On the measure of axial symmetry with respect to folding for parallelograms

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Abstract Let C_m be a subset of a planar convex body C cut off by a straight line m , which remains in C after folding it along m . The maximum $\text{masf}(C)$ of the ratio of the double area of C_m to the area of C over all straight lines m is a measure of axial symmetry of C . We prove that $\text{masf}(P) > \frac{1}{2}$ for every parallelogram P and that this inequality cannot be improved.

Keywords Convex body · Folding · Measure of axial symmetry · Parallelogram

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[Lassak \(2003\)](#) proposes the measure

$$\text{masf}(C) = 2 \cdot \frac{\max_m \text{area}(C_m)}{\text{area}(C)}$$

of axial symmetry of a convex body C in the Euclidean plane E^2 . We call it *measure of axial symmetry with respect to folding*. Here C_m denotes a subset of C cut off by a straight line m , which remains in C after folding it along m . Moreover, in [Lassak \(2003\)](#) there is proved that for every convex body $C \subset E^2$ we have $\text{masf}(C) \geq \frac{1}{4}$ (see Theorem 3 there and the Erratum). We conjecture that $\text{masf}(C) > \frac{1}{2}$ for every centrally symmetric convex body $C \subset E^2$, and that this inequality cannot be improved in general for centrally symmetric convex bodies. We prove this inequality for parallelograms and show that it cannot be improved in general for such 4-gons.

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A final estimate of axial symmetry of parallelograms for another kind of axial symmetry is given in [Lassak and Nowicka \(2010\)](#). Some other measures of axial symmetry are considered by [Chakerian and Stein \(1965\)](#), [deValcourt \(1966a,b\)](#), and [Nohl \(1962\)](#). A survey on measures of symmetry of convex bodies is given by [Grünbaum \(1963\)](#). See also Part 4.2 of the article by [Heil and Martini \(1993\)](#).

We omit the easy proofs of the following claims.

Claim 1 *For every axially symmetric convex body $C \subset E^2$ the straight line m for which $\text{area}(C_m)$ is maximal is an axis of symmetry of C .*

Claim 2 *For every convex body $C \subset E^2$ we have $\text{masf}(C) = 1$ if and only if C is axially symmetric.*

We use the same symbol for an angle and its measure, and we simply say angle instead of measure of an angle. Denote by a, b, c and d successive vertices of the parallelogram P . Assume that $\angle cba \leq \frac{\pi}{2}$ and $|ab| \leq |bc|$. By o we denote the center of symmetry of P . Put $\alpha = \angle cba$ and $\beta = \angle acb$. By $\mathcal{A}$ (respectively, by $\mathcal{B}$) we mean the pencil of supporting lines through a (respectively, through b). The symbol $\text{bi}(\angle ef g)$ stands for the straight line containing the bisector of the angle $\angle ef g$. By a *strip* we mean the convex hull of two parallel lines.

Let $l \in \mathcal{A} \cup \mathcal{B}$. Take an arbitrary straight line m parallel to l which has nonempty intersection with P . By R_m we denote the subset of P which is contained in the strip between m and l . Moreover, by T_m we mean the image of R_m after folding it along m . Notice that T_m is the mirror image of R_m in the line m .

Lemma 1 *Consider a parallelogram P and the family of all straight lines m parallel to a fixed straight line l which have nonempty intersection with P . The union of the lines for which $T_m \subset P$ is a strip.*

Proof The easy proof of the case when P is axially symmetric is left to the reader. From now on assume that P does not have an axis of symmetry. Let $l \in \mathcal{A}$ (for $l \in \mathcal{B}$, the consideration is analogous).

