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The order of L^1-approximation by elements of the disc algebra

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Abstract

We prove that the order of L^1-approximation by elements of the disc algebra given by Khavinson, Pérez-González and Shapiro is precise.

Let Δ be the unit disc, $\mathbf{T}$ its boundary and consider the disk algebra $\mathcal{A}$ of those continuous functions on $\overline{\Delta}$ that are holomorphic in Δ. In $\mathcal{A}$ the norm is the supremum norm

$$\|f\|_\infty = \sup_t |f(e^{it})|,$$

but we shall also consider the L^1 norm given by

$$\|f\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|dt.$$

In connection with approximation in L^1-norm by elements of a uniform algebra D. Khavinson, F. Pérez-González and H. Shapiro proved the following theorem (see [2, Theorem 3.3]).

Theorem *Let f be a continuous function on $\mathcal{T}$ with $\|f\|_\infty = 1$. Assume there exists an H^1-function G such that*

$$\|f - G\|_1 \le \varepsilon.$$

Then there exists a function G^* in the disk algebra $\mathcal{A}$ such that $\|G^*\|_\infty \le 1$ and

$$\|f - G^*\|_1 \le C\varepsilon \log\frac{1}{\varepsilon}, \tag{1}$$

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where C is a constant independent of f.

See [2] for the motivation of this result and its connection to a theorem of Hoffman and Wermer on homomorphisms of uniform algebras.

The authors of [2] also verified that in (1) the bound $C\varepsilon \log \frac{1}{\varepsilon}$ cannot be replaced by $C\varepsilon$ ([2, Theorem 3.4]), but the problem if the order $O(\varepsilon \log \frac{1}{\varepsilon})$ in (1) can be improved at all, i.e., if it is precise or not, remained open and stated explicitly in Remark (i) in [2]. That problem was communicated to us by D. Khavinson [1]. In this note we prove that the stated order is, indeed, precise.

Theorem 1 *There is a constant $c > 0$ with the property that for every sufficiently small $\varepsilon > 0$ there is a continuous function $f = f_\varepsilon$, $\|f\|_\infty = 1$, such that*

$$\|f - G\|_1 \le \varepsilon$$

for some $G \in \mathcal{A}$, but for any G^* in $\mathcal{A}$ with $\|G^*\|_\infty \le 1$ we have

$$\|f - G^*\|_1 \ge c\varepsilon \log \frac{1}{\varepsilon}.$$

Proof It will be convenient to verify the claim with ε replaced by ε^2.

Let

$$u(z) + iv(z) = \frac{\varepsilon^2}{1 + \varepsilon - z},$$

where $u(z)$ and $v(z)$ are real. Using that

$$(1 + \varepsilon - \cos t)^2 + \sin^2 t = \varepsilon^2 + 4(1 + \varepsilon)\sin^2(t/2),$$

for $z = e^{it}$ we have[1]

$$\Re \frac{1}{1 + \varepsilon - z} = \frac{1 + \varepsilon - \cos t}{(1 + \varepsilon - \cos t)^2 + \sin^2 t} \sim \begin{cases} 1/\varepsilon & \text{if } |t| \le \varepsilon, \\ 1 + \varepsilon/t^2 & \text{if } \varepsilon \le |t| \le \pi, \end{cases}$$

while

$$\left|\Im \frac{1}{1 + \varepsilon - z}\right| = \frac{|\sin t|}{(1 + \varepsilon - \cos t)^2 + \sin^2 t} \sim \begin{cases} |t|/\varepsilon^2 & \text{if } |t| \le \varepsilon, \\ 1/|t| & \text{if } \varepsilon \le |t| \le \pi/2, \\ \pi - |t| & \text{if } \pi/2 \le |t| \le \pi. \end{cases}$$

Indeed, these are easy consequences of the inequality

$$\frac{2}{\pi} u \le \sin u \le u, \qquad 0 \le u \le \pi/2,$$

[1] In what follows $A \sim B$ means that A/B lies in between two positive absolute constants, and $A \preceq B$ and $B \succeq A$ stand for A/B being bounded.

i.e. of

$$\sin u \sim u, \qquad 0 \le u \le \pi/2.$$

For example, for $z = e^{it}$, $\varepsilon \le |t| \le \pi$ $(0 < \varepsilon \le 1)$, we obtain

$$\begin{aligned}\Re\frac{1}{1+\varepsilon-z} &= \frac{1+\varepsilon-\cos t}{(1+\varepsilon-\cos t)^2+\sin^2 t} = \frac{2\sin^2(t/2)+\varepsilon}{\varepsilon^2+4(1+\varepsilon)\sin^2(t/2)}\\ &\sim \frac{t^2+\varepsilon}{\varepsilon^2+t^2} \sim \frac{t^2+\varepsilon}{t^2} = 1+\varepsilon/t^2\end{aligned}$$

as was claimed above.

