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Geometric characterization of interpolation in the space of Fourier–Laplace transforms of ultradistributions of Roumieu type

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Abstract We give a geometric characterization of interpolating varieties for the algebra of Fourier–Laplace transforms of ultradistributions of Roumieu type with compact support on the real line in the non-quasianalytic case. In particular, such a characterization is found in the case of classical Gevrey classes. We also provide a relation between interpolating varieties in this case and in the case of Hörmander algebras related to ultradistributions of Beurling type.

Keywords Interpolation of holomorphic functions · Interpolating variety · Ultradifferentiable functions of Roumieu type · Ultradistributions of Roumieu type · Entire functions · Fourier–Laplace transform

Mathematics Subject Classification (2000) Primary 30E05; Secondary 46F05 · 42A85

1 Introduction

Let $\mathscr{E}_{\{\omega\}}(\mathbb{R})$ be the space of ultradifferentiable functions of Roumieu type and $\mathscr{E}'_{\{\omega\}}(\mathbb{R})$ be its dual, the space of ultradistributions of Roumieu type with compact support (see for instance [11,12]). The best known and widely applicable examples are the Gevrey classes for $\omega(t) = t^{1/d}, d \geq 1$ (see [26]). These classes cover also most of the Denjoy-Carleman classes in the sense of Komatsu [16] (see [8, Theorem 14]). This type of ultradifferentiable functions and ultradistributions plays a crucial role in the study of partial differential operators (see for instance [10,13,14,17,22,24,26,27] etc.). By the Paley-Wiener type theorems $\hat{\mathscr{E}}'_{\{\omega\}}(\mathbb{R})$ is isomorphic via Fourier–Laplace transform to the space $A_{\{\omega\}}$ of entire functions

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with prescribed growth determined by some weight function ω (see [12,27] for more on this relationship and Sect. 2 below for precise definitions). The structure of these spaces was considered, for instance, in [6,7].

A multiplicity variety $\{(\lambda, m_\lambda)\}$, where λ runs through a discrete set $\Lambda \subset \mathbb{C}$, m_λ are positive integers, is called an interpolating variety for $A_{\{\omega\}}$ if for any doubly indexed sequence of complex numbers $\{v_{\lambda,l}\}$ with suitable growth there exists $f \in A_{\{\omega\}}$ satisfying

$$\frac{f^{(l)}(\lambda)}{l!} = v_{\lambda,l}$$

for any $\lambda \in \Lambda$ and $0 \le l < m_\lambda$.

The problem of interpolation, i.e., characterization of interpolating varieties, has been thoroughly studied for Hörmander algebras A_p and a complete description of interpolating varieties in case of radial p has been obtained by Berenstein and Li [4]. For radial weights Berenstein, Li and Vidras [5] described interpolating varieties for entire functions of minimal type A_p^0. Compare also [25] and references therein.

A special role is played by non-radial weights $p(z) = |\mathrm{Im}\, z| + \omega(|z|)$. In this case $A_p =: A_{(\omega)}$ is the Fourier–Laplace image of the space of ultradistributions of Beurling type with compact support $\hat{\mathscr{E}}'_{(\omega)}(\mathbb{R})$ (see [12]). The space of standard distributions with compact support belongs to this class of spaces. A geometric characterization of interpolating varieties for $\hat{\mathscr{E}}'_{(\omega)}(\mathbb{R})$ was given in [18] (for earlier partial results see [28,29]).

We give a geometric description of interpolating varieties for $A_{\{\omega\}}$, where ω is a non-quasianalytic weight, in terms of geometric properties of the varieties and we find a relation between being an interpolating variety for $A_{\{\omega\}}$ and being an interpolating variety for some Hörmander algebra $A_{(\omega)}$.

Ultradifferentiable functions and the interpolation problem appear naturally in the study of convolution operators. On the side of Fourier–Laplace transforms a convolution operator T_μ becomes just a multiplication operator M_μ which multiplies by an entire function $\hat{\mu}$. The image of this operator consists of functions in $A_{\{\omega\}}$ vanishing at zeros of $\hat{\mu}$ (in some cases all such functions). If the set of zeros is an interpolating variety then $A_{\{\omega\}}/\,\mathrm{Im}\, M_{\hat{\mu}}$ can be identified with the space of sequences with suitable growth, which is in turn isomorphic to the dual of the kernel of T_μ. This idea lies behind results in [9,15,17,20,24], where sequential descriptions of kernels of convolution operators are given, surjectivity characterizations of convolution operators are obtained or existence of right inverses for convolution operators is established. This methodology was developed for instance in [19] (comp. [21]) and it is connected with the study of ideals in Hörmander algebras.