Let w (respectively, v) be the intersection point of the segment bc with the circle of radius $|ab|$ (respectively, $|ad|$) centered at b (respectively, d). Denote by l_1 and l_2 the straight lines from $\mathcal{A}$ which are parallel to $\text{bi}(\angle adv)$ and $\text{bi}(\angle wba)$, respectively. Since $w \in bc$, we see that $\text{bi}(\angle wba) = \text{bi}(\angle cba)$. The symbols a_m, b_m, c_m and d_m stand for the points correspondingly symmetric to the points a, b, c and d with respect to m . Denote by a' the intersection of the straight line through a perpendicular to l with the segments bc or cd . Let a'' be the midpoint of aa' . The points of intersection of m with the boundary of P are denoted by p_m and q_m . The notation is chosen such that $p_m \in ad \cup cd$ and $q_m \in ab \cup bc$.

Case 1, when the angle between l and the straight line containing the segment ad is at most the angle between l_1 and this straight line.

Denote by k, k_1, k_2 and k_3 the straight lines parallel to l which pass through d, a'', b and c , respectively. By S, S_1, S_2 and S_3 we mean the strips between l and k, k and k_1, k_1 and k_2 and k_2 and k_3 , respectively.

Observe that for $m \in S$ we have $T_m = a_m p_m q_m$, where $p_m \in ad$ and $q_m \in ab$ (see Fig. 1). Since p_m, q_m, a_m belong to P , we get $T_m \subset P$.

Fig. 1 The situation in Case 1 for $m \in S$

Fig. 2 The situation in Case 1 for $m \in S_1$

Let $m \in S_1 \cup S_2 \cup S_3$. If $m \in S_1$ (see Fig. 2) or $m \in S_2$, then $T_m = a_m d_m p_m q_m$, where $p_m \in cd$ and $q_m \in ab$. If $m \in S_3$, then $T_m = a_m d_m p_m q_m b_m$, where $p_m \in cd$ and $q_m \in bc$. For every $m \neq k$ we have $d_m \notin P$. Consequently, $T_m \not\subset P$.

Case 2, when the angle between the straight line containing ad and l is greater than the angle between the straight line containing ad and l_1 and simultaneously at most the angle between the line containing ad and l_2 .

Assume that the distance between b and l is smaller than the distance between d and l . In the opposite situation the considerations are analogous. The symbols k, k_1, k_2

and k_3 stand for the straight lines parallel to l passing through a'' , b , d and c , respectively. The strips between l and k , k and k_1 , k_1 and k_2 , k_2 and k_3 are denoted by S , S_1 , S_2 , S_3 , respectively.

For $m \in S$ we have $T_m = a_m p_m q_m$. From $p_m \in ad$, $q_m \in ab$ and $a_m \in P$, we obtain $T_m \subset P$.

Notice that $T_m = a_m p_m q_m$ if $m \in S_1$, and $T_m = a_m p_m q_m b_m$ if $m \in S_2$. Moreover, if $m \in S_3$, then $T_m = a_m d_m p_m q_m b_m$. In the first case $p_m \in ad$ and $q_m \in ab$, in the second case $p_m \in ad$ and $q_m \in bc$, and in the third case $p_m \in cd$ and $q_m \in bc$. Observe that for every $m \neq k$ we have $a_m \notin P$. Hence $T_m \not\subset P$.

Case 3, when the angle between the straight line containing ad and l is greater than the angle between the straight line containing ad and l_2 .

We repeat the considerations of Case 1 taking this time instead of k , k_1 , k_2 and k_3 the straight lines parallel to l which pass through b , a'' , d and c , respectively. $\square$

From now on, for every $l \in \mathcal{A} \cup \mathcal{B}$ we denote by S_l the strip of all straight lines m parallel to l for which T_m remains in P .

