Remarks on global sub-Lorentzian geometry

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Abstract This paper aims at being a starting point for the investigation of the global sub-Lorentzian (or more generally sub-semi-Riemannian) geometry, which is a subject completely not known. We present an adaptation to the fiber bundles setting (and state its elementary proof in the real analytic case) of the decomposition result for indefinite metrics. Using it and the result by Sussmann on global spanning of distributions, we state and prove some global facts on sub-Lorentzian manifolds. We also construct a global control system describing nonspacelike curves and study its controllability properties in case of compact manifolds.

Keywords Vector bundles · Indefinite metrics on vector bundles · Distributions

1 Introduction

The motivation of this note is the lack of global theorems in the sub-Lorentzian or more generally sub-semi-Riemannian geometry. A *sub-semi-Riemannian manifold* is, by definition, a triplet (M, H, g) where M is a smooth (smooth means of class C^∞ in this paper) connected and paracompact manifold, H is a smooth bracket generating vector distribution of constant rank on M , $\text{rank } H < \dim M$, and g is a semi-Riemannian metric on H (H is bracket generating if for every $x \in M$ there exists a basis $X_0, \dots, X_k$ of H defined near x such that the fields $X_0, \dots, X_k$ and all their successive commutators evaluated at x span the whole tangent space $T_x M$). A couple (H, g) is called a *sub-semi-Riemannian metric or structure* on M . The *index of a metric* g at a point x is the maximal dimension of all linear subspaces in H_x on which

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$g_x = g|_{H_x \times H_x}$ is negative definite. As in the semi-Riemannian geometry [23] the index is independent of a base point x . If the index of g is equal to 1 then g is a Lorentzian metric, (H, g) is called a *sub-Lorentzian metric* on M , and (M, H, g) is a *sub-Lorentzian manifold*. In the sub-semi-Riemannian geometry we can ask similar question and try to solve similar problems as in the classical semi-Riemannian geometry. There have appeared lately several papers concerning the local sub-Lorentzian geometry—see e.g. [12–14], showing some significant differences between the Lorentzian and sub-Lorentzian case. There are also papers treating global problems but only in flat cases—see [8, 17, 18]. The problem is the little understanding of global phenomena that occur in the sub-geometries on general manifolds. For instance, as far as the author of this paper is concerned, there is only one known global fact from the general sub-Lorentzian geometry proved in [10], and saying that any two causally related points on a time-oriented globally hyperbolic sub-Lorentzian manifold can be joined by a length maximizing nonspacelike future directed curve.

The main problem in proving global theorems is the fact that the appropriate tools are missing. The aim of this paper is twofold. First is to draw the reader's attention to two results (Theorems 1.1, 1.2) which can serve as a starting point for global studies. Examples of global facts that can be obtained in this way may be found in Sect. 3. Before we state the mentioned theorems let us fix some notation. If $\pi : E \rightarrow M$ is a vector bundle and $x \in M$ then by E_x we denote the fiber of E over x , i.e. $E_x = \pi^{-1}(x)$.

Theorem 1.1 *Let $\pi : E \rightarrow M$ be a smooth vector bundle. Suppose that there exists a smooth fiber metric g of index l on E . Then E can be represented in the form of the Whitney sum $E = E^- \oplus E^+$ of smooth subbundles, where E^- is of rank l and the restriction of g to E^- is negative definite, while the restriction of g to E^+ is positive definite. Moreover the sum can be chosen to be orthogonal, i.e. for each $x \in M$ the subspaces E_x^- and E_x^+ are g_x -orthogonal.*

The decomposition $E = E^- \oplus E^+$ is not unique. The converse statement to Theorem 1.1 is of course true. If a (smooth) vector bundle admits a decomposition $E = E^- \oplus E^+$, where E^- is of rank l , then clearly there exists a metric of index l on E which is negative definite on E^- .

If E is equipped with an indefinite fiber metric g , then a vector $v \in E_x$ is called *timelike* if $g_x(v, v) < 0$ and is called *spacelike* if $g_x(v, v) > 0$.

Theorem 1.2 *Let $\pi : E \rightarrow M$ be a smooth vector bundle of rank k over an n -dimensional manifold M . Suppose that there exists a fiber metric g of index l on E . Then there exist global sections $\sigma_1, \dots, \sigma_d, \tau_1, \dots, \tau_e : M \rightarrow E$, where $d \leq l(n+1)$, $e \leq (k-l)(n+1)$, and such that for every $x \in M$*

- (i) *each of the vectors $\sigma_1(x), \dots, \sigma_d(x)$ is either timelike or equal to zero,*
- (ii) *each of the vectors $\tau_1(x), \dots, \tau_e(x)$ is either spacelike or equal to zero,*
- (iii) *$g_x(\sigma_i(x), \tau_r(x)) = 0$, $i = 1, \dots, d$, $r = 1, \dots, e$, and*
- (iv) *$E_x = \text{Span}\{\sigma_1(x), \dots, \sigma_d(x), \tau_1(x), \dots, \tau_e(x)\}$.*

This theorem is a corollary of Sussmann's theorem on global spanning of distributions [25] and the decomposition result from Theorem 1.1.