The preceding relations show that

$$u(e^{it}) \sim \begin{cases} \varepsilon & \text{if } |t| \le \varepsilon,\\ \varepsilon^2+\varepsilon^3/t^2 & \text{if } \varepsilon \le |t| \le \pi,\end{cases} \tag{2}$$

while

$$|v(e^{it})| \sim \begin{cases} |t| & \text{if } |t| \le \varepsilon,\\ \varepsilon^2/|t| & \text{if } \varepsilon \le |t| \le \pi/2\\ \varepsilon^2(\pi-|t|) & \text{if } \pi/2 \le |t| \le \pi.\end{cases} \tag{3}$$

Therefore, if we set $F_\varepsilon = F(z) = \exp(\varepsilon^2/(1+\varepsilon-z))$, $|z| \le 1$, then for small ε

$$\begin{aligned}\Re F(e^{it}) - 1 &= e^{u(e^{it})}\cos v(e^{it}) - 1 \ge (1+u(e^{it}))\cos v(e^{it}) - 1\\ &= u(e^{it})\cos v(e^{it}) - 2\sin^2(v(e^{it})/2) > 0.\end{aligned}$$

This is so, because we subtract from a term $\sim \varepsilon$ $(|t| \le \varepsilon)$ resp $\succeq \varepsilon^3/t^2$ $(|t| \ge \varepsilon)$ a term that is at most $\preceq \varepsilon^2$ resp. $\preceq \varepsilon^4/t^2$. Now the preceding inequality implies in view of the maximum principle that $\Re F(z) - 1$ is positive in the unit disk.

Let $f_\varepsilon = f = F/|F|$, for which we have for small ε

$$\begin{aligned}\|f-F\|_1 &= \left\|\frac{F}{|F|}(|F|-1)\right\|_1 = \||F|-1\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{u(e^{it})}-1)dt\\ &\sim \int_{-\pi}^{\pi} u(e^{it})dt \sim \varepsilon^2,\end{aligned} \tag{4}$$

where we used that $u \le e^u - 1 \le 2u$ provided $0 \le u \le 1/2$ (cf. (2)).

Note that F is in the disk algebra and f is a continuous function with $\|f\|_\infty = 1$. Now let $G^* \in \mathcal{A}$, $\|G^*\|_\infty \le 1$, be any function. We are going to show that

$$\|f - G^*\|_1 \ge c\varepsilon^2 \log\frac{1}{\varepsilon} \tag{5}$$

with some $c > 0$ independent of ε, and that will prove the theorem (with ε replaced by ε^2 and $f_{\alpha\varepsilon}$ resp. $F_{\alpha\varepsilon}$ replacing f resp. G in it, where α is a constant for which $\|f_{\alpha\varepsilon} - F_{\alpha\varepsilon}\|_1 \leq \varepsilon^2$; see (4)).

For the L^1 distance of F and G^* we have

$$\|F - G^*\|_1 \leq \|F - f\|_1 + \|f - G^*\|_1 \leq C_1\varepsilon^2 + \|f - G^*\|_1. \tag{6}$$

The real part of

$$g_1(z) := 1 - G^*(z) + i\Im G^*(0)$$

is clearly nonnegative. Now to the pairs $g_1(z)$ and $g_2(z) := F(z) - 1$ with nonnegative real part in Δ and with imaginary part $= 0$ at the origin we can apply the “reverse triangle inequality"

$$\|g_1\|_1 + \|g_2\|_1 \leq C_0\|g_1 + g_2\|_1,$$

proved in [2, Lemma 3.5], where C_0 is an absolute constant. This yields

$$\begin{aligned}\|F - 1\|_1 &\leq \|g_1\|_1 + \|g_2\|_1 \leq C_0\|(F - 1) + (1 - G^* + i\Im G^*(0))\|_1 \\ &= C_0\|F - G^* + i\Im G^*(0)\|_1 \leq C_0(\|F - G^*\|_1 + |i\Im G^*(0)|).\end{aligned}$$

On the right

$$|i\Im G^*(0)| = |\Im(F - G^*)(0)| \leq \|F - G^*\|_1,$$

so, in view of (6),

$$\|F - 1\|_1 \leq 2C_0\|F - G^*\|_1 \leq 2C_0C_1\varepsilon^2 + 2C_0\|f - G^*\|_1 \tag{7}$$

follows. Since on the left

$$\begin{aligned}\|F - 1\|_1 \geq \|\Im F\|_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{u(e^{it})}|\sin v(e^{it})|dt \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin v(e^{it})|dt \\ &\succeq \frac{1}{2\pi}\int_{\varepsilon}^{\pi/2} |v(e^{it})|dt \sim \varepsilon^2 \log\frac{1}{\varepsilon} = \frac{1}{2}\varepsilon^2 \log\frac{1}{\varepsilon^2}\end{aligned}$$

(where, for the $\sim$ relation we used (3)), the inequality (5) follows from (7) for all sufficiently small ε. □

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.[2]

[2] This statement was requested to be included during the submission process.

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Declarations

Conflict of interest There is no conflict of interest.

References

1. Khavinson, D.: Personal communication
2. Khavinson, D., Pérez-González, P., Shapiro, H.: Approximation in L^1-norm by elements of a uniform algebra. Constr. Approx. **14**, 401–410 (1998)