Lemma 2 *Let P be a parallelogram and let $l \in \mathcal{A} \cup \mathcal{B}$. Assume that m_1 and m_2 are the straight lines parallel to l such that $m_1, m_2 \in S_l$. If the distance between m_1 and o is at least the distance between m_2 and o , then $\text{area}(R_{m_1}) \leq \text{area}(R_{m_2})$.*

Proof Without loss of generality assume that $l \in \mathcal{A}$. The proof of the case when P is axially symmetric is left to the reader. We look at Cases 1–3 of Lemma 1 and we conclude that $T_{m_1} = a_{m_1} p_{m_1} q_{m_1}$ and $T_{m_2} = a_{m_2} p_{m_2} q_{m_2}$, where p_{m_1} and p_{m_2} are the intersections of the segment ad with m_1 and m_2 , respectively. Analogously, q_{m_1} and q_{m_2} are the intersections of the segment ab with m_1 and m_2 , respectively. Recall that T_{m_1} and T_{m_2} are the images of R_{m_1} and R_{m_2} after folding them along m_1 and m_2 , respectively. Hence $R_{m_1} = a q_{m_1} p_{m_1}$ and $R_{m_2} = a q_{m_2} p_{m_2}$. Since m_1, m_2 are parallel and since the distance between m_1 and o is at least the distance between m_2 and o , we see that $|ap_{m_1}| \leq |ap_{m_2}|$ and $|aq_{m_1}| \leq |aq_{m_2}|$. These inequalities and the fact that $\angle q_{m_1} a p_{m_1} = \angle q_{m_2} a p_{m_2}$ imply that $\text{area}(R_{m_1}) \leq \text{area}(R_{m_2})$. $\square$

From Lemmas 1 and 2 we get the following corollary.

Corollary 1 *For a given strip S_l the maximum area of R_m is attained for that straight line m bounding S_l which is closer to o .*

Below we find the straight lines cutting off from P the subsets of the largest areas which remain in P after folding them along those straight lines. Observe that, if P is axially symmetric, then by Claim 1 they are the axes of symmetry of P . Hence in Lemma 3 we assume that P is not an axially symmetric parallelogram. Without loss of generality, we assume that in P we have $|bc| = 1$. For further considerations we need the Cartesian coordinate system Oxy in which $a = (p, q)$, $b = (0, 0)$, $c = (1, 0)$ and $d = (p + 1, q)$, where $p \in (0, 1)$ and $q \in (0, \sqrt{1 - p^2})$.

Lemma 3 *Let P be a parallelogram which is not axially symmetric and whose longer side has length 1. Then $\max_m \{\text{area}(R_m); T_m \subset P\} = \max\{\text{area}(R_{\text{bi}(\angle adv)}), \text{area}(R_{\text{bi}(\angle wba)})\} = \max \left\{ \frac{1}{2(1-p+\sqrt{1-q^2})}, \frac{\sqrt{p^2+q^2}}{2} \right\} \cdot \text{area}(P)$.*

Proof Let $l \in \mathcal{A} \cup \mathcal{B}$. Denote by φ the angle between the straight line l and the axis Ox . Having in mind the Corollary, we limit the consideration to a straight line $m \in S_l$ bounding S_l which is closer to o . Of course, the angle between m and Ox is also φ .

Case 1, when $\varphi \in [0, \alpha]$.

Subcase 1.1, when $\varphi \leq \frac{1}{2}\angle adv$.

Clearly, S_l is the strip between l and the parallel straight line m passing through d . Hence $R_m = aq_md$, where $q_m \in ab$. Observe that if φ increases from 0 to $\frac{1}{2}\angle adv$, then $|dq_m|$ and $\angle adq_m = \varphi$ are increasing and $|ad|$ is constant. Consequently, $\text{area}(R_m)$ increases and attains its greatest value for $m = \text{bi}(\angle adv)$.

Subcase 1.2, when $\frac{1}{2}\angle adv \leq \varphi \leq \frac{1}{2}\angle wba$.