Suppose that (M, H, g) is a sub-Lorentzian manifold. A curve $\gamma : [a, b] \rightarrow M$ is called horizontal if it is absolutely continuous and $\dot{\gamma}(t) \in H_{\gamma(t)}$ a.e. on $[a, b]$. By a *time orientation* of (H, g) we mean a continuous timelike vector field on M . Suppose that (M, H, g) is time-oriented by a vector field X . A horizontal curve $\gamma : [a, b] \rightarrow M$ is said to be nonspacelike future directed if $g(\dot{\gamma}(t), \dot{\gamma}(t)) \leq 0$ and $g(X(\gamma(t)), \dot{\gamma}(t)) < 0$ a.e. on $[a, b]$. Nonspacelike curves on a given (sub-)Lorentzian manifold are responsible for its causal structure, so it is of highest importance to have as precise description of nonspacelike curves as possible. On certain time-oriented sub-Lorentzian manifolds (for instance on all compact), Theorem 1.2 facilitates this task, namely it allows one to construct a global affine control system having as its trajectories a class of nonspacelike future directed curves. This system can be written as

$$\dot{x} = \sigma + \sum_{r=1}^e u_r \tau_r, \quad (1.1)$$

where σ and τ_r 's are taken from Theorem 1.2, σ is a time orientation, the set of control parameters is equal to

$$\mathcal{U} = \left\{ u \in \mathbb{R}^e : \sum_{r=1}^e u_r^2 \leq \frac{1}{R} \right\},$$

$R = \max \{ |c_{rs}^+(x)| : x \in M, r, s = 1, \dots, e \}$, and $c_{rs}^+(x) = g_x(\tau_r(x), \tau_s(x))$, $r, s = 1, \dots, e$ —all details are contained in Sect. 3 below. As we said, the set of all trajectories of the system (1.1) is contained in the class of all nonspacelike future directed curves on (M, H, g) , and in some particular cases the two sets of curves are equal. The second aim of this note is to prove some results concerning controllability of the system (1.1). In order to state the result we need more definitions. So, if M is a manifold and H is a distribution of constant rank on M , then by $H^\perp$ we denote the annihilator of H i.e. $H^\perp = \{ \lambda \in T^*M : \langle \lambda, \xi \rangle = 0 \text{ for every } \xi \in H_{\pi(\lambda)} \}$; here $\pi : T^*M \rightarrow M$ is the canonical projection. Now, a horizontal curve $\gamma : [a, b] \rightarrow M$ is called *abnormal* if there exists an absolutely continuous lift to the cotangent bundle (an *abnormal lift*) $\lambda : [a, b] \rightarrow T^*M$ (i.e. $\gamma = \pi \circ \lambda$) such that $\lambda([a, b]) \subset H^\perp$, $\lambda(t) \neq 0$ for every $t \in [a, b]$, and finally $\Omega_{\lambda(t)}(\dot{\lambda}(t), \zeta) = 0$ for almost every $t \in [a, b]$ and every $\zeta \in T_{\lambda(t)}H^\perp$, where Ω is the restriction to $H^\perp$ of the standard symplectic structure on T^*M . An abnormal curve $\gamma : [a, b] \rightarrow M$ is a *Goh curve* if it has an abnormal lift $\lambda(t)$ that additionally satisfies $\lambda([a, b]) \subset (H^2)^\perp$ (this is the so-called *Goh condition*), where H^2 is the distribution spanned by the fields tangent to H and their commutators of first order. Goh condition is a second-order necessary condition for optimality. Goh curves appear in the sub-Riemannian and sub-Lorentzian geometry, and often their presence is accompanied by some singular phenomena (cf. e.g. [1, 14, 20]). The second definition that we need is as follows. Let $\mathcal{U}$ be a subset of $\mathbb{R}^k$. A control system $(\Sigma) \dot{x} = f(x, u)$, $x \in M$, $u \in \mathcal{U}$, is called *controllable* if for every $x, y \in M$ there exists a trajectory of (Σ) joining x to y . (Σ) is *controllable at a point* x if there exists a neighbourhood U of x such that (Σ) is controllable on U ; such a neighbourhood is called a *controllable neighbourhood*.

Theorem 1.3 *Suppose that (M, H, g) is a compact time-oriented sub-Lorentzian manifold. Then the induced control system (1.1) has closed trajectories, among which there are closed timelike curves. Moreover, at every point belonging to any closed timelike curve which is a trajectory of (1.1) and which is not a Goh curve with respect to H , the system is locally controllable. More precisely, every closed timelike trajectory of (1.1), that is not a Goh curve, possesses a controllable neighbourhood for (1.1).*

The analogous reasoning leads to the following similar but a little different in its character result.

Theorem 1.4 *Suppose that (M, H, g) is a compact time-oriented sub-Lorentzian manifold. Then there exist closed timelike curves on M . Any such curve that is not a Goh curve (with respect to H) possesses a neighbourhood U with the following property: for every $x, y \in U$ there exists a timelike future directed curve contained in U and joining x to y .*

Closed timelike curves from Theorem 1.4 need not be trajectories of (1.1).

The method of the proofs of Theorems 1.3, 1.4 presented in this paper slightly generalizes considerations from [22], as it will be explained at the end of this paper. Theorem 1.4 has an immediate corollary:

Corollary 1.1 *Let (M, H, g) be a time-oriented sub-Lorentzian manifold of step two, i.e. $H^2 = TM$. Then any closed timelike curve possesses a neighbourhood described in the assertion of Theorem 1.4.*

The paper is organized as follows. In Sect. 2 we discuss proofs of Theorems 1.1 and 1.2. In Sect. 3 we recall basic notions from the sub-Lorentzian geometry. Then we present some results concerning global sub-Lorentzian geometry; part of them are corollaries of the mentioned theorems. In Sect. 4 we prove Theorems 1.3, 1.4 and show their possible generalizations.

2 Proofs of Theorems 1.1, 1.2

In this section all objects are smooth or real analytic. Let $\pi : E \rightarrow M$ be a rank k bundle. By a fiber metric on E we mean a section g of the bundle $E^* \otimes E^* \rightarrow M$ such that $g_x : E_x \times E_x \rightarrow \mathbb{R}$ is a nondegenerate symmetric bilinear form for every $x \in M$. If the index of g , $\text{ind}(g)$, is 0 then g is called a Riemannian metric, if $\text{ind}(g) = 1$, g is called Lorentzian. Generally, g is called semi-Riemannian if the index is greater than 0 and less than $\text{rank } E$. It is well known that Riemannian metrics exist on every vector bundle. In a smooth case it follows from the existence of partitions of unity, in a real analytic case from the imbedding theorem—see [9]. The existence of semi-Riemannian metrics is a more subtle thing. Adapting for instance arguments from [24] to the fiber bundles setting, one can prove that: *there exists a metric of index l on a vector bundle $E \rightarrow M$ if and only if E possesses a rank l subbundle $L \rightarrow M$.* This theorem however does not say anything about the behaviour of the metric on L ,

and from the point of view of applications one needs a more precise statement—such as Theorem 1.1.