For two functions f, g given on some set Ω we use the notation $f \lesssim g$ if there exists some constant $C > 0$ such that $f \le Cg$ on Ω. If $g \lesssim f \lesssim g$ we write $f \simeq g$. We also use the standard "big O" and "small o" notation always using it with the argument tending to infinity. By $\mathbb{H}_+$ we denote the upper half-plane and by $\mathbb{H}_-$ the lower half-plane of $\mathbb{C}$. For non-explained notions from functional analysis we refer to [23], for complex analysis see [2,3].

2 Definitions and results

We begin with the standard definition of a weight function (see [12]).

Definition 2.1 (Weight function) We consider the following conditions for a function $\omega\colon [0,\infty) \to [0,\infty)$:

$$(\alpha)\quad \omega(2t) = O(\omega(t)), \qquad (\delta)\quad \varphi\colon t \to \omega(e^t) \text{ is convex},$$

$$(\beta)\quad \int_0^\infty \frac{\omega(t)}{1+t^2}dt < \infty, \qquad (\varepsilon)\quad \omega(t) = O(t),$$

$$(\gamma)\quad \ln t = o(\omega(t)).$$

We call ω a weight function if it is continuous, increasing and satisfies (α), (γ), (δ) and (ε). We say that a weight is non-quasianalytic if it additionally satisfies (β) and quasianalytic otherwise.

Lemma 2.2 *Every weight ω satisfies the following condition:*

$$\exists\, C > 0\, \forall\, z, \xi \in \mathbb{C}, r > 0\colon |z - \xi| \le r \Rightarrow \omega(|z|) \le C(\omega(|\xi|) + r + 1). \tag{2.1}$$

Proof Using [12, Lemma 1.2]

$$\omega(|z|) \le \omega(|\xi| + r) \lesssim 1 + \omega(|\xi|) + \omega(r) \le 1 + \omega(|\xi|) + Ar + B$$

where in the last inequality we used the continuity of ω and condition (ε). □

Throughout the rest of the paper X always denotes a multiplicity variety $\{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ where Λ is a discrete subset of $\mathbb{C}$, m_λ are positive integers. We always consider non-quasianalytic weights ω. We extend ω on $\mathbb{C}$ by the formula $\omega(z) = \omega(|z|)$. It is well known that such a function is subharmonic provided ω is increasing and satisfies (δ) (see [2, Lemma 4.4.18]).

We consider the usual Hörmander algebras. For $p\colon \mathbb{C} \to [0, \infty)$ we define

$$A_p = \left\{ f \in H(\mathbb{C}) |\, \exists\, C > 0 : \sup_{z\in\mathbb{C}} |f(z)| e^{-Cp(z)} < \infty \right\}.$$

Next we introduce the algebras related with the spaces of ultradistributions of Roumieu type.

Definition 2.3 Define

$$\|f\|_{M,m} = \sup_{z\in\mathbb{C}} |f(z)| e^{-M|\mathrm{Im}\, z| - \frac{1}{m}\omega(z)}$$

and

$$A_{\{\omega\}} := \left\{ f \in H(\mathbb{C}) |\, \exists\, M > 0\ \forall\, m > 0\colon \|f\|_{M,m} < \infty \right\}.$$

In order to make the notation more uniform we will write $A_{(\omega)} = A_{|\mathrm{Im}\, z| + \omega(z)}$. The brackets in subscript reflect the origin of these spaces. The spaces $A_{(\omega)}$, $A_{\{\omega\}}$ are equal to the Fourier–Laplace transform image of the spaces of ultradistributions with compact support of Beurling $\mathscr{E}'_{(\omega)}(\mathbb{R})$ and Roumieu $\mathscr{E}'_{\{\omega\}}(\mathbb{R})$ type (see [12, Definition 4.1] for precise definitions and [12, Theorem 7.4], [27, Satz 2.19] for proofs of the equalities), respectively. Moreover, the Fourier–Laplace transform is also a topological isomorphism of duals of Denjoy-Carleman classes $\mathscr{E}'_{(M_p)}(\mathbb{R})$ and $\mathscr{E}'_{\{M_p\}}(\mathbb{R})$ (see definitions in [16]) onto $A_{(\omega)}$ and $A_{\{\omega\}}$ where $\omega(t) = \sup_{p\in\mathbb{N}} \ln \frac{t^p}{M_p}$ (see [8]), provided certain assumptions are fulfilled.