Since $\frac{1}{2}\angle adv > 0$ and $\frac{1}{2}\angle wba = \frac{\alpha}{2}$, we conclude that $\varphi \in (0, \frac{\alpha}{2}]$. In this subcase S_l is the strip between l and the parallel straight line m which passes through the midpoint a'' of aa' . Observe that $R_m = aq_mp_m$ and $T_m = a_m p_m q_m$, where p_m and q_m are the points of intersection of m with ad and ab , respectively. Assume that the equation of m is $y = Ax + B$. From $a_m \in bc$ we obtain $a_m = (Aq + p, 0)$. Of course, $B = \frac{1}{2}(q - 2Ap - A^2q)$. Having in mind that $p_m \in ad$ and providing easy calculation, we obtain that $p_m = (\frac{q-B}{A}, q)$. Since $q_m \in ab$, we have $q_m = (\frac{B}{q/p-A}, q/p \frac{B}{q/p-A})$. We see that the area of R_m equals

$$f(A) = \frac{q^3}{8p} \frac{(A^2 + 1)^2}{(q/p - A)A}. \quad (1)$$

Clearly, $A = \frac{\sin \varphi}{\cos \varphi}$. Moreover, it is easy to show that $p = \frac{\cos \alpha \sin \beta}{\sin(\alpha + \beta)}$ and $q = \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)}$. Substituting these three equalities into (1) and providing elementary, but time-consuming calculation, we obtain

$$f(\varphi) = \frac{\sin^3 \alpha \sin^2 \beta}{\sin^2(\alpha + \beta)} \cdot \frac{1}{2 \sin \alpha \sin 2\varphi + \cos(\alpha - 4\varphi) - \cos \alpha}.$$

Consider the function $g(\varphi) = 2 \sin \alpha \sin 2\varphi + \cos(\alpha - 4\varphi) - \cos \alpha$. Since $g''(\varphi) = -8 \sin \alpha \sin 2\varphi - 16 \cos(\alpha - 4\varphi)$ is negative in the interval $(0, \frac{\alpha}{2}]$, we see that $g(\varphi)$ is concave in this interval. Hence the function $f(\varphi)$ is convex in the interval $(0, \frac{\alpha}{2}]$. This and $[\frac{1}{2}\angle adv, \frac{1}{2}\angle wba] \subset (0, \frac{\alpha}{2}]$ imply that $f(\varphi)$ attains the greatest value at one or both endpoints of the interval $[\frac{1}{2}\angle adv, \frac{1}{2}\angle wba]$. So the area of R_m may be maximal only for $m = \text{bi}(\angle adv)$, or only for $m = \text{bi}(\angle wba)$, or for both $m = \text{bi}(\angle adv)$ and $m = \text{bi}(\angle wba)$.

Subcase 1.3, when $\varphi \geq \frac{1}{2}\angle wba$.

Now S_l is the strip between l and the parallel straight line m passing through b . Of course, $R_m = abp_m$, where $p_m \in ad$. When φ increases from $\frac{1}{2}\angle wba$ to $\angle cba$, then $|bp_m|$ and $\angle p_mba = \alpha - \varphi$ decrease and $|ab|$ is constant. We conclude that $\text{area}(R_m)$ decreases.

Case 2, when $\varphi \in [\alpha, \pi]$.

Similarly like in Case 1 we show that the area of R_m attains its maximum for $m = \text{bi}(\angle bad)$.

From Cases 1 and 2 and next by $\text{area}(R_{\text{bi}(\angle wba)}) = \text{area}(R_{\text{bi}(\angle bad)})$ we obtain that

$$\begin{aligned}\max_m \text{area}(R_m) &= \max\{\text{area}(R_{\text{bi}(\angle adv)}), \text{area}(R_{\text{bi}(\angle wba)}), \text{area}(R_{\text{bi}(\angle bad)})\} \\ &= \max\{\text{area}(R_{\text{bi}(\angle adv)}), \text{area}(R_{\text{bi}(\angle wba)})\}.\end{aligned}$$