Theorem 1.1 is known for Lorentzian metrics on manifolds, i.e. when E is a tangent bundle TM and $\text{ind}(g) = 1$ —see e.g. [5]. It is also known for general semi-Riemannian manifolds. The proof may be found in [3] and relies on the following observation. The presence on M of a metric of index l is equivalent to the existence of $O(l, n)$ -structure in the bundle LM of linear frames over M . In turn, the structure group $O(l, n)$ can be reduced to its maximal compact subgroup $O(l) \times O(n-l)$ (reduction to a maximal compact subgroup is always possible). Denote by $(P, M, O(l) \times O(n-l))$ the reduced bundle. Now we use properties of universal bundles. If ω_G denotes the universal principle G -bundle $EG \rightarrow BG$, then it is known that the universal principle $O(l) \times O(n-l)$ -bundle $\omega_{O(l) \times O(n-l)}$ is isomorphic to the product bundle $\omega_{O(l)} \times \omega_{O(n-l)}$. Now, if $f : M \rightarrow BO(l) \times BO(n-l)$ is the classifying map for P , then $P = P_1 \oplus P_2$, where $P_1 = (pr_1 \circ f)^* \omega_{O(l)}$, $P_2 = (pr_2 \circ f)^* \omega_{O(n-l)}$. The decomposition of P results in the decomposition of the tangent bundle $TM = \xi_1 \oplus \xi_2$, where $\xi_1 = P_1 \times_{O(l)} \mathbb{R}^l$ and $\xi_2 = P_2 \times_{O(n-l)} \mathbb{R}^{n-l}$ (i.e. for instance ξ_1 is a vector bundle with typical fiber $\mathbb{R}^l$, associated to P_1). It remains to check that the metric g is negative on ξ_1 and positive on ξ_2 .

In the fiber bundle setting the proof might go along similar lines. Instead of this we will present elementary and fully constructive proof of real analytic version of Theorem 1.1.

2.1 Proof of Theorem 1.1 in a real analytic case

During the proof all objects are supposed to be real analytic. Let $\pi : E \rightarrow M$ be a rank k vector bundle equipped with a fiber metric g of index l . Introduce a Riemannian fiber metric h on E . For each point $x \in M$ we can define a linear mapping $f_x : E_x \rightarrow E_x$ by formula

$$g_x(v, w) = h_x(f_x(v), w) \quad (2.1)$$

for every $v, w \in E_x$. Fix a point x_0 and let $s_1, \dots, s_k : U \rightarrow E$ be a local basis of sections of E defined on a neighbourhood U of x_0 . If $g_{ij}(x) = g(s_i(x), s_j(x))$, $h_{ij}(x) = h(s_i(x), s_j(x))$, and $(h^{ij}(x))$ is the inverse matrix to $(h_{ij}(x))$ then one can easily make sure that

$$f_x(v) = \sum_{\beta} \sum_{i,j} h^{\beta j}(x) g_{ij}(x) v^i s_{\beta}(x) \quad (2.2)$$

where $v = \sum v^i s_i(x) \in E_x$. By (2.2) it is seen that f_x depends analytically on x . Next, using (2.1) it is seen that

$$h_x(f_x(v), w) = h_x(v, f_x(w))$$

which means that there exists a basis $v_1(x), \dots, v_k(x)$ of eigenvectors for f_x which is orthonormal with respect to h . Clearly such a basis is orthogonal with respect to g .

Let $\lambda_1(x), \dots, \lambda_k(x)$ denote the corresponding eigenvalues. Since $g(v_i(x), v_i(x)) = h(f_x(v_i(x)), v_i(x)) = \lambda_i(x)$ and $\text{ind}(g) = l$, it means that, possibly after reordering $\lambda_i(x)$'s, we have

$$\lambda_1(x), \dots, \lambda_l(x) < 0, \quad \lambda_{l+1}(x), \dots, \lambda_k(x) > 0 \quad (2.3)$$

for $x \in U$. Finally, for $x \in U$ let us set

$$E_x^- = \text{Span}\{v_1(x), \dots, v_l(x)\}, \quad E_x^+ = \text{Span}\{v_{l+1}(x), \dots, v_k(x)\}.$$

Notice that the spaces E_x^-, E_x^+ have an invariant meaning for f_x , i.e. they do not depend on the choice of the basis $v_1(x), \dots, v_k(x)$, and moreover E_x^+ is the g_x -orthogonal complement of E_x^- . To end the proof we must show that the spaces E_x^-, E_x^+ depend analytically on x in a neighbourhood of x_0 .

Let

$$P(x; \lambda) = \det(f_x - \lambda Id) = (-1)^k (\lambda - \lambda_1(x)) \dots (\lambda - \lambda_k(x)). \quad (2.4)$$

The case $l = 1$ is easy, because then $\lambda_1(x)$ is a root of multiplicity 1, and hence, by implicit function theorem, is analytic (or smooth in a smooth category). Consequently, $E_x^- = \ker(f_x - \lambda_1(x)Id)$ is an analytic subbundle.

Now suppose that $l > 1$. $P(x; \lambda)$ obviously is a pseudopolynomial, i.e. a polynomial with respect to the powers of λ with coefficients being analytic functions of x . Without changing the set of roots of P we may assume that the discriminant of P , which is an analytic function of x , does not vanish identically [21] in a neighbourhood of x_0 (we just get rid of multiple factors). If x_0 does not belong to the discriminant set of P , then clearly roots are of multiplicity 1, so they are real analytic functions around x_0 .

