Laborde projection in Madagascar cartography and its recovery in WGS84 datum

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Abstract The International Ellipsoid 1924 was adopted in Madagascar in 1925 for the definition of the Tananarive 1925 datum, while Laborde projection have been used for Madagascar cartography in 1926. Direct and inverse Laborde formulas are reviewed to convert cartographic and geodetic coordinates. The Hotine approximation of the Laborde projection is also discussed. From the geodetic point of view, there is the need to recover the Madagascar cartographic heritage in the WGS84 datum; datum transformations in the Molodensky form are known by literature, with the three dimensional accuracy of some meters. A new set of Helmert parameters has been estimated taking in account geoid correction and giving metric accuracy.

Keywords Laborde projection · Oblique azimuthal equidistant · Madagascar

History of Madagascar cartography

Madagascar is the fourth largest island in the world for size; its length is nearly 1,600 km and its maximum width is 576 km. The island is obliquely oriented with respect to the cardinal points, roughly in the southwest–northeast direction, and this is crucial for the development of cartography of the island in the choice of map projection (Snyder 1993). The capital of the Republic of Madagascar is Antananarivo,

in the center of the island. Among the first topographical observations, we were reminded by the campaigns between the years 1824 and 1825 by Owen, the British Royal Navy, and the subsequent campaigns of the French Navy in 1830 and 1850. Geodetic triangulation for the hydrographic surveying of the Diego Suarez Bay was initiated by colonial engineer M. Grégoire in 1887. A base line of 1,044.295 m was measured with great accuracy, and the repetition of the measure differs by only 1 mm.

The first datum was established in 1887 in the pillars of Antsirana with coordinates $\phi_0 = -12^\circ 16' 25.5''$ and $\lambda_0 = 46^\circ 57' 36.2''$ east of Paris. The orientation was established between Antsirana and the signal of the landmark of Oranjia with azimuth $\alpha_0 = 79^\circ 12' 19''$. The Antsirana 1887 datum refers to the Clarke 1880 ellipsoid, with major semi axis $a = 6,378,249.145$ m and $1/f = 293.465$. The datum used in Madagascar is presented in the Table 1.

Note that the first three datum located in the northwest of Madagascar was associated with the Hatt azimuthal equidistant projection to map the respective hydrographic surveys. Recall that the equidistant azimuthal projection has been used mainly for the polar regions, centered on the North Pole and more rarely on the South Pole. The equatorial azimuthal equidistant projection is considered but with minor interest, unless it is necessary to represent without deformation distances and directions in the area of the equator. It was used in 1895 for a map of Africa by Debes. An oblique azimuthal equidistant map projection was instead used by G. Buchanan in 1816 for a map of the hemisphere centered on London. Similarly, Debes, in 1895, still used it for a map of Europe. The equations for the ellipsoid were drawn by Philippe Hatt in 1886 using the series expansion for the calculation of the rectangular

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Table 1 Madagascar datum from the origin up today

Datum	Year	Origin		Orientation		Ellipsoid	Notes
		φ_0	λ_0 (Parigi)		α_0		
Antsirana	1887	−12° 16′ 25.5″	46° 57′ 36.2″	Oronjia	79° 12′ 19″	Clarke 1880	Hydrographic surveys
Hellville	1888	−13° 24′ 20.7″	45° 57′ 05.0″	Tany–Kely	197° 25′ 18″	Clarke 1880	Hydrographic surveys
Mojanga	1888	−15° 43′ 24.2″	43° 58′ 54.4″	?	?	Clarke 1880	Hydrographic surveys
Antsirana	1906	−12° 16′ 20.3″	46° 57′ 36.2″				
Nosy Vé	1907	−23° 39′ 00.2″	41° 16′ 44.2″				Hydrographic surveys
Hellville	1926	−13° 24′ 20.7″	45° 56′ 14.9″				
Tananarive	1925	−18° 55′ 02.1″	45° 12′ 52.4″			Int. 1924	

coordinates by azimuth and distances. This projection has been adopted by the Hellenic Military Geographic Service for the mapping of Greece to the reduction scales 1:100.000, 1:50.000, and 1:5.000 in sheets of 30′×30″, and uses a plane tangent to the emanation point on the ellipsoid according to the Map Sheet Center. The coverage of large land areas requires the definition of a large number of projection plans, each with its own system of Cartesian coordinates, whose origin coincides with the point of tangency. With the advantage of limiting the angular and areal deformations and of keeping the distances from the emanation point undistorted, there is a big disadvantage of the definition of a large number of local systems. In Madagascar, a system of quadrants for Hatt projection was implemented, but always for hydrographic surveys only. Simultaneously, however, the French army mapping agency began to use the Bonne pseudo-conic projection for topographic maps of Madagascar. The Bonne projection was a standard in Europe at that time, and it was the basis for the series of French maps *Depot de la Guerre*. The system of quadrants was used again in Madagascar, both for maps and for the calculation of the triangulations.

