Approximation of convex bodies by inscribed simplices of maximum volume

Marek Lassak

Received: 19 March 2010 / Published online: 3 May 2011

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Abstract The Banach-Mazur distance between an arbitrary convex body and a simplex in Euclidean n -space E^n is at most $n + 2$. We obtain this estimate as an immediate consequence of our theorem which says that for an arbitrary convex body C in E^n and for any simplex S of maximum volume contained in C the homothetical copy of S with ratio $n + 2$ and center in the barycenter of S contains C . In general, this ratio cannot be improved, as it follows from the example of any double-cone.

Keywords Approximation · Banach–Mazur distance · Convex body · Double-cone · Simplex · Volume

Mathematics Subject Classification (2000) 52A21 · 52A10 · 46B20

Denote by $\mathcal{C}^n$ the family of all convex bodies and by $\mathcal{M}^n$ the family of all centrally symmetric convex bodies in Euclidean n -dimensional space E^n .

Finding simplices of large volume in convex bodies has a long history. See, for instance, the papers by Blaschke (1917) on the maximum area of triangles in planar convex bodies, by MacKinney McKinney (1974) on simplices of maximum volume in centrally symmetric convex bodies, and the survey article Hudelson et al. (1996) by Hudelson, Klee and Larman on simplices of large volume in cubes.

Let A be an $(n - 1)$ -dimensional convex body and let $I = ab$ be a perpendicular to the hyperplane containing A non-degenerated segment centered at a point of A . We call the convex hull $\text{conv}(A \cup I)$ the *double pyramid with base A and apices a and b* . If B is a ball of E^{n-1} in place of A and if the centers of B and I coincide, the double-pyramid is called a *double-cone*.

M. Lassak (✉)

Institute of Mathematics and Physics, University of Technology, 85-789 Bydgoszcz, Poland
e-mail: lassak@utp.edu.pl

Theorem For arbitrary $C \in \mathcal{C}^n$ and for any simplex S of maximum volume contained in C we have $C \subset (n + 2)S$. For $n \geq 2$ the ratio $n + 2$ is best possible and, in particular, it cannot be lessened for every double-cone in part of C .

For the proof of the Theorem we need Lemmas 1 and 2 presented below. We give the proof of the Theorem after them.

The first lemma is very easy to show, see the proof of Proposition in Lassak (2006).

Lemma 1 Let $C \in \mathcal{C}^n$, and let S be a simplex of maximum volume contained in C . Then S is inscribed in C and for every vertex v of S the hyperplane through v parallel to the opposite facet of S supports C .

This lemma immediately implies the well known property that *the only simplices of maximum volume contained in a ball are the regular inscribed simplices*. We will need this fact in the proof of Lemma 2.

Lemma 2 A simplex S contained in a double-cone $D \subset E^n$, where $n \geq 3$, with base B is of maximum possible volume if and only if a vertex of S is at an apex of D and the remaining vertices of S are vertices of a regular $(n - 1)$ -dimensional simplex inscribed in B . For $n = 2$ only the “if” part holds true.

Proof By Lemma 1 every simplex of maximum volume contained in D is inscribed in D .

I. Arbitrary simplex P inscribed in D with two vertices at two apices of D is not a simplex of maximum volume in D .

Let v_0 be a vertex of P at an apex of D , and denote the remaining vertices of P by $v_1, \dots, v_n$ (see Fig. 1). Denote by G the hyperplane containing them. Support D by the hyperplane H parallel to G such that S is contained in the strip between G and H . Since a vertex from amongst $v_1, \dots, v_n$ is the other apex of D , we see that H supports D at a point w of the sphere bounding B and that this is the only point of support. Consequently, the distance of w from G is greater than the distance of v_0 from G . Thus the simplex with vertices $v_1, \dots, v_n$ and w has volume greater than the volume of P . Hence P is not a simplex of maximum volume in D .