Suppose then that x_0 belongs to the discriminant set of P . Consider a complexification of (2.4), i.e. a pseudopolynomial $P(z; \lambda)$ where $z \in \tilde{U}$, $\tilde{U}$ is an open subset of $\mathbb{C}^n$ such that $\tilde{U} \cap \mathbb{R}^n = U$, $\lambda \in \mathbb{C}$. From now on we suppose a neighbourhood U (resp. $\tilde{U}$) of x_0 to be sufficiently small. Using continuity of roots of pseudopolynomials (see [21]) and (2.3), we can find an $\varepsilon > 0$ such that

$$\lambda_1(z), \dots, \lambda_l(z) \in \{\lambda \in \mathbb{C} : \text{Re} \lambda < -\varepsilon\}, \quad \lambda_{l+1}(z), \dots, \lambda_k(z) \in \{\lambda \in \mathbb{C} : \text{Re} \lambda > \varepsilon\} \quad (2.5)$$

for $z \in \tilde{U}$. This proves that $P(z; \lambda)$ is reducible. Indeed, by [6], all roots of an irreducible pseudopolynomial $H(z, \lambda) = \lambda^k + A_1(z)\lambda^{k-1} + \dots + A_k(z)$, $z \in \tilde{U}$, form a closed system, that is to say any root can be obtained from any other root by analytic continuation in $\tilde{U}$. In our case by (2.5) this cannot be true. Let

$$P(z; \lambda) = H_1(z; \lambda) \dots H_m(z; \lambda) \quad (2.6)$$

be a decomposition into irreducible pseudopolynomials. Using what we have said above, after suitable reordering of factors in (2.6) we are led to the situation where

for a certain number s , $1 \leq s \leq l$, the pseudopolynomial $H_1(z; \lambda) \dots H_s(z; \lambda)$ has only negative roots when restricted to U while $H_{s+1}(z; \lambda) \dots H_m(z; \lambda)$ has only positive roots when restricted to U . In this way we have come to the conclusion that $\{\lambda_1(x), \dots, \lambda_l(x)\}$ is the set of all roots of a real pseudopolynomial. In particular any function of the form $x \rightarrow G(\lambda_1(x), \dots, \lambda_l(x))$, where $G(y_1, \dots, y_l)$ is analytic and symmetric, is analytic in a neighbourhood of x_0 . Now, the observation

$$E_x^- = \ker((f_x - \lambda_1(x)Id) \circ \dots \circ (f_x - \lambda_l(x)Id)),$$

$x \in U$, ends the proof, since $(f_x - \lambda_1(x)Id) \circ \dots \circ (f_x - \lambda_l(x)Id)$ is an analytic vector bundle morphism of constant rank.

The decomposition in the hypothesis of Theorem 1.1 is of course not unique—it essentially depends on the choice of h .

2.2 Proof of Theorem 1.2

Having in mind the decomposition result, now it is enough to apply to $E^\pm$ the result by Sussmann [25]:

Let $E \rightarrow M$ be a (in general singular) subbundle of the vector bundle $V \rightarrow M$. Let $r = \max_{x \in M} \dim E_x$. Then there exist global sections $s_1, \dots, s_m : M \rightarrow E$, $m \leq r(n+1)$, such that $E_x = \text{Span}\{s_1(x), \dots, s_m(x)\}$ for every $x \in M$.

3 Applications to the global sub-Lorentzian geometry

For all details and facts concerning the sub-Lorentzian geometry the reader is referred to [12] and its reference section, and also to [17]. Let (M, H, g) be a fixed sub-Lorentzian manifold. Recall that a time orientation of (H, g) is, by definition, a continuous timelike vector field on M (not every sub-Lorentzian metric admits a time orientation). Suppose our (M, H, g) to be time-oriented by a field X . A time orientation permits to divide all non-zero nonspacelike vectors into two disjoint classes, namely a nonspacelike $v \in H_q$ is called *future directed* if $g(v, X(q)) < 0$, and is called *past directed* if $g(v, X(q)) > 0$. Now, an absolutely continuous curve $\gamma : [a, b] \rightarrow M$ is called nonspacelike (future, past directed) if $\dot{\gamma}(t)$ is a nonspacelike (future, past directed) a.e. on $[a, b]$. Analogously one defines timelike and null (future, past directed) curves. Such curves can be applied to the study of affine control systems—see [15]. For a fixed point $x_0 \in M$ and its neighbourhood U we will denote by $J^+(x_0, U)$ the (future) nonspacelike reachable set from x_0 which, by definition, is equal to the set of all points $x \in U$ that can be reached from x_0 by a nonspacelike future directed curve contained in U . If we replace the word ‘nonspacelike’ with the word ‘timelike’ (resp. ‘null’) then we obtain the definition of the (future) timelike (resp. null) reachable set from x_0 which are denoted respectively by $I^+(x_0, U)$ and $N^+(x_0, U)$. In order to ensure better properties of reachable sets we assume that U is a normal neighbourhood of x_0 , that is to say if we extend, around x_0 , a metric g to a Lorentzian metric $\tilde{g}$, then U is a normal convex neighbourhood of x_0 with respect to $\tilde{g}$ and its closure $\overline{U}$ is contained in some other normal convex neighbourhood of

x_0 with respect to $\tilde{g}$; see also [13] for a constructive definition of normal neighbourhoods. Now, if U is a normal neighbourhood of x_0 , then $J^+(x_0, U)$ is closed with respect to U , $J^+(x_0, U) = cl_U(int I^+(x_0, U)) = cl_U(int N^+(x_0, U))$, and finally $\partial_U J^+(x_0, U) = \partial_U I^+(x_0, U) = \partial_U N^+(x_0, U)$, where cl_U is the closure and ∂_U is the boundary, both taken with respect to U —see [12] for proofs. Replacing ‘future directed’ by ‘past directed’ we obtain dual notions, for instance in an obvious manner we define $J^-(x_0, U)$, the (past) nonspacelike reachable set from x_0 . In case $U = M$ we use the notation $I^+(x) = I^+(x, M)$, $J^+(x) = J^+(x, M)$, $I^-(x) = I^-(x, M)$ etc. If $S \subset M$ is a set then we define $I^+(S) = \bigcup_{x \in S} I^+(x)$, and similarly for $J^+(S)$, $I^-(S)$, $J^-(S)$.

