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On a conjecture of Gustafsson and Lin concerning Laplacian growth

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Abstract
Gustafsson and Lin recently published a significant result concerning Laplacian growth problems that start from a simply connected planar domain. However, the validity of their result depends on the verification of a particular conjecture. This paper provides the missing proof.

Keywords Laplacian growth · Partial balayage · Potential · Starshaped

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1 Introduction

A recent book of Gustafsson and Lin [4] explores the evolution of domains under a Laplacian growth process that starts from a simply connected planar domain with smooth boundary. A key result of theirs, Theorem 5.1, states that this process can be continued indefinitely as a family of simply connected domains on a suitable branched Riemann surface. However, their theorem relies on the validity of a lemma which they believe to be true but are unable to prove. (See also section 8 of [3].) The purpose of this note is to verify their conjecture and so complete the proof of their result.

Let g be a holomorphic function on a connected neighbourhood ω of $\overline{\mathbb{D}}$, where $\mathbb{D}$ denotes the unit disc, and let λ denote planar Lebesgue measure. (We assume that

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$g \not\equiv 0$ and assign g the value 1, say, outside ω to make it globally defined.) For each $t > 0$ we define $\Omega(t) = \{u_t > 0\}$, where

$$u_t = \inf\{w \in C(\mathbb{R}^2\backslash\{0\}) : w \geq 0,\ \ \Delta w \leq |g|^2\, \lambda|_{\mathbb{R}^2\backslash\mathbb{D}} - t\delta_0\} \tag{1}$$

in the sense of distributions and δ_0 is the unit measure at 0. The conjecture of Gustafsson and Lin is that the domains $\Omega(t)$ are simply connected for all sufficiently small $t > 0$. Their difficulty in verifying it arises when the function g has one or more zeros on $\partial\mathbb{D}$. Indeed, they remark that the same issue was also left unresolved in earlier work of Sakai [7]. We prove their conjecture below.

Theorem 1 *There exists $\delta > 0$ such that the domains $\Omega(t)$ $(0 < t < \delta)$ are all starshaped about* 0*, and so in particular are simply connected.*

Our proof of Theorem 1 remains valid if we replace $|g|^2$ in (1) by any C^1 function $f > 0$ on a neighbourhood of $\overline{\mathbb{D}}$. (Indeed, with minor modifications, it also yields the corresponding result in higher dimensions for such functions f.) However, the result may fail if f is allowed to have even one zero, as we now illustrate.

Example 2 There is a C^∞ function $f : \mathbb{R}^2 \to [0, \infty)$ with precisely one zero such that, if $|g|^2$ is replaced by f in (1), then there are arbitrarily small values of $t > 0$ for which $\Omega(t)$ is multiply connected.

Thus the geometrical character of $\Omega(t)$ for small $t > 0$ is highly sensitive to the nature of this function f.

We will establish Theorem 1 and Example 2 in Sects. 3 and 4, respectively, following a brief review of the technique of partial balayage, on which these arguments rely. A survey of related topics, including quadrature domains and free boundary problems, may be found in [6].

2 Partial balayage

If μ is a (positive) measure with compact support in $\mathbb{R}^2$, then we define the logarithmic potential

$$U\mu(x) = -\frac{1}{2\pi}\int \log|x - y|\, d\mu(y) \quad (x \in \mathbb{R}^2)$$

and note that $-\Delta U\mu = \mu$ (in the sense of distributions). Let $f : \mathbb{R}^2 \to [0, \infty)$ be a continuous function such that $f \geq 1$ outside some compact set. The following construction, known as partial balayage, was developed by Gustafsson and Sakai [5] and also expounded by the authors in [2].

