. 3 The distribution of axial non-dimensional temperature along the finite-length fin for different values of m

The solution of this equation, when $p \rightarrow 1$, will be as follows:

$$\theta(X) = \theta_0(X) + \theta_1(X)$$

Hence, the equation answer to set boundary conditions will be:

$$\left[-\frac{4}{3} \frac{e^6 + 3e^4 - 3e^2 - 1}{(1 + 2e^2 + e^4)(1 + e^2)} \right] \quad (17)$$

5 Results and Discussion

The governing Eq. (1) is a non-linear second-order ordinary differential equation. The equation is solved numerically

Fig. 4 The variation of the ratio of porous fin to solid fin heat transfer rate with K_r

using the by HPM. Depending on the tip condition of the fin, we have three different types of cases, but in this research, we only study finite-length fin with insulated tip.

From Fig. 2, we can see that the value of dimensionless temperature decreases along the fin length. It should be noted that as the value of S_h increases, the temperature decreases rapidly and the fin quickly reaches the surrounding temperature.

Figure 3 shows the results for the effect of variation of m on dimensionless temperature distribution. The values of m were varied while keeping $S_h = 1$. Figure 4 shows porous fin to solid fin heat transfer rate with convection and without convection. The figure also shows the variation ratio of porous fin to solid fin heat transfer rate with K_r . Variation of ratio of porous fin to solid fin heat transfer rate increases as K_r increases for both the cases.

6 Conclusions

This work introduces a simple method to analysis the performance of a porous fin. It is found that the problem of heat transfer through the porous fin is governed by a second-order nonlinear, ordinary, differential equation. In this letter, we have studied porous fin problem with a small parameter with the homotopy perturbation method. As shown in Eq. (6), the homotopy perturbation method does not need a small parameter. This thermal analysis was performed on one type of fin case: The finite-length fin with insulated tip. The effect of these all two parameters m and S_h has been investigated.

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