By (α) the space $A_{\{\omega\}}$ is closed under differentiation and by (γ) it contains all polynomials. The natural topology on $A_{\{\omega\}}$ is an (LF)-topology, i.e.,

$$A_{\{\omega\}} = \operatorname*{ind}_{M\in\mathbb{N}} A_{\{\omega\},M}, \quad A_{\{\omega\},M} = \left\{ f \in H(\mathbb{C}) |\, \forall\, m \in \mathbb{N}\colon \|f\|_{M,m} < \infty \right\}$$

where the latter space is equipped with the Fréchet space topology given by the system of seminorms $(\|\cdot\|_{M,m})_{m\in\mathbb{N}}$.

Definition 2.4 (Interpolating Variety) We call X an interpolating variety for $A_{\{\omega\}}$ if for every sequence of values $\{v_{\lambda,l}|\ \lambda \in \Lambda,\ 0 \le l < m_\lambda\}$ satisfying

$$\exists\, M \in \mathbb{N} \quad \forall\, m \in \mathbb{N}\colon \|v\|_{M,m} = \sup_{\lambda\in\Lambda} \sum_{l=0}^{m_\lambda-1} |v_{\lambda,l}| e^{-M|\operatorname{Im}\lambda| - \frac{1}{m}\omega(\lambda)} < \infty. \tag{2.2}$$

there exists $f \in A_{\{\omega\}}$ with

$$\frac{f^{(l)}(\lambda)}{l!} = v_{\lambda,l}, \qquad \lambda \in \Lambda, \quad 0 \le l < m_\lambda.$$

Condition (2.2) is necessary for such a function to exist since, by an application of Cauchy inequalities, every function $f \in A_{\{\omega\}}$ satisfies

$$\exists\, M \in \mathbb{N} \quad \forall\, m \in \mathbb{N}\colon \sup_{z\in\mathbb{C}} \sum_{l=0}^{m_\lambda-1} \left| \frac{f^{(l)}(z)}{l!} \right| e^{-M|\operatorname{Im} z| - \frac{1}{m}\omega(z)} < \infty.$$

Definition 2.5 We denote by $A_{\{\omega\}}(X)$ the space of all sequences satisfying condition (2.2) equipped with its natural (LF)-topology.

Definition 2.6 (Restriction operator)

$$R\colon A_{\{\omega\}} \to A_{\{\omega\}}(X), \qquad f \mapsto \left\{ \frac{f^{(l)}(\lambda)}{l!} \right\}_{\lambda\in\Lambda,\ 0\le l<m_\lambda}.$$

Using this notion we can say that X is interpolating for $A_{\{\omega\}}$ if, and only if, the operator R is surjective. An application of Cauchy inequalities and condition (2.1) shows that R is continuous.

To express geometric properties of interpolating varieties we use Nevanlinna counting functions

$$n(z, r, X) = \sum_{\lambda\in\overline{D}(z,r)\cap\Lambda} m_\lambda$$

and

$$N(z, r, X) = \int_0^r \frac{n(z, t, X) - n(z, 0, X)}{t}\, dt + n(z, 0, X) \ln r.$$

The argument X will be omitted whenever it leads to no misunderstanding.

The geometric conditions already considered (comp. [18]) as well as some new ones are presented in the following definition.

Definition 2.7 Let $p\colon \mathbb{C} \to [0, \infty)$. We introduce the following conditions on X:

(a) There exists $C > 0$ such that

$$N(\lambda, p(\lambda)) \le C p(\lambda)$$

for all $\lambda \in \Lambda$.

(b)

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\operatorname{Im}\lambda| > \omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty.$$

(A) $\exists M \in \mathbb{N} \forall m \in \mathbb{N} \exists C_m \in \mathbb{R} \forall r > 0, \quad \lambda \in \Lambda$:

$$N(\lambda, r) \leq C_m + M|\mathrm{Im}\,\lambda| + \frac{1}{m}\omega(\lambda) + Mr,$$

(B) $\exists M \in \mathbb{N} \forall m \in \mathbb{N} \exists C_m \in \mathbb{R} \forall U \subset \mathbb{H}_+, \quad \mathbb{H}_- \forall x \in \mathbb{R}$:

$$\sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x-\lambda|^2} \leq \frac{C_m + M|\mathrm{Im}\,\lambda_x| + \frac{1}{m}\omega(\lambda_x)}{|\mathrm{Im}\,\lambda_x|}$$

where $\lambda_x \in U \cap \Lambda$ is such that $d(x, \lambda_x) = d(x, U \cap \Lambda)$.