We define, for $t > 0$,

$$V_{t,f} = \sup\left\{v \in C(\mathbb{R}^2\backslash\{0\}) : -\Delta v \leq f\lambda|_{\mathbb{R}^2\backslash\mathbb{D}},\, v \leq tU\delta_0\right\}$$

and $u_{t,f} = tU\delta_0 - V_{t,f}$, whence $u_{t,f} \geq 0$. Then

$$-\Delta V_{t,f} = f\lambda|_{\Omega_f(t)\backslash\mathbb{D}}, \quad \text{where} \quad \Omega_f(t) = \{u_{t,f} > 0\} \supset \overline{\mathbb{D}}, \tag{2}$$

and so $V_{t,f} = U(f\lambda|_{\Omega_f(t)\setminus\mathbb{D}})$. It follows easily, using the assumption that $f \geq 1$ outside a compact set, that $\Omega_f(t)$ is bounded. Also,

$$\int_{\Omega_f(t)\setminus\mathbb{D}} f(y)d\lambda(y) = t, \tag{3}$$

since $tU\delta_0 = V_{t,f}$ outside $\Omega_f(t)$.

Here are some more basic properties that we will need.

Proposition 3 *Let $t > 0$ and $f, f_n : \mathbb{R}^2 \to [0,\infty)$ $(n \geq 1)$ be continuous functions that exceed 1 outside some compact set.*

(a) If $f_1 \leq f_2$, then $V_{t,f_1} \leq V_{t,f_2}$, $u_{t,f_1} \geq u_{t,f_2}$ and $\Omega_{f_2}(t) \subset \Omega_{f_1}(t)$.
(b) If (f_n) decreases to f, then $V_{t,f_n} \to V_{t,f}$, $u_{t,f_n} \to u_{t,f}$ and

$$\cup_{n=1}^{\infty}\Omega_{f_n}(t) = \Omega_f(t).$$

(c) If (f_n) increases to f, then $V_{t,f_n} \to V_{t,f}$, $u_{t,f_n} \to u_{t,f}$,

$$\Omega_f(t) \subset \cap_{n=1}^{\infty}\Omega_{f_n}(t) \quad \textit{and} \quad \int_{\cap_{n=1}^{\infty}\Omega_{f_n}(t)\setminus\Omega_f(t)} f d\lambda = 0.$$

Proof (a) This follows immediately from the definition of $V_{t,f}$.

(b) By part (a) the sequence (u_{t,f_n}), which equals $\left(tU\delta_0 - U\left(f_n\lambda|_{\Omega_{f_n}(t)\setminus\mathbb{D}}\right)\right)$, increases to the limit

$$v = tU\delta_0 - U(f\lambda|_{(\cup_n\Omega_{f_n}(t))\setminus\mathbb{D}}),$$

where

$$0 \leq v \leq u_{t,f} = tU\delta_0 - U\left(f\lambda|_{\Omega_f(t)\setminus\mathbb{D}}\right).$$

Since $v = u_{t,f}$ outside $\Omega_f(t)$, this equality must hold everywhere. The other assertions follow immediately.

(c) The argument is similar to part (b), except that $(\Omega_{f_n}(t))$ is now decreasing. □

Let

$$D_r(w) = \{z \in \mathbb{C} : |z - w| < r\} \quad (w \in \mathbb{C}, r > 0)$$

and $D_r = D_r(0)$, so that $\mathbb{D} = D_1$. We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. The function g in Sect. 1 is holomorphic on a neighbourhood ω of $\overline{\mathbb{D}}$. We choose $R > 1$ such that $\overline{D}_R \subset \omega$ and g has no zeros in $\overline{D}_R\setminus\overline{\mathbb{D}}$. In the next section we choose f such that $f = |g|^2$ on $\overline{D}_R$ and $f = 1$ outside D_{R+1}, and will drop the symbol f from the subscripts in the notation $V_{t,f}$, $u_{t,f}$, $\Omega_f(t)$ where no confusion can arise. We claim that there exists $\varepsilon > 0$ such that