Using Theorem 1.1 we obtain the following

Proposition 3.1 *Let (M, H, g) be a sub-Lorentzian manifold. Moreover let $H = H^- \oplus H^+$ be the decomposition from Theorem 1.1. Then (H, g) is time-orientable if and only if H^- is orientable as a vector bundle.*

Proof Since a rank 1 vector bundle is orientable if and only if it is trivial, orientability of H^- implies time-orientability of (H, g) . Now suppose that (H, g) is time-orientable and let $X : M \rightarrow H$ be a time orientation. Denote by $P_x : H_x \rightarrow H_x^-$ the orthogonal (in sense of g_x) projection. The formula $Y(x) = P_x(X(x))$ defines a smooth global non-vanishing vector field with values in H^- , hence H^- is an orientable vector bundle. $\square$

Since every bundle over a simply-connected base is orientable we have

Corollary 3.1 *Every simply-connected sub-Lorentzian manifold is time-orientable.*

Remark 3.1 Note in this place that in the similar manner we can speak about time-orientability of sub-semi-Riemannian manifolds (cf. [19]; see [23] for the semi-Riemannian case). To be more precise, if (M, H, g) is a sub-semi-Riemannian manifold, and $H = H^- \oplus H^+$ is the corresponding decomposition from Theorem 1.1, then we can say that (M, H, g) is time orientable if H^- is an orientable vector bundle. It is easy to see that such a notion does not depend on the particular choice of the decomposition. Indeed, if $H^- \oplus H^+ = \tilde{H}^- \oplus \tilde{H}^+$ are two decompositions, then the orthogonal projection $P : H \rightarrow H^-$ defines a vector bundle isomorphism $P|_{\tilde{H}^-} : \tilde{H}^- \rightarrow H^-$. Now, as a corollary we have: every simply-connected sub-semi-Riemannian manifold is time-orientable.

Next immediate consequence of Theorem 1.1 is the following result.

Proposition 3.2 *Every simply-connected sub-Lorentzian manifold (M, H, g) with $\dim M = 3$ is parallelizable.*

Proof Indeed, $TM = H \oplus L$, where L is a suitable rank 1 subbundle of TM . Now $TM = H^- \oplus H^+ \oplus L$ where, due to our assumptions, H^- , H^+ , and L are rank 1 trivial bundles. $\square$

Thus, for instance, sub-Lorentzian metrics do not exist on the manifold $\mathbb{R} \times S^2$, although Lorentzian ones do exist. Let us state another fact that easily follows from Theorem 1.1.

Proposition 3.3 *Let (M, H, g) be a simply-connected sub-Lorentzian manifold. If $\text{rank } H = 2$ then (H, g) admits a global basis.*

Proof Pick a time orientation X . Then $H = \text{Span}\{X\} \oplus H^+$ where, by assumption, H^+ is trivial. $\square$

Below we will need the following known fact from control theory that can be found for instance in [16], Proposition 1.1 (or in any textbook devoted to the subject).

Lemma 3.1 *Suppose that we are given a control system $(\Sigma) \dot{x} = f(x, u)$, $u \in \mathcal{U}$, on a manifold M . Let $x \in M$ and denote by $\mathcal{A}(x)$ the reachable set from x for (Σ) . Finally, let $\gamma : [0, T] \rightarrow M$ be an arbitrary trajectory of (Σ) such that $\gamma(0) = x$. If $\gamma(t_0) \in \text{int } \mathcal{A}(x)$ for a $t_0 \in [0, T)$ then $\gamma(t) \in \text{int } \mathcal{A}(x)$ for all $t \in (t_0, T]$.*

Next we present some facts concerning reachable sets which are well-known in the Lorentzian geometry.

Proposition 3.4 *Let $x \in M$. If $y \in \text{int } I^+(x)$ then there exists a timelike future directed curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x$, $\gamma(1) = y$, $\gamma((0, 1)) \subset \text{int } I^+(x)$.*

Proof The result is by no means obvious since the structure of the sets $I^+(x)$ from the global point of view is not known even in the Lorentzian case; in addition, in the sub-Lorentzian case the sets $I^+(x)$ need not be open. We come to the proof now. It is known by Krener's theorem (see e.g. [1]) that the sets $\text{int } I^+(x)$, $\text{int } I^-(x)$ are non-empty for every x . Moreover, it follows e.g. from [10] that if U is a normal neighbourhood of x then the set of points $z \in I^+(x, U)$ that can be reached from x by a timelike future directed curve that is not abnormal, is dense and open in $I^+(x, U)$. Let $\eta : [0, 1] \rightarrow M$ be a timelike future directed curve such that $\eta(0) = x$, $\eta(1) = y$. Take a normal neighbourhood U of x . Then $\eta(t_1) \in U$ for a $t_1 \in (0, 1)$. Next there exists a point $z \in U \cap I^-(\eta(t_1))$ that can be reached from x by a timelike future directed curve which is not abnormal. Denote this curve by σ . Since every timelike curve starting from x and contained in $\partial_U I^+(x, U)$ is abnormal (it is enough to apply Pontriagin maximum principle and the definition of abnormal curves—see [12]), it follows that $\sigma \setminus \{x\} \subset \text{int } I^+(x, U)$. To sum up, there exists a timelike future directed curve joining x to y such that its initial segment is contained in $\text{int } I^+(x, U) \subset \text{int } I^+(x)$. To terminate the proof we use Lemma 3.1. $\square$

Note that unlike the Lorentzian case a version of Proposition 3.4 for the sets $N^+(x)$ is not true. Consider as an example the Heisenberg sub-Lorentzian metric on $\mathbb{R}^3$ which is defined by supposing the fields $X = \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} - \frac{1}{2}x \frac{\partial}{\partial z}$ to be an orthonormal frame with the time orientation X . As it was computed in [11], $N^+(0) = J^+(0) = \{\eta_1 \leq 0, x \geq 0, z \geq 0\} \cup \{\eta_2 \leq 0, x \geq 0, z \leq 0\}$ with $\eta_1 = z - \frac{1}{4}(x^2 - y^2)$, $\eta_2 = -z - \frac{1}{4}(x^2 - y^2)$, and as it is not difficult to check, $\text{int } N^+(0)$ contains no null future directed curves initiating at 0—an initial segment of any such curve must be contained in $\partial N^+(0)$.