II. Consider all simplices with a vertex at an apex of D and with the remaining vertices in the boundary of B . A simplex from this family has the greatest volume amongst simplices from this family if and only if its vertices lying in the boundary of B are vertices of an $(n - 1)$ -dimensional regular simplex.

This follows from the $(n - 1)$ -dimensional version of the property presented immediately after Lemma 1.

III. Every simplex inscribed in D which is not of the form considered in Parts I and II is not a simplex of maximum volume contained in D .

Assume the opposite: there exists a simplex Q_1 not of the form considered in Parts I and II, which is a simplex of maximum volume contained in D .

From the assumptions of Parts I and II we conclude that Q_1 has a vertex x_1 which is not in B and not in an apex of D (see Fig. 2). Clearly, x_1 belongs to a segment ab_1 , where a is an apex of D and b_1 is a boundary point of B . There is a unique hyperplane V_1 supporting D at x_1 . By this uniqueness and by the second statement of Lemma 1,

Fig. 1 Illustration to Part I of the proof of Lemma 2

the facet F_1 of S opposite to x_1 is parallel to V_1 . Denote by H_1 the carrying hyperplane of F_1 . Since $x_1, b_1 \in V_1$, the distances from x_1 and b_1 to H_1 are equal. Hence the simplex Q_2 obtained by exchanging x_1 into b_1 has the same volume as Q_1 . If there exists a vertex x_2 of Q_2 which is not in B and not at an apex of D , by analogous procedure we can exchange it by a point b_2 from the boundary of B obtaining a new simplex Q_3 of equal volume. Then we repeat this procedure, as long as it is possible. At the last k -th stage, from a simplex Q_{k-1} we get a simplex Q_k with n vertices in the boundary of B . Clearly, one vertex of Q_k must be at an apex of D since in the opposite case Q_k would not be of maximum volume.

The maximality of the volume of Q_k together with Part II imply that the convex hull of its n vertices which are in the boundary of B is a regular $(n-1)$ -dimensional simplex inscribed in B . This determines the shape of Q_k .

Our procedure implies that the volumes of Q_{k-1} and Q_k are equal. On the other hand this is not true since from the established above shape of Q_k we see that its vertex b_{k-1} is farther than the point x_{k-1} from the hyperplane H_{k-1} containing the vertices of Q_k different from b_{k-1} (the shape of Q_k shows that the parallel hyperplane to H_{k-1} through b_{k-1} supports D only at this point). A contradiction.

IV. From Parts I–III we conclude the thesis of Lemma 2 for $n \geq 3$. An easy proof of the thesis for $n = 2$ is left to the reader as an exercise. $\square$

The homothetical image of a simplex S with ratio $\lambda > 0$ and the center in the barycenter of S is denoted by λS .

Fig. 2 Illustration to Part III of the proof of Lemma 2

Proof of the Theorem. Denote by $v_0, \dots, v_n$ the vertices and by c the barycenter of S . Let P_i be the hyperplane containing all the vertices of S different from v_i . We provide the parallel hyperplane O_i through v_i and the parallel hyperplane R_i such that P_i is in equal distances from O_i and R_i . Thus, if a point p of C is on the opposite side of R_i than S , the simplex S_i with one vertex at p and with the remaining vertices at the vertices of S different from v_i has volume greater than the volume of S . This is impossible since $S_i \subset C$ and since S is a simplex of maximum volume in C . Consequently, every point of C is in the half-space T_i bounded by R_i which contains S . So C is contained in the simplex being the intersection of the half-spaces $T_0, \dots, T_n$. Since c is $n + 2$ times farther from R_i than from P_i , this simplex is nothing else but $(n + 2)S$. We see that $C \subset (n + 2)S$.