Below we calculate $\text{area}(R_{\text{bi}(\angle adv)})$. Having in mind that v is the intersection point of the segment bc with the circle of radius $|ad| = 1$ centered at d , straightforward calculation shows that $v = (p+1-\sqrt{1-q^2}, 0)$. Since $\text{bi}(\angle adv)$ contains the bisector of the angle $\angle adv$ we conclude that its slope equals $\frac{1-\sqrt{1-q^2}}{q}$. Taking $A = \frac{1-\sqrt{1-q^2}}{q}$ in (1) we have $f\left(\frac{1-\sqrt{1-q^2}}{q}\right) = \frac{q}{2(1-p+\sqrt{1-q^2})}$. Consequently, $\text{area}(R_{\text{bi}(\angle adv)}) = \frac{q}{2(1-p+\sqrt{1-q^2})}$.

Similar considerations like above show that $\text{area}(R_{\text{bi}(\angle wba)}) = \frac{q\sqrt{p^2+q^2}}{2}$.

From the preceding three paragraphs and from $\text{area}(P) = q$ we get $\max_m \text{area}(R_m) = \max\{\text{area}(R_{\text{bi}(\angle adv)}), \text{area}(R_{\text{bi}(\angle wba)})\} = \max\left\{\frac{1}{2(1-p+\sqrt{1-q^2})}, \frac{\sqrt{p^2+q^2}}{2}\right\} \cdot \text{area}(P)$. $\square$

By the definition of $\text{masf}(C)$ and from Lemma 3 we obtain the following proposition.

Proposition *For every parallelogram P which is not axially symmetric and whose longer side has length 1 we have $\text{masf}(P) = \max\left\{\frac{1}{1-p+\sqrt{1-q^2}}, \sqrt{p^2+q^2}\right\}$, where p and q are defined just like before Lemma 3.*

Theorem *For every parallelogram P we have $\text{masf}(P) > \frac{1}{2}$. In general, this inequality cannot be improved.*

Proof Since the measure of axial symmetry is invariant under homotheties, below we consider only parallelograms defined as before Lemma 3. Of course, our family does not contain axially symmetric parallelograms (from Claim 2 it follows that $\text{masf}(P) = 1$ for each of them, and thus Theorem holds true for them). It is easy to show that $\frac{1}{1-p+\sqrt{1-q^2}}$ and $\sqrt{p^2+q^2}$ decrease provided $q \in (0, \sqrt{1-p^2})$ decreases and p is fixed. Observe that $\lim_{q \rightarrow 0} \frac{1}{1-p+\sqrt{1-q^2}} = \frac{1}{2-p}$ and $\lim_{q \rightarrow 0} \sqrt{p^2+q^2} = p$. Moreover for every $p \in (0, 1)$ we have $\max\left\{\frac{1}{2-p}, p\right\} \geq \frac{1}{2-p} > \frac{1}{2}$. In other words, $\lim \text{masf}(P) = \frac{1}{2}$ provided $p \rightarrow 0$ and $q \rightarrow 0$.

Consider the family of parallelograms for which $p = \mu$, $q = \sqrt{2\mu - \mu^2}$ and $\mu \in \left(0, \frac{3-\sqrt{5}}{4}\right)$. Then $\text{masf}(P) = \max\left\{\frac{1}{2-2\mu}, \sqrt{2\mu}\right\}$. For every μ from $\left(0, \frac{3-\sqrt{5}}{4}\right)$ we have $\frac{1}{2-2\mu} > \sqrt{2\mu}$. So $\text{masf}(P) = \frac{1}{2-2\mu}$. If $\mu \rightarrow 0$, then $\text{masf}(P) \rightarrow \frac{1}{2}$. Hence this family of parallelograms shows that, in general, the inequality presented in the Theorem cannot be improved. $\square$

Remark In order to show the second part of the Theorem we may also take some other families of parallelograms for which p and q tend to 0, e.g., the family for which $p = q$.

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