Proposition 3.5 *On every compact time-oriented sub-Lorentzian manifold there exists a closed timelike future directed curve.*

Proof See e.g. [2], page 58 for the proof in the Lorentzian case. In our case, consider the family of open (non-empty) sets of the form $\text{int } I^+(x, U)$ where $x \in U$ and U is a normal neighbourhood of x . First let us notice that this family is indeed a covering of M . To this end take a point $x \in M$. Let us issue a timelike past directed curve $\sigma(t)$ from x , $\sigma(0) = x$, which is not abnormal. Denote by U a normal neighbourhood of x . Supposing that $t > 0$ is sufficiently small we can assume that U is a normal neighbourhood of $\sigma(t)$ —this follows from the construction of normal neighbourhoods in the semi-Riemannian case. Now, the reasoning from the last part of the proof of Proposition 3.4 and Lemma 3.1 show that $x \in \text{int } I^+(\sigma(t), U)$.

Chose a finite subcovering $\text{int } I^+(x_1, U_1), \dots, \text{int } I^+(x_m, U_m)$. Then $x_1 \in \text{int } I^+(x_{i_1}, U_{i_1})$ for a certain $i_1 \in \{1, \dots, m\}$. Next $x_{i_1} \in \text{int } I^+(x_{i_2}, U_{i_2})$, again $i_2 \in \{1, \dots, m\}$ etc. In this way we obtain a sequence of points $x_{i_1}, x_{i_2}, \dots, x_{i_k}, \dots$ but since there is only a finite number of distinct points, repetitions must occur, i.e. we will have a situation like this: $x_{i_k} \in \text{int } I^+(x_{i_{k+1}}, U_{i_{k+1}})$, $x_{i_{k+1}} \in \text{int } I^+(x_{i_{k+2}}, U_{i_{k+2}}), \dots$, $x_{i_{k+l-1}} \in \text{int } I^+(x_{i_{k+l}}, U_{i_{k+l}})$, $x_{i_{k+l}} \in \text{int } I^+(x_{i_k}, U_{i_k})$ for certain k and l . So $x_{i_{k+l}}$ can be joined from x_{i_k} by a timelike future directed curve contained in U_{i_k} ; $x_{i_{k+l-1}}$ can be joined from $x_{i_{k+l}}$ by a timelike future directed curve in $U_{i_{k+l}}, \dots$, and finally x_{i_k} can be joined from $x_{i_{k+1}}$ by a timelike future directed curve in $U_{i_{k+1}}$. The proof is over. $\square$

In the proof of Proposition 3.5 we could cover M with sets of the form $\text{int } I^+(x)$, $x \in M$. However, coverings by sets $\text{int } I^+(x, U)$, where U is a normal neighbourhood of x , are needed in the proofs below.

Corollary 3.2 *On every compact time-oriented sub-Lorentzian manifold there exists a closed null future directed curve.*

Proof This follows from the previous proof and from the fact that $\text{int } I^+(x, U) = \text{int } N^+(x, U) \subset N^+(x, U)$ (see [12], theorems 3.1, 3.2). $\square$

A sub-Lorentzian manifold (M, H, g) is contact, if H is a contact distribution on M . It means that $\dim M = 2n + 1$, $\text{rank } H = 2n$, and locally H can be described as the kernel of a 1-form ω such that $\omega \wedge (d\omega)^n$ is non-degenerate.

Corollary 3.3 *Suppose that (M, H, g) is a compact and contact time-oriented sub-Lorentzian manifold. Then there exists a closed piecewise smooth null future directed curve on M (with a finite number of non-smooth points).*

Proof Since M may be covered by a finite number of normal neighbourhoods, it is enough, in view of Proposition 3.5, to make the following observation. Suppose that $y \in I^+(x, U)$ for a normal neighbourhood U of x . Then y can be reached from x by a piecewise smooth null future directed curve with a finite number of non-smooth points. Indeed, take any smooth null past directed vector field Z on U . Send a trajectory of Z from y . After some time it reaches $\partial_U I^+(x, U)$ (cf. [10]). According to [12], $\partial_U I^+(x, U) = \partial_U J^+(x, U)$ is made up of piecewise smooth null future directed curves starting from x and having a finite number of non-smooth points. $\square$

Remark now important properties of closed causal curves.

Corollary 3.4 *Let Γ be a closed timelike curve. Then, for any $x \in M$, $\Gamma \cap I^+(x) \neq \emptyset$ (resp. $\Gamma \cap I^-(x) \neq \emptyset$) if and only if $\Gamma \subset I^+(x)$ (resp. $\Gamma \subset I^-(x)$). Also, for any $x, y \in \Gamma$, $I^+(x) = I^+(y)$, $I^-(x) = I^-(y)$, and in fact $I^+(x) = I^+(\Gamma)$, $I^-(x) = I^-(\Gamma)$.*

Corollary 3.5 *Let Γ be a closed null curve. Then, for any $x \in M$, $\Gamma \cap N^+(x) \neq \emptyset$ (resp. $\Gamma \cap N^-(x) \neq \emptyset$) if and only if $\Gamma \subset N^+(x)$ (resp. $\Gamma \subset N^-(x)$). Also, for any $x, y \in \Gamma$, $N^+(x) = N^+(y)$, $N^-(x) = N^-(y)$, and in fact $N^+(x) = N^+(\Gamma)$, $N^-(x) = N^-(\Gamma)$.*

Let us also note a result which is a straightforward corollary of known results in control theory (see e.g. [1], page 112).