The main theorem of the paper is stated as follows.

Theorem 2.8 *The following conditions are equivalent:*

(i) *X is interpolating for $A_{\{\omega\}}$,*
(ii) *there exists a weight $\sigma = o(\omega)$ such that X is interpolating for $A_{(\sigma)}$,*
(iii) *there exists a weight $\sigma = o(\omega)$ such that X satisfies (a) for $p(z) = |\mathrm{Im}\,z| + \sigma(z)$ and (b) for σ,*
(iv) *X satisfies (A) and (B).*

Remark 2.9 The result reflects the fact that $\mathscr{E}_{\{\omega\}}(\mathbb{R}) = \bigcap \mathscr{E}_{(\sigma)}(\mathbb{R})$ where the intersection is taken over all $\sigma = o(\omega)$ (see [1, Corollary 4.6]) and is similar to some results relating properties of $\mathscr{E}_{\{\omega\}}$ with those of $\mathscr{E}_{(\sigma)}$ (see [22, Corollary 3.12]).

Corollary 2.10 *A variety X is interpolating for the Fourier–Laplace transform image of the Gevrey ultradistributions with compact support $\hat{G}'_d(\mathbb{R})$, $d > 1$ if, and only if, X satisfies (A) and (B) with $\omega(t) = t^{1/d}$. Note that*

$$G_d(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) | \; \forall M \in \mathbb{N} \exists m > 0 : \sup_{x \in [-M,M]} \sup_{l \in \mathbb{N}_0} \frac{|f^{(l)}(x)|}{m^l (l!)^d} < \infty \right\}.$$

Example 2.11 It is obvious that (B) does not imply (A) since it does not say anything about the behaviour of X on the real line.

(1) $X = \{(in, 1)\}_{n \in \mathbb{N}}$ shows that the converse does not hold as well.
(2) $X = \{(n, 1)\}_{n \in \mathbb{N}}$ satisfies (A) and (B) hence it shows that rotating a variety which is not interpolating may change it into an interpolating variety and vice versa.
(3) $X = \{(n + ia_n, 1), (n + e^{-a_n - \sigma(n)} + ia_n, 1)\}_{n \in \mathbb{Z}}$ where (a_n) is a sequence of positive reals and $\sigma = o(\omega)$ shows that $|\mathrm{Im}\,\lambda|$ and $\omega(\lambda)$ cannot be omitted in (A). Condition (B) is satisfied whenever (a_n) increases slower than $\omega(n)$.
(4) $X = \{(2^n + i, m_n)\}_{n \in \mathbb{Z}}$ with multiplicities (m_n) growing sufficiently fast shows that the series in (B) is not always uniformly bounded over all $x \in \mathbb{R}$. Condition (A) is satisfied provided (m_n) is chosen such that $n(\lambda, r)$ grows at most linearly with r.

Mixing examples (3) and (4) we obtain an interpolating variety using $|\mathrm{Im}\,\lambda|$ and $\omega(\lambda)$ in the estimate in (A) and with not uniformly bounded series in (B). These examples with different σ and (m_n) also distinguish the sets of interpolating varieties for different spaces $A_{\{\omega\}}$.

From Theorem 2.8 (iv) the following corollary is easily seen.

Corollary 2.12 *The property of being an interpolating variety is monotone with respect to the weight, i.e., if X is interpolating for $A_{\{\sigma\}}$ then it is interpolating for any $A_{\{\omega\}}$ with $\sigma = O(\omega)$.*

Remark 2.13 As an immediate consequence of [4, Corollary 3.5] we see that similar property holds for spaces $A_{(\omega)}$. For weights ω, σ with $\sigma = O(\omega)$, if X is an interpolating variety for $A_{(\sigma)}$ then it is interpolating for $A_{(\omega)}$. As a special case it holds for $\sigma(t) = \ln(1+t^2)$, which corresponds to $\hat{\mathscr{E}}'(\mathbb{R})$, the space of distributions with compact support on the real line.

It is worth emphasizing that Corollary 2.12 is highly non-trivial since for the proof it requires the obtained geometric characterization of interpolating varieties.