$$\Omega(t) \subset D_R \quad (0 < t < \varepsilon).$$

To see this we note that, if $1 < r_1 < r_2 < R$, then there exists $c \in (0, 1]$ such that $f \geq c$ on the set $A = \left(D_{r_2}\setminus D_{r_1}\right) \cup \left(\mathbb{R}^2\setminus D_{R+1}\right)$. Hence $\Omega_f(t) \subset \Omega_{c\chi_A}(t)$. The latter set is of the form $D_{\rho(t)}$ for some $\rho(t) > 1$, and $\rho(t) \to r_1$ as $t \to 0+$, in view of (3). Indeed, there exists $r(t) > 1$ such that $r(t) \to 1$ as $t \to 0+$ and $\Omega_f(t) \subset D_{r(t)}$.

3 Proof of Theorem 1

Let g, f and R be as described above.

Lemma 4 *Let $x_1, x_2, \ldots, x_k$ denote the zeros (if any) of g on $\partial\mathbb{D}$. Then, for each $i \in \{1, 2, \ldots, k\}$, there exist $r_i \in (0, R-1)$ and a positive constant C_i such that*

$$\nabla f(x) \cdot x \geq -C_i f(x) \quad (x \in D_{r_i}(x_i)\backslash\overline{\mathbb{D}}).$$

Proof Suppose that g has a zero of order m at x_i. Then $f(x) = |x - x_i|^{2m} h(x)$ on ω, where $h \geq 0$ is smooth and $h(x_i) > 0$. It follows that

$$\begin{aligned}\nabla f(x) \cdot x &= 2m|x - x_i|^{2m-2} h(x)(x - x_i) \cdot x + |x - x_i|^{2m} \nabla h(x) \cdot x \\ &= h(x)\, |x - x_i|^{2m} \left(2m \frac{(x - x_i) \cdot x}{|x - x_i|^2} + \frac{\nabla h(x) \cdot x}{h(x)}\right) \\ &\geq f(x) \frac{\nabla h(x) \cdot x}{h(x)} \quad (x \in D_R\backslash\overline{\mathbb{D}}),\end{aligned}$$

since

$$(x - x_i) \cdot x = |x|^2 - x_i \cdot x > 0 \quad (|x| > |x_i| = 1).$$

The result follows on noting that $h > 0$ on a neighbourhood of x_i. □

Lemma 5 *There exists $C_0 > 0$ such that*

$$\nabla f(x) \cdot x + (C_0 + 2) f(x) \geq 0 \quad (x \in D_R\backslash\overline{\mathbb{D}}).$$

Proof Let x_i, r_i, C_i $(i = 1, \ldots, k)$ be as in Lemma 4 and define

$$A = D_R\backslash \left(\overline{\mathbb{D}} \cup D_{r_1}(x_1) \cup \cdots \cup D_{r_k}(x_k)\right).$$

Clearly $\inf_A f > 0$. The result follows on choosing C_0 large enough so that $C_0 + 2 \geq C_i$ $(i = 1, \ldots, k)$ and

$$\inf_{x \in A} \nabla f(x) \cdot x + (C_0 + 2) \inf_A f \geq 0.$$

□

Proof of Theorem 1 Let

$$v_t(x) = \nabla u_t(x) \cdot x + C_0 u_t(x) \quad (t > 0),$$

where u_t is as in Sect. 2 and C_0 is as in Lemma 5. We choose $R > 1$ and $\varepsilon > 0$ as in Sect. 2, whence $\Omega(t) \subset D_R$ when $0 < t < \varepsilon$. Since

$$\begin{aligned}\Delta\left(\nabla u_t(x) \cdot x\right) &= 2\Delta u_t(x) + (\nabla \Delta u_t(x)) \cdot x \\ &= 2f(x) + \nabla f(x) \cdot x \quad (x \in \Omega(t)\backslash\overline{\mathbb{D}}),\end{aligned}$$

the function v_t is subharmonic in $\Omega(t)\backslash\overline{\mathbb{D}}$.