Proposition 3.6 *Let (M, H, g) be a time-oriented sub-Lorentzian manifold. If, for a point $x \in M$, $M = \overline{I^+(x)}$, then $M = I^+(x) = J^+(x)$.*

As the next application of Theorems 1.1, 1.2 we will construct a global affine control system describing nonspacelike curves.

Let (M, H, g) be a sub-Lorentzian manifold with $\dim M = n$ and $\text{rank } H = k$. At the beginning of this section we defined nonspacelike curves. Now we will slightly generalize this definition. So a curve $\gamma : (a, b) \rightarrow M$, defined on a finite or infinite interval (a, b) , is nonspacelike if it is absolutely continuous on every compact subinterval $[\alpha, \beta] \subset (a, b)$, and $\dot{\gamma}(t) \in H_{\gamma(t)}$, $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \leq 0$, both relations taking place a.e. on (a, b) .

Let $H = H^- \oplus H^+$ be a decomposition from Theorem 1.1. By Theorem 1.2 there exist global sections $\sigma_1, \dots, \sigma_d$ of H^- , $d \leq n+1$, and global sections $\tau_1, \dots, \tau_e$ of H^+ , $e \leq (k-1)(n+1)$, which taken all together span H_x at any point $x \in M$.

Let us consider smooth functions $c_{ij}^-, c_{rs}^+ : M \rightarrow \mathbb{R}$ defined by: $c_{ij}^-(x) = g_x(\sigma_i(x), \sigma_j(x))$, $i, j = 1, \dots, d$, and $c_{rs}^+(x) = g_x(\tau_r(x), \tau_s(x))$, $r, s = 1, \dots, e$. Let $\gamma : (a, b) \rightarrow M$ be a nonspacelike curve. Then we have a global representation

$$\dot{\gamma}(t) = \sum_{i=1}^d \eta_i(t) \sigma_i(\gamma(t)) + \sum_{r=1}^e \xi_r(t) \tau_r(\gamma(t)) \quad (3.1)$$

where $\eta_i, \xi_r : (a, b) \rightarrow \mathbb{R}$ are locally integrable, and

$$\sum_{i,j=1}^d c_{ij}^-(\gamma(t)) \eta_i(t) \eta_j(t) + \sum_{r,s=1}^e c_{rs}^+(\gamma(t)) \xi_r(t) \xi_s(t) \leq 0$$

a.e. on (a, b) . Suppose further that (M, H, g) is time-oriented by a vector field $\sigma : M \rightarrow H \subset TM$. After modifying σ as in the proof of Proposition 3.1 and normalizing it, we can suppose that σ generates H^- and $g(\sigma, \sigma) = -1$. In such a case, (3.1) becomes

$$\dot{\gamma}(t) = \eta(t) \sigma(\gamma(t)) + \sum_{r=1}^e \xi_r(t) \tau_r(\gamma(t)) \quad (3.2)$$

with $\eta, \xi_r : (a, b) \longrightarrow \mathbb{R}$ locally integrable, and such that

$$-\eta^2(t) + \sum_{r,s=1}^e c_{rs}^+(\gamma(t)) \xi_r(t) \xi_s(t) \leq 0$$

a.e. on (a, b) . Finally, if in addition Γ is future directed, then (3.2) simplifies to

$$\dot{\gamma}(t) = \sigma(\gamma(t)) + \sum_{r=1}^e \xi_r(t) \tau_r(\gamma(t)) \quad (3.3)$$

with $\xi_r : (a, b) \longrightarrow \mathbb{R}$ locally integrable, and such that

$$\sum_{r,s=1}^e c_{rs}^+(\gamma(t)) \xi_r(t) \xi_s(t) \leq 1 \quad (3.4)$$

a.e. on (a, b) .

Let us observe that in general the expression (3.3) is not a control system, since the set of control parameters

$$\mathcal{U}_x = \left\{ \xi \in \mathbb{R}^e : \sum_{r,s=1}^e c_{rs}^+(x) \xi_r \xi_s \leq 1 \right\} \quad (3.5)$$

depends on a state point, and not only on time (control systems with the set of control parameters depending on time are treated for instance in [7]). Supposing that M is compact we can help it in the following way. The matrix $(c_{rs}^+(x))_{r,s=1,\dots,e}$ defines a non-negative definite quadratic form, hence there exists a number $R > 0$ such that for each $\xi \in \mathbb{R}^e$

$$0 \leq \sum_{r,s=1}^e c_{rs}^+(x) \xi_r \xi_s \leq R \left(\xi_1^2 + \dots + \xi_e^2 \right).$$

In this way we are led to the affine control system

$$\dot{x} = \sigma + \sum_{r=1}^e u_r \tau_r, \quad u \in \mathcal{U} \quad (3.6)$$

with $\mathcal{U} = \{u \in \mathbb{R}^e : \sum_{r=1}^e u_r^2 \leq \frac{1}{R}\}$. Obviously, any trajectory of the system (3.6) is a nonspacelike future directed curve on (M, H, g) . In some cases, however, for instance when $\text{rank } H = 2$ and M is time-oriented (but not necessarily compact), the trajectories of (3.6) (with $R = 1$) coincide with the set of all nonspacelike future directed curves—this follows from Proposition 3.3.

4 Proofs of Theorems 1.3, 1.4

We start from the proof of Theorem 1.4. Suppose (M, H, g) to be a compact time-oriented sub-Lorentzian manifold. The existence of closed timelike orbits was already established in Proposition 3.5. Next we have

Lemma 4.1 *Let Γ be a closed timelike curve which is not a Goh curve. Then for any $x \in \Gamma$, $x \in \text{int } I^+(x)$. In particular, $\Gamma \subset \text{int } I^+(\Gamma)$.*