3 Proof of the main result

To prove the implication (iv) $\Rightarrow$ (iii) we fix constants C_m in conditions (A) and (B) and then use [12, Lemma 1.7] for $g(t) = \inf_{m\in\mathbb{N}}(C_m + \frac{1}{m}\omega(t))$ and ω obtaining a weight σ with $g = o(\sigma)$ and $\sigma = o(\omega)$. Putting $r = |\text{Im}\,\lambda| + \sigma(\lambda)$ in (A) we obtain (a) for $p(z) = |\text{Im}\,z| + \sigma(z)$. Using (B) for $U = \{z \in \mathbb{C} \colon |\text{Im}\,z| > \sigma(z)\}$ and the discreteness of X we obtain (b) for σ.

The implication (iii) $\Rightarrow$ (ii) is just one part of the main theorem of [18] where the weight is assumed to be subadditive. Careful investigation of the proof shows that it is enough for the weight to satisfy $\omega(x+y) \lesssim \omega(x) + \omega(y) + 1$.

3.1 The proof of (ii) $\Rightarrow$ (i)

Lemma 3.1 *In the algebraic sense*

$$A_{\{\omega\}} = \bigcup_{\sigma = o(\omega)} A_{(\sigma)} \quad \textit{and} \quad A_{\{\omega\}}(X) = \bigcup_{\sigma = o(\omega)} A_{(\sigma)}(X).$$

Proof Let $f \in A_{\{\omega\}}$. Then $|f(z)| \le e^{C_m + M|\text{Im}\,z| + \frac{1}{m}\omega(z)}$. Using [12, Lemma 1.7] for $g(t) = \inf_{m\in\mathbb{N}}(C_m + \frac{1}{m}\omega(t))$ and ω we obtain a weight $\sigma = o(\omega)$ such that $f \in A_{(\sigma)}$. The converse inclusion follows from the observation that if $\sigma = o(\omega)$ then for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sigma(t) \le C_m + \frac{1}{m}\omega(t)$ for all $t \in [0, \infty)$. The proof of the second equality is identical. □

Suppose that X is $A_{(\nu)}$ interpolating for some $\nu = o(\omega)$ and let $v \in A_{\{\omega\}}(X)$. By the previous lemma $v \in A_{(\mu)}(X)$ for some $\mu = o(\omega)$. Since $\max(\nu, \mu) = o(\omega)$ we can, once again using [12, Lemma 1.7], obtain another weight σ between $\max(\nu, \mu)$ and ω. Then by Remark 2.13, X is $A_{(\sigma)}$ interpolating. Moreover, $v \in A_{(\sigma)}(X)$ hence we obtain $f \in A_{(\sigma)}$ satisfying $R(f) = v$. One more application of Lemma 3.1 completes the proof.

3.2 The proof of (i) $\Rightarrow$ (iv)

Throughout this section we assume that X is $A_{\{\omega\}}$ interpolating. In the proof of (A) we follow the lines of reasoning presented in [29] and in the proof of (B) we use ideas of [18].

A standard feature of the A_p spaces is that interpolation can be done in a uniform way (see [2, Lemma 2.2.6]). Although the natural topology of $A_{\{\omega\}}$ as (LF)-space is more complex, uniform interpolation remains feasible.

Lemma 3.2 (Uniform interpolation) *If R is surjective then*

$$\forall N \in \mathbb{N}\, \exists M \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists n \in \mathbb{N}, \quad r > 0:$$
$$B_{N,n}(0,1) \subset R(B_{M,m}(0,r))$$

where $B_{M,m}(z,r)$ denotes a ball in the norm $\|\cdot\|_{M,m}$ in suitable space centered at z with radius r.

Proof Denote $E = A_{\{\omega\}}$, $F = A_{\{\omega\}}(X)$ and for fixed $M, \quad N \in \mathbb{N}$

$$E_M = \left\{f \in H(\mathbb{C}) \mid \forall\, m \in \mathbb{N}\colon \|f\|_{M,m} < \infty\right\},$$
$$F_N = \left\{v \in \mathbb{C}^{\mathbb{N}} \mid \forall\, n \in \mathbb{N}\colon \|v\|_{N,n} < \infty\right\}.$$

As R is surjective we have $F = \bigcup_{M\in\mathbb{N}} R(E_M)$. By Grothendieck's factorization theorem for every $N \in \mathbb{N}$ there exist $M \in \mathbb{N}$ such that $F_N \subset R(E_M)$. Denote by j the inclusion from F_N to $R(E_M)$. Consider on $R(E_M)$ the topology τ induced from E_M by R (the quotient topology). Then $R_M = j_M \circ R\colon E_M \to (R(E_M), \tau)$, where $j_M\colon E_M \to E$ is the inclusion, and $j\colon F_N \to (R(E_M), \tau)$ are continuous by the closed graph theorem. Hence $j^{-1}(R(B_{M,m}(0,1)))$, where $B_{M,m}(0,1) = \{f \in H(\mathbb{C})\colon \|f\|_{M,m} < 1\}$, is a 0-neighbourhood in F_N, which completes the proof. □

As a corollary we derive a standard form of uniform interpolation.