We know that u_t, and hence v_t, vanishes outside $\Omega(t)$. Next, we will show that $v_t \leq 0$ on $\partial\mathbb{D}$ for all sufficiently small t. Suppose that $x \neq 0$. Since

$$u_t(x) = -\frac{t}{2\pi}\log|x| + \frac{1}{2\pi}\int_{\Omega(t)\backslash\mathbb{D}} \log|x-y|\, f(y)d\lambda(y), \tag{4}$$

we see that

$$\begin{aligned}
\nabla u_t(x)\cdot x &= -\frac{t}{2\pi}\frac{x}{|x|^2}\cdot x + \frac{1}{2\pi}\int_{\Omega(t)\backslash\mathbb{D}} \frac{x-y}{|x-y|^2}\cdot x f(y)d\lambda(y)\\
&= -\frac{t}{2\pi} + \frac{1}{2\pi}\int_{\Omega(t)\backslash\mathbb{D}} \frac{x-y}{|x-y|^2}\cdot (x-y) f(y)d\lambda(y)\\
&\quad + \frac{1}{2\pi}\int_{\Omega(t)\backslash\mathbb{D}} \frac{x-y}{|x-y|^2}\cdot y f(y)d\lambda(y)\\
&= \frac{1}{2\pi}\int_{\Omega(t)\backslash\mathbb{D}} \frac{x-y}{|x-y|^2}\cdot y f(y)d\lambda(y),
\end{aligned} \tag{5}$$

by (3). This last integrand is negative when $|x| = 1$, since $(x-y)\cdot y = x\cdot y - |y|^2$ and $|y| > 1$. Let

$$A_{x,t} = \{y \in \Omega(t)\backslash\mathbb{D} : x\cdot y \leq 0\} \quad (x \in \partial\mathbb{D}, t > 0).$$

Then

$$\frac{x-y}{|x-y|^2}\cdot y \leq -\frac{|y|^2}{|x-y|^2} \leq -\frac{1}{4} \quad (y \in A_{x,t}),$$

and so

$$\int_{\Omega(t)\backslash\mathbb{D}} \frac{x-y}{|x-y|^2}\cdot y f(y)d\lambda(y) \leq -\frac{1}{4}\int_{A_{x,t}} f d\lambda \leq -\frac{1}{4}\inf_{z\in\partial\mathbb{D}}\int_{A_{z,t}} f d\lambda. \tag{6}$$

There exists $c > 0$ such that $\Omega(t) \supset D_{1+ct}$, because f is bounded above. Since f has only finitely many zeros on $\partial\mathbb{D}$, there exists $C_* > 0$ such that

$$\inf_{z\in\partial\mathbb{D}}\int_{A_{z,t}} f d\lambda \geq C_* t \quad (0 < t < \varepsilon),$$

so we now see from (5) and (6) that

$$\nabla u_t(x)\cdot x \leq -\frac{C_*}{8\pi}t < 0 \quad (x \in \partial\mathbb{D}, 0 < t < \varepsilon). \tag{7}$$

Also, it follows from (4) and (3) that the family $\{u_t/t : 0 < t < \varepsilon\}$ of subharmonic functions on $\mathbb{R}^2\backslash\{0\}$ is locally uniformly bounded above. Since

$$\limsup_{t\to 0+} \frac{u_t(x)}{t} = 0 \quad (x \in \mathbb{R}^2\backslash\overline{\mathbb{D}}),$$

this upper limit is bounded above by $-(\log|x|)/2\pi$ on $\overline{\mathbb{D}}$. It follows from Corollary 5.7.2 of [1] that $u_t(x)/t \to 0$ uniformly on $\partial\mathbb{D}$ as $t \to 0+$. Hence, by (7), there exists $\delta \in (0, \varepsilon)$ such that

$$\nabla u_t(x) \cdot x \leq -\frac{C_*}{8\pi}\frac{t}{u_t(x)} u_t(x) \leq -C_0 u_t(x) \quad (x \in \partial\mathbb{D}, 0 < t < \delta),$$

and so $v_t \leq 0$ on $\partial\mathbb{D}$ when $0 < t < \delta$, as claimed.