Proof Since any subarc of a Goh curve is a Goh curve, it follows that there exists an $x \in \Gamma$ and a normal neighbourhood U of x such that $\Gamma \cap U$ is not a Goh curve in U . Then using [12] we know that $\Gamma \cap I^+(x, U) \subset \text{int } I^+(x, U)$. Fix a point $z \in \Gamma \cap \text{int } I^+(x, U)$ and take an open set V such that $z \in V \subset I^+(x)$. Now, for any $y \in V$ it is easy to join y from z by a timelike future directed curve: we first travel from z to x along a suitable piece of Γ , and then from x to y . Thus $z \in \text{int } I^+(z)$. Using Lemma 3.1 this last condition is equivalent to the statement that $I^+(z)$ is open. This proves the assertion since $I^+(x) = I^+(z)$ for any $x \in \Gamma$ by Corollary 3.4. $\square$

Replacing ‘future directed’ by ‘past directed’ we obtain

Lemma 4.2 *Let Γ be a closed timelike curve which is not a Goh curve. Then for any $x \in \Gamma$, $x \in \text{int } I^-(x)$. In particular, $\Gamma \subset \text{int } I^-(\Gamma)$.*

Corollary 4.1 *If Γ is a closed timelike curve which is not a Goh curve then the sets $I^+(\Gamma)$, $I^-(\Gamma)$ are open.*

Now, we finish the proof of Theorem 1.4. Fix a closed timelike curve Γ . We will show that $U = I^+(\Gamma) \cap I^-(\Gamma)$ is a neighbourhood of Γ having the desired properties. Indeed, take $x, y \in U$. Since $y \in I^+(\Gamma)$, there exists a timelike future directed curve joining a point of Γ to y ; also, since $x \in I^-(\Gamma)$, there exists a timelike future directed curve joining x to a point of Γ . Finally, any two points of Γ can be connected by a timelike future directed curve, namely a suitable piece of Γ . In this way we have constructed a timelike future directed curve joining x to y , and from the very definition of the sets $I^+(\Gamma)$, $I^-(\Gamma)$ this curve stays in U .

We proved that any two points from U can be joined by a timelike future directed curve contained in U . Let us comment now on how Theorem 1.4 can be strengthened.

First of all, instead of assuming that our closed timelike curve Γ is not a Goh curve, we can introduce a weaker condition, namely we suppose that there exist $x_1, x_2 \in \Gamma$ such that

$$\Gamma \cap I^+(x_1) \subset \text{int } I^+(x_1), \quad \Gamma \cap I^-(x_2) \subset \text{int } I^-(x_2). \quad (4.1)$$

This is exactly the condition used in the proof of lemmas 4.1, 4.2, and this is indeed a weaker assumption—see [14] for examples of Goh timelike curves satisfying condition (4.1).

Next we can replace a closed timelike future directed curve which is not a Goh curve with a closed null future directed curve Γ (which exists by Corollary 3.2) which satisfies the modification of condition (4.1): there exist points $x_1, x_2 \in \Gamma$ such that

$$\Gamma \setminus \{x_1\} \cap N^+(x_1) \subset \text{int } N^+(x_1), \quad \Gamma \setminus \{x_2\} \cap N^-(x_2) \subset \text{int } N^-(x_2). \quad (4.2)$$

Now we can achieve local controllability by null curves by repeating the above arguments, where instead of the sets $\text{int } I^\pm(y)$ we use the sets $\text{int } N^\pm(y)$. In this way we prove that $N^+(y) = N^+(\Gamma)$, $N^-(y) = N^-(\Gamma)$ are open, whenever y belongs to Γ , and that $U = N^+(\Gamma) \cap N^-(\Gamma)$ is such a neighbourhood of Γ that if $y, z \in U$ are arbitrary then there exists a null future directed curve contained in U connecting y to z . Note that condition (4.2) is never satisfied on sub-Lorentzian manifolds with rank $H = 2$.

We come to the proof of Theorem 1.3 now. Consider the control system

$$\dot{x} = -\sigma + \sum_{r=1}^e u_r(-\tau_r), \quad u \in \mathcal{U}, \quad (4.3)$$

and denote by $\mathcal{A}^+(x)$ (resp. by $\mathcal{A}^-(x)$) the reachable set from x for (1.1) (resp. for (4.3)), that is to say $\mathcal{A}^+(x)$ (resp. $\mathcal{A}^-(x)$) is the set of endpoints of all trajectories of (1.1) (resp. (4.3)) that start from x and are generated by measurable and bounded controls. Due to bracket generating hypothesis, Krener's theorem ensures that $\text{int } \mathcal{A}^+(x)$, $\text{int } \mathcal{A}^-(x)$ are non-empty for every $x \in M$. Now it is enough to repeat all above arguments replacing the sets $\text{int } I^\pm(y)$ with $\text{int } \mathcal{A}^\pm(x)$. As a result we obtain a controllable neighbourhood $U = \mathcal{A}^+(\Gamma) \cap \mathcal{A}^-(\Gamma)$ for the system (1.1). Note that if Γ is not a Goh curve then one can find points $x_1, x_2 \in \Gamma$ such that

$$\Gamma \setminus \{x_1\} \cap \mathcal{A}^+(x_1) \subset \text{int } \mathcal{A}^+(x_1), \quad \Gamma \setminus \{x_2\} \cap \mathcal{A}^-(x_2) \subset \text{int } \mathcal{A}^-(x_2).$$

Finally, remark that the compactness of M is needed only in proving the existence of closed timelike (null) curves. The proof of local controllability around a closed orbit does not require the compactness assumption.

At the very end of this paper we will explain why the proof of local controllability in Theorem 1.3 slightly generalizes the results from [22]. So, at first notice that, unlike our considerations, in [22] only smooth closed orbits are studied. Moreover, using e.g. [4] (see the formula for the image under the differential of the endpoint map) it is known that the assumptions made in [22] imply that a closed orbit in question is not abnormal with respect to H . But first of all, it is not abnormality of a closed orbit that can spoil local controllability around it but the fact that the orbit is a Goh curve (any trajectory of (1.1) that lies on $\partial \mathcal{A}^+(x)$ and is generated by controls from the set $\{u \in \mathbb{R}^e : \sum_{r=1}^e u_r^2 < \frac{1}{R}\}$ is a Goh curve for H), and secondly, these are only necessary conditions, so their satisfaction does not forjudge whether local controllability holds or does not.

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