Lemma 3.3 *If X is an interpolating variety for $A_{\{\omega\}}$ then there exists $M > 0$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ such that for all $\lambda \in \Lambda$, $0 \le l < m_\lambda$ there is a function $f_{\lambda,l} \in A_{\{\omega\}}(\mathbb{C})$ satisfying*

$$\frac{f^{(k)}_{\lambda,l}(\eta)}{k!} = \begin{cases} 1 & \text{if } \eta = \lambda \text{ and } k = l \\ 0 & \text{otherwise} \end{cases}$$

for all $\eta \in \Lambda$, $0 \le k < m_\eta$ and

$$\|f_{\lambda,l}\|_{M,m} \le C_m.$$

We are now ready to prove that X satisfies (A). Let $f_\lambda := f_{\lambda,m_\lambda-1}$ be the functions given by Lemma 3.3. For every $\lambda \in \Lambda$ define

$$g_\lambda(z) = \frac{f_\lambda(z)}{(z-\lambda)^{m_\lambda-1}}$$

By Jensen's formula for g_λ in the disc $D(\lambda, r)$ we get

$$\sum_{i=1}^{N} \ln\left(\frac{r}{|\alpha_i - \lambda|}\right) + m_\lambda \ln r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})|d\theta + \ln r. \tag{3.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are the zeros of g_λ which lie in $D(\lambda, r)$ counted according to their multiplicities. But every point λ' of Λ not equal λ contained in $D(\lambda, r)$ is a zero of g_λ with multiplicity at least $m_{\lambda'}$ thus

$$\sum_{i=1}^{N} \ln\left(\frac{r}{|\alpha_i - \lambda|}\right) + m_\lambda \ln r = \int_0^r \frac{n(\lambda, t, Z(g_\lambda))}{t}\,dt + m_\lambda \ln r$$
$$\ge \int_0^r \frac{n(\lambda, t, X) - n(\lambda, 0, X)}{t}\,dt + m_\lambda \ln r.$$

From (3.1) we get then

$$N(\lambda, r, X) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})| d\theta + \ln r.$$

Since by the assumption on the norms of f_λ there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\|f_\lambda\|_{M,m} \leq C_m$, hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|C_m e^{M|\mathrm{Im}(\lambda + re^{i\theta})| + \frac{1}{m}\omega(\lambda + re^{i\theta})}| d\theta$$

$$\leq \ln C_m + \sup_{z \in \partial D(\lambda, r)} \left(M|\mathrm{Im}\, z| + \frac{1}{m}\omega(z) \right).$$

Therefore using condition (2.1) we get

$$N(\lambda, r, X) \leq \ln C_m + C + M|\mathrm{Im}\, \lambda| + \tfrac{C}{m}\omega(\lambda) + \ln r + Mr + \tfrac{C}{m} r$$

for all $\lambda \in \Lambda$ and $r > 0$. This completes the proof of condition (A).

To prove condition (B) we need a standard estimate for the multiplicities of X (compare [29]). Substituting $r = e$ in condition (A) we obtain the following.

Lemma 3.4 *If X is an interpolating variety for $A_{\{\omega\}}$ then there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ such that*

$$m_\lambda \leq C_m + M|\mathrm{Im}\, \lambda| + \frac{1}{m}\omega(\lambda).$$

for all $\lambda \in \Lambda$.

A very important part of the proof of (B) is the construction of a non-zero holomorphic function with a prescribed growth. To shorten the notation instead of $\mathbb{H}_+$ or $\mathbb{H}_-$ we write $\mathbb{H}_*$.

Lemma 3.5 *There exist a holomorphic function $H: \mathbb{H}_* \to \mathbb{C}$ with $H(z) \neq 0$ for all $z \in \mathbb{H}_*$ and such that*

$$\frac{1}{4}\omega(|z|) \leq \ln|H(z)| \leq C(\omega(|z|) + |\mathrm{Im}\, z| + 1)$$

for some $C > 0$ and all $z \in \mathbb{H}_*$.