We can now apply the maximum principle to the subharmonic function v_t on $\Omega(t)\backslash\overline{\mathbb{D}}$ to see that $v_t < 0$ there. Hence

$$\nabla u_t(x) \cdot x \leq -C_0 u_t(x) < 0 \quad (x \in \Omega(t)\backslash\overline{\mathbb{D}}, 0 < t < \delta),$$

and we also know that $\nabla u_t(x) \cdot x = 0$ on $\mathbb{R}^2\backslash\Omega(t)$. Since $\overline{\mathbb{D}} \subset \{u_t > 0\} = \Omega(t)$, and u_t is decreasing in the radial direction from 0 at each point of $\Omega(t)\backslash\mathbb{D}$, it follows that $\Omega(t)$ is starshaped about 0, as required. □

4 Details of Example 2

Let

$$f_e(x) = \begin{cases} \exp\left(-|x - y_0|^{-2}\right) & (x \in \mathbb{R}^2\backslash\{y_0\}) \\ 0 & (x = y_0) \end{cases},$$

where y_0 is the point $(1, 0)$, and let $\psi : \mathbb{R}^2 \to [0, 1]$ be a C^∞ function such that $\psi(x) = 0$ when $|x| \in [\frac{1}{2}, \frac{3}{4}]$ and $\psi(x) = 1$ when $|x| \in [0, \frac{1}{4}] \cup [1, \infty)$. For each n in $\mathbb{N}$ we define

$$x_n = \left(\cos\frac{\pi}{n}, \sin\frac{\pi}{n}\right) \quad \text{and} \quad r_n = \frac{1}{n(n+1)},$$

whence the discs $\overline{D}_{r_n}(x_n)$ are pairwise disjoint, and the closed annulus

$$A_n = \overline{D}_{3r_n/4}(x_n)\backslash D_{r_n/2}(x_n).$$

We further define

$$\psi_n(x) = \psi\left(\frac{x - x_n}{r_n}\right), \quad \psi_{n,m}(x) = \frac{\psi_n(x) + 1/m}{1 + 1/m} \quad (m \in \mathbb{N})$$

and

$$f_0 = f_e \prod_{n\geq 1} \psi_n.$$

Since $\int_{\Omega_{f_0}(t)\backslash D_1} f_0 d\lambda = t$ and

$$\int_{D_{r_1/4}(x_1)\backslash D_1} f_0 d\lambda = \int_{D_{r_1/4}(x_1)\backslash D_1} f_e d\lambda > 0,$$

we can choose $t_1 > 0$ small enough to ensure that

$$D_{r_1/4}(x_1)\backslash\Omega_{f_0}(t_1) \neq \emptyset.$$

In view of (2) the nonnegative function u_{t_1,f_0} is nonconstant and harmonic on the domain $\left(D_1 \cup A_1^\circ\right)\backslash\{0\}$, and so is strictly positive there. Further, u_{t_1,f_0} cannot take the value 0 at any point y of ∂A_1, since this would imply that $\nabla u_{t_1,f_0}(y) = 0$, which contradicts the Hopf lemma. Hence

$$A_1 \subset \Omega_{f_0}(t_1)$$

and the constant $c_1 = \left(\inf_{A_1} u_{t_1,f_0}\right)/2$ is strictly positive. We define

$$f_{1,m} = f_e \psi_{1,m} \prod_{n\geq 2} \psi_n \quad (m \in \mathbb{N})$$

and note that the sequence $(f_{1,m})$ decreases to f_0, whence by Proposition 3 the sequences $(\Omega_{f_{1,m}}(t_1))$ and $(u_{t_1,f_{1,m}})$ are increasing,

$$\lim_{m\to\infty} u_{t_1,f_{1,m}} = u_{t_1,f_0} \quad \text{and} \cup_m \Omega_{f_{1,m}}(t_1) = \Omega_{f_0}(t_1).$$