The Antsirana 1906 datum was an update provided by M. Lesage of the French Navy at the pillar of Antsirana, whose latitude was determined by Driencourt in 1904; while for the longitude, the measure by Fave in 1887 was considered reliable. A later document, with date unknown, relates about a Hatt local coordinate system, with a very close origin ($\varphi_0 = -12^\circ 16' 26.148''$, $\lambda_0 = 46^\circ 55' 34.669''$ east of Paris, and large translations of 80 km in east and 30 km in north directions). Even for the execution of hydrographic surveys of the area of the port of Tuléar, in southwest of Madagascar, the Nosy Vé 1907 datum was established.

Even with the Hellville 1888 datum, the Hatt azimuthal equidistant projection was used as a system of quadrants. In 1926, this datum was redefined in terms of longitude with $\lambda_0 = 48^\circ 16' 28.95''$ east of Greenwich. Presumably, this

modification was identified as Hellville 1926 datum. Although the reference origin had not been preserved, it has been observed that the origin has coordinates $x = -28.1$ and $y = -2.6$ m (note: x, y s.i.c.).

Laborde projection for Madagascar

In 1928, Laborde, a French artillery colonel, wrote, in critical terms, about the Bonne projection chosen by the Geographical Service of Madagascar that it was a bad choice as it was already in France. "Beginning in 1924, the defects of the Bonne Projection became totally inconvenient for use on the first part in geodetic work for the calculation of geodetic coordinates on the ellipsoid, and on the other part, the rectangular topographic surveys accumulated a substantial error. In 1925, regions quite removed from the Bonne Axis of projection, notably the delta region of Mangoky in the southwest part of Madagascar, attained errors of 22′ for angles and $\pm 1/580$ for lengths. It was decided to find another solution urgently so that the quality would not be compromised for the survey campaign of 1925. Pressed for time, a local provisional projection was adopted in which the distortions were practically negligible within the region of application. Furthermore, difficulty was encountered in recruiting specialized personnel, and in 1926, the triangulation operations were constrained for lack of officers. Since training had to be furnished to the personnel because of the isolation of the survey parties do to the topography; the field calculations had to be simple to employ and therefore rectangular coordinates were used. Since the Bonne Projection was useless, provisional local coordinates systems were improvised. Between 1925–26 four local projections were put into service based on the Gauss Conformal Projection.¹ The

¹ The mathematic model used for the four projections was the Gauss–Schreiber transverse Mercator.

Laborde projection, adopted in 1926, was put into full use for the reduction of the calculations of the 1926 geodetic survey campaign and for the topographic survey campaign of 1927.” (Laborde 1928).²

The projection of Laborde is a triple conformal projection, designed to have two lines represented in real size oblique to the meridian. The projection of the ellipsoid on to the plane is obtained through three steps: the first is the projection of the ellipsoid on the sphere. The second step is a Gauss–Schreiber transverse Mercator projection of the sphere on a secant cylinder. The third step of the Laborde projection is a conformal distortion of the plan through a rotation. Starting from the first step, the projection of the ellipsoid onto the sphere, Laborde followed Gauss. Gauss had shown that the ellipsoid could be mapped conformally on the sphere in such a way that the distortions in the vicinity of some point are of the third order or higher and negligible. Note also that the indirect or double projection through the sphere is not exactly the same as the direct projection because the series expansion of the formulas differs in higher order terms. The conformal projection of the ellipsoid on the sphere was used by the Prussian Land Survey between the years 1876 and 1923 as the foundation of a double projection, and is currently being used in the calculations for cadastral retracement.

This triple Laborde projection, using the Gauss sphere to obtain an oblique Mercator projection, is fundamentally different from the method used 20 and more years later by Martin Hotine. Most commercial programs simply ignore this. In reference to the four provisional coordinate systems cited by Laborde, he wrote later what relations existed between the local systems and the Laborde projection. His discussion referred to the triangulation of 1925–1926 in the region of Bemolanga–Morafenobe. The work was carried out over an area of 4,200 km² and comprised of 127 geodetic triangulation points calculated on the provisional coordinate system obtained by Gauss–Schreiber projection. Two points of a previous reconnaissance triangulation were chosen as the basis for a local grid, the grid was “une projection conforme de Gauss limitée”, a Gauss conformal projection of limited extent. We know from other Laborde notes that the transformation from this provisional system to Laborde oblique Mercator was simply a translation, a rotation, and a change of scale “sans aucune déformation”. This is one of the most remarkable properties of the conformal projection: it preserves the shapes without distortion through simple three-parameter transformations (translation in x , translation in y , and rotation). The

application of the Laborde projection for Madagascar was then a considerable simplification of the work.