Proof By [12, Lemma 2.2] the harmonic extension $P_\omega: \mathbb{H}_* \to [0, \infty)$ of $\omega(|t|)$ given by the formula

$$P_\omega(x + iy) = \frac{|y|}{\pi} \int_{\mathbb{R}} \frac{\omega(|t|)}{(x - t)^2 + y^2}\, dt$$

satisfies

(1) $P_\omega(x + iy) \geq \frac{1}{4}\omega(|x + iy|)$ for all $x + iy \in \mathbb{H}_*$,
(2) there exists $C > 0$ such that $P_\omega(x + iy) \leq C(\omega(x) + |y| + 1)$ for all $x + iy \in \mathbb{H}_*$.

Define $H(z) = e^{P_\omega(z) + i\tilde{P}_\omega(z)}$ where $\tilde{P}_\omega$ is a harmonic conjugate of P_ω. Then H is holomorphic and $H(z) \neq 0$ for all $z \in \mathbb{H}_*$. Since $\ln|H(z)| = P_\omega(z)$ this completes the proof. □

Having this function we are able to prove the following estimate.

Lemma 3.6 *If X is an interpolating variety for $A_{\{\omega\}}$ then there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ such that*

$$\sum_{\substack{\lambda \in \Lambda \cap \mathbb{H}_* \\ \lambda \neq \lambda'}} m_\lambda \ln \left| \frac{\lambda' - \overline{\lambda}}{\lambda' - \lambda} \right| \leq C_m + M|\operatorname{Im} \lambda'| + \frac{1}{m}\omega(\lambda')$$

for all $\lambda' \in \Lambda \cap \mathbb{H}_*$.

Proof We only give the proof for the upper half-plane since the proof for the lower half-plane is nearly identical. Let $\lambda' \in \Lambda \cap \mathbb{H}_+$, $f_{\lambda'} = f_{\lambda',0}$ be the function given by Lemma 3.3 and H be the function from Lemma 3.5. Define

$$h_{\lambda'}(z) = \frac{f_{\lambda'}(z) e^{iMz}}{H^{\frac{1}{m}}(z)}$$

with constants $M, m \in \mathbb{N}$ to be chosen later. We have

$$\ln|h_{\lambda'}(z)| = \frac{1}{m} \ln \left| \frac{(f_{\lambda'}(z)e^{iMz})^m}{H(z)} \right|.$$

Since the function under the module is holomorphic in the upper half-plane, $\ln|h_{\lambda'}(\cdot)|$ is subharmonic there. We get

$$\begin{aligned}
&\exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \exists\, C_n > 0 : \\
&|h_{\lambda'}(z)| = \left| \frac{f_{\lambda'}(z)e^{iMz}}{H^{\frac{1}{m}}(z)} \right| \leq C_n e^{N|\operatorname{Im} z| + \frac{1}{n}\omega(z)} e^{-M|\operatorname{Im} z|} e^{-\frac{1}{m}\ln|H(z)|} \\
&\qquad \leq C_n e^{N|\operatorname{Im} z| + \frac{1}{n}\omega(z)} e^{-M|\operatorname{Im} z|} e^{-\frac{1}{4m}\omega(z)} \\
&\qquad = C_n e^{(N-M)|\operatorname{Im} z| + (\frac{1}{n} - \frac{1}{4m})\omega(z)}
\end{aligned}$$

and

$$\begin{aligned}
|h_{\lambda'}(\lambda')| &= e^{-M|\operatorname{Im} \lambda'| - \frac{1}{m}\ln|H(\lambda')|} \\
&\geq e^{-M|\operatorname{Im} \lambda'| - \frac{1}{m}(C\omega(\lambda') + C + C|\operatorname{Im} \lambda'|)} \\
&= e^{-C/m - (\frac{C}{m} + M)|\operatorname{Im} \lambda'| - \frac{C}{m}\omega(\lambda')}.
\end{aligned}$$

Fix M satisfying $M \geq N$. Then for every $m \in \mathbb{N}$ we can choose $n > 4m$. This yields that $h_{\lambda'}$ is a bounded holomorphic function. Applying Poisson–Jensen formula for subharmonic functions in the upper half-plane for $\ln|h_{\lambda'}(\cdot)|$ we get

$$\ln|h_{\lambda'}(\lambda')| = \int_{\mathbb{R}} P(\lambda', x) \ln|h_{\lambda'}(x)|\, dx - \sum_{\substack{z \in \mathbb{H}_+ \\ f_{\lambda'}(z) = 0, z \neq \lambda'}} m_z \ln \left| \frac{\lambda' - \overline{z}}{\lambda' - z} \right|.$$