Now we show that the ratio $n + 2$ cannot be diminished for every double-cone D . Let us stay with the notation of the first part of this proof. If $n \geq 3$, the position of every simplex S of maximum volume in D is described in Lemma 2, and for $n = 2$ take a triangle S being the convex hull of any three vertices of the parallelogram D . So one vertex of S , say v_0 , is an apex of D , and the convex hull of the remaining n vertices is a regular $(n - 1)$ -dimensional simplex inscribed in the base B of D . Of course, $B \subset P_0$. Since the other apex of D is in the hyperplane R_0 , the ratio $n + 2$ cannot be lessened. $\square$

Denote by D_s the double-pyramid whose base is a regular $(n - 1)$ -dimensional simplex inscribed in B and whose apices are the apices of D . *The estimate $n + 2$ in the*

Theorem cannot be lessened also for every convex body K such that $D_S \subset K \subset D$. This is an obvious consequence of the second statement of this theorem.

The second aim of this short note is the below presented Corollary on the Banach-Mazur distance between an arbitrary convex body and a simplex.

Recall that the Banach-Mazur distance of bodies $C_1, C_2 \in \mathcal{C}^n$ is the number

$$\delta(C_1, C_2) = \inf_{t, h_\lambda} \{ \lambda; \ t(C_2) \subset C_1 \subset h_\lambda(t(C_2)) \},$$

where t stands for an affine transformation, and h_λ stands for any homothety with a positive ratio λ . For more than 70 years this notion with $C_1, C_2 \in \mathcal{M}^n$ plays an important role in functional analysis. A survey of results is given by Tomczak-Jaegermann [Tomczak-Jaegermann \(1989\)](#). In the last 20 years also the general situation, when the convex bodies C_1 and C_2 are not necessarily centrally symmetric, is considered in many papers. For a larger context see the survey article [Szarek \(2006\)](#) by Szarek. From the celebrated theorem of John [John \(1948\)](#) that $\delta(C, B) \leq n$ for every convex body $C \in \mathcal{C}^n$ and the ball $B \subset E^n$ we immediately conclude that $\delta(C_1, C_2) \leq n^2$ for arbitrary $C_1, C_2 \in \mathcal{C}^n$. For instance, Rudelson [Rudelson \(2000\)](#) proved that there exists a constant k such that $\delta(C_1, C_2) \leq k \cdot n^{4/3} \log^9 n$ for every positive integer n and every $C_1, C_2 \in \mathcal{C}^n$, but this gives no concrete information about the upper estimate on Banach-Mazur distance of arbitrary convex bodies of a fixed dimension n .

From the first statement of the Theorem we obtain the following Corollary.

Corollary *For an arbitrary $C \in \mathcal{C}^2$ and a simplex S we have*

$$\delta(C, S) \leq n + 2.$$

Recall that the author presented a similar proof of this estimate during his talk for the Sixth International Conference on Geometry in Nasholim (Israel) in 1991. This estimate is presented without a proof in the abstract [Lassak \(1992\)](#) of this conference.

For $n = 2$ the successive upper estimates 2.5, 2.34 and 2.25 on $\delta(C, S)$ were presented in [Lassak \(1992\)](#); [Bálint et al. \(1993\)](#) and [Fleicher et al. \(1992\)](#). We conjecture that $\delta(C, S) \leq 1 + \frac{1}{2}\sqrt{5}$ (≈ 2.118) for every $C \in \mathcal{C}^2$ with the equality only for the regular pentagon P . For a position of $t(S)$ with respect to P such that $t(S) \subset P \subset (1 + \frac{1}{2}\sqrt{5})t(S)$ see Fig. 3 in [Lassak \(1992\)](#). Novotný [Novotný \(1994\)](#) proved that for every $C \in \mathcal{C}^3$ we have $\delta(C, S) \leq \frac{13}{3}$.

For every $C \in \mathcal{M}^n$ we have $\delta(C, S) = n$ (see [Grünbaum 1963](#)). This and the Theorem imply the following property: *if S is a simplex of maximum volume contained in $C \in \mathcal{M}^n$, then the smallest m such that $C \subset mS$ fulfills the inequalities $n \leq m \leq n + 2$.*

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