By compactness we can choose $m_1 \in \mathbb{N}$ such that $A_1 \subset \Omega_{f_{1,m_1}}(t_1)$ and $\inf_{A_1} u_{t,f_{1,m_1}} > c_1$, and then define

$$f_1 = f_{1,m_1} = f_e \psi_{1,m_1} \prod_{n\geq 2} \psi_n.$$

Since $f_1 \geq f_0$ we note that

$$D_{r_1/4}(x_1)\backslash\Omega_{f_1}(t_1) \supset D_{r_1/4}(x_1)\backslash\Omega_{f_0}(t_1) \neq \emptyset.$$

Next, arguing as above, we choose $t_2 \in (0, t_1/2)$ small enough to ensure that

$$D_{r_2/4}(x_2)\backslash\Omega_{f_1}(t_2) \neq \emptyset$$

and, noting that $f_1 = f_0$ outside $D_{r_1}(x_1)$, observe that

$$A_2 \subset \Omega_{f_1}(t_2).$$

Let c_2 denote the positive constant $\left(\inf_{A_2} u_{t_2,f_1}\right)/2$. We define

$$f_{2,m} = f_e \psi_{1,m_1} \psi_{2,m} \prod_{n \geq 3} \psi_n \quad (m \in \mathbb{N})$$

and note that $(f_{2,m})$ decreases to f_1. As before, we can choose $m_2 \in \mathbb{N}$ such that

$$A_j \subset \Omega_{f_{2,m_2}}(t_j) \quad \text{and} \quad \inf_{A_j} u_{t_j, f_{2,m_2}} > c_j \quad (j = 1, 2).$$

We define

$$f_2 = f_{2,m_2} = f_e \psi_{1,m_1} \psi_{2,m_2} \prod_{n \geq 3} \psi_n$$

and note that $\Omega_{f_2}(t) \subset \Omega_{f_1}(t)$ $(t > 0)$, whence

$$D_{r_1/4}(x_1) \backslash \Omega_{f_2}(t_1) \neq \emptyset \quad \text{and} \quad D_{r_2/4}(x_2) \backslash \Omega_{f_2}(t_2) \neq \emptyset.$$

Proceeding inductively in this way, we obtain a sequence of numbers (t_j) decreasing to 0, a sequence of positive numbers (c_j), and an increasing sequence of functions (f_k) such that

$$A_j \subset \Omega_{f_k}(t_j), \quad D_{r_j/4}(x_j) \backslash \Omega_{f_k}(t_j) \neq \emptyset \quad \text{and} \quad u_{t_j, f_k} > c_j \quad \text{on} \quad A_j \quad (1 \leq j \leq k).$$

We define

$$f = \lim_{j \to \infty} f_j = f_e \prod_{j \geq 1} \psi_{j, m_j}.$$

Clearly

$$D_{r_j/4}(x_j) \backslash \Omega_f(t_j) \neq \emptyset \quad (j \in \mathbb{N}).$$

By Proposition 3 again we note that (u_{t,f_k}) decreases to $u_{t,f}$ as $k \to \infty$ for every $t > 0$. Since $u_{t_j,f_k} \geq c_j$ on A_j for all $j \leq k$, we see that $u_{t_j,f} \geq c_j$ on A_j for all j, and so $A_j \subset \Omega_f(t_j)$ for each j. Thus $\Omega_f(t_j)$ is multiply connected for each $j \in \mathbb{N}$. Finally, f vanishes precisely at y_0 and, since

$$\inf \left\{ \frac{r_j}{|x - y_0|^2} : x \in D_{r_j}(x_j),\, j \geq 1 \right\} > 0,$$

we see that $f \in C^\infty(\mathbb{R}^2)$.

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Declarations

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