Therefore

$$\sum_{\substack{\lambda \in \Lambda \cap \mathbb{H}_+ \\ \lambda \neq \lambda'}} m_\lambda \ln \left| \frac{\lambda' - \overline{\lambda}}{\lambda' - \lambda} \right| \leq \sup_{x \in \mathbb{R}} \ln|h_{\lambda'}(x)| - \ln|h_{\lambda'}(\lambda')|$$

where by $\ln|h_{\lambda'}(x)|$ we denote the non-tangential limit of $\ln|h_{\lambda'}(\cdot)|$ in x. We have that $\sup_{x\in\mathbb{R}} \ln|h_{\lambda'}(x)| < D_m$ for some constant $D_m > 0$ depending on m.

Finally we get

$$\sum_{\substack{\lambda\in\Lambda\cap\mathbb{H}_+\\ \lambda\neq\lambda'}} m_\lambda \ln\left|\frac{\lambda'-\overline{\lambda}}{\lambda'-\lambda}\right| \leq D_m + C + (C+M)|\operatorname{Im}\lambda'| + \tfrac{C}{m}\omega(\lambda').$$

□

The following lemma is strictly technical and holds for any multiplicity variety.

Lemma 3.7 *Let $U \subset \mathbb{H}_*$. Then for every $x \in \mathbb{R}$*

$$\sum_{\lambda\in U\cap\Lambda} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \leq \frac{m_{\lambda_x}}{|\operatorname{Im}\lambda_x|} + \frac{4}{|\operatorname{Im}\lambda_x|} \sum_{\substack{\lambda\in U\cap\Lambda\\ \lambda\neq\lambda_x}} m_\lambda \ln\left|\frac{\lambda_x-\overline{\lambda}}{\lambda_x-\lambda}\right|$$

where $\lambda_x \in U\cap\Lambda$ is such that $d(x,\lambda_x) = d(x, U\cap\Lambda)$.

Proof Given $x \in \mathbb{R}$ let $\lambda_x \in U\cap\Lambda$ be the closest element to x. Then for all $\lambda \in U\cap\Lambda$

$$|\lambda_x - \overline{\lambda}| \leq |\lambda_x - x| + |x-\overline{\lambda}| \leq |\lambda - x| + |x - \overline{\lambda}| = 2|x-\lambda|.$$

Therefore

$$\sum_{\lambda\in U\cap\Lambda} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \leq 4 \sum_{\lambda\in U\cap\Lambda} m_\lambda \frac{|\operatorname{Im}\lambda|}{|\lambda_x-\overline{\lambda}|^2}.$$

The following inequality holds for any complex numbers:

$$\frac{|\operatorname{Im}\lambda||\operatorname{Im}\lambda_x|}{|\lambda_x-\overline{\lambda}|^2} \leq 1 - \frac{|\lambda_x-\lambda|}{|\lambda_x-\overline{\lambda}|}$$

and we omit its elementary proof. We have $1-t \leq \ln t^{-1}$ for $t \in (0,1)$ thus

$$\frac{|\operatorname{Im}\lambda||\operatorname{Im}\lambda_x|}{|\lambda_x-\overline{\lambda}|^2} \leq 1 - \frac{|\lambda_x-\lambda|}{|\lambda_x-\overline{\lambda}|} \leq \ln\left|\frac{\lambda_x-\overline{\lambda}}{\lambda_x-\lambda}\right|.$$

This yields

$$\begin{aligned}\sum_{\lambda\in U\cap\Lambda} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} &\leq 4 \sum_{\lambda\in U\cap\Lambda} m_\lambda \frac{|\operatorname{Im}\lambda|}{|\lambda_x-\overline{\lambda}|^2}\\ &= \frac{m_{\lambda_x}}{|\operatorname{Im}\lambda_x|} + 4\sum_{\substack{\lambda\in U\cap\Lambda\\ \lambda\neq\lambda_x}} m_\lambda \frac{|\operatorname{Im}\lambda|}{|\lambda_x-\overline{\lambda}|^2}\\ &\leq \frac{m_{\lambda_x}}{|\operatorname{Im}\lambda_x|} + \frac{4}{|\operatorname{Im}\lambda_x|} \sum_{\substack{\lambda\in U\cap\Lambda\\ \lambda\neq\lambda_x}} m_\lambda \ln\left|\frac{\lambda_x-\overline{\lambda}}{\lambda_x-\lambda}\right|.\end{aligned}$$

□

Using consecutively Lemmata 3.7, 3.4 and 3.6 we get that X satisfies (B).

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