The contribution of Laborde to the Service Géographique de Madagascar also led to establish a unified coordinate system for the colony. After extensive observations, the coordinates of the capital, Antananarivo, were used as the source of the Madagascar datum of 1925, where, $\varphi_0 = -18^\circ 55' 02.10''$ and $\lambda_0 = 47^\circ 33' 06.45''$ east of Greenwich. Of course, the reference ellipsoid was the International 1924. The defining parameters were published by the retired General Jean Laborde in the seminal work, “Traité des Projections des Cartes Géographiques a L’Usage des Cartographes et des Geodésiens”, by L. Driencourt and J. Laborde, Paris, 1932. Those defining parameters are:³

$\varphi_c = -21$ grad	latitude of the projection centre
$\lambda_c = 49$ grad	longitude of the projection centre (east of Paris)
$\alpha_c = 21$ grad	azimuth (true) of the initial line passing through the projection centre
$k_c = 0.9995$	scale factor
$FE = 400$ km	false easting at the natural origin
$FN = 800$ km	false northing at the natural origin

All angular units are in centesimal degrees and should be converted to radians prior to use; all longitudes should be reduced to the Paris Meridian using the Paris Longitude of 2.5969212963 grads ($2^\circ 20' 14.025''$ E) east of Greenwich. The reduced longitude is given by

$$\lambda_P = \lambda_G - \pi \cdot \frac{2.5969212963}{200}$$

where, λ_G is the longitude east of Greenwich expressed in radians. The reference ellipsoid for Madagascar is the International 1924 ellipsoid whose parameters are:

$$\begin{aligned} a &= 6,378,388 \text{ m} \\ e &= 0.081991890 \\ 1/f &= 297 \\ e^2 &= 0.006722670 \end{aligned}$$

and oriented on Tananarive (Tananarive 1925 datum).

² Translation from French language in (Mugnier J. C. 2000).

³ If the oblique Mercator method is used as an approximation to the Laborde Madagascar, the additional parameter required by that method, the angle from the rectified grid to the skew (oblique) grid γ_c takes the same value as the azimuth of the initial line passing through the projection centre, α_c .

From these defining parameters, the following constants for the map projection may be calculated (Mugnier 1982 and OGP 2007):

$$B = \sqrt{1 + \frac{e^2 \cos^4 \varphi_c}{1 - e^2}}$$

$$\varphi_s = \arcsin\left(\frac{\sin \varphi_c}{B}\right)$$

$$R = a \cdot k_c \left(\frac{\sqrt{1 - e^2}}{1 - e^2 \sin^2 \varphi_c} \right)$$

$$C = \ln \left[\tan \left(\frac{\pi}{4} + \frac{\varphi_s}{2} \right) \right] - B \cdot \ln \left[\tan \left(\frac{\pi}{4} + \frac{\varphi_c}{2} \right) \cdot \left(\frac{1 - e \sin \varphi_c}{1 + e \sin \varphi_c} \right)^{e/2} \right]$$

Direct case

To compute the cartographic coordinates (E, N) from the given ellipsoidal coordinates (φ , λ):

$$L = B(\lambda - \lambda_c)$$

$$q = C + B \cdot \ln \left[\tan \left(\frac{\pi}{4} + \frac{\varphi}{2} \right) \cdot \left(\frac{1 - e \sin \varphi}{1 + e \sin \varphi} \right)^{e/2} \right]$$

$$P = 2 \cdot \arctan(e^q) - \frac{\pi}{2}$$

where, e is the base of natural logarithms

$$U = \cos P \cdot \cos L \cdot \cos \varphi_s + \sin P \cdot \sin \varphi_s$$

$$V = \cos P \cdot \cos L \cdot \sin \varphi_s - \sin P \cdot \cos \varphi_s$$

$$W = \cos P \cdot \sin L$$

$$d = \sqrt{U^2 + V^2}$$

$$\begin{aligned} \text{if } d \neq 0, \text{ then } L' &= 2 \cdot \arctan\left(\frac{V}{U+d}\right) \text{ and } P' = \arctan\left(\frac{W}{d}\right) \\ \text{if } d = 0, \text{ then } L' &= 0 \text{ and } P' = \text{sign}(W) \cdot \frac{\pi}{2} \end{aligned}$$

$$H = -L' + i \cdot \ln \left[\tan \left(\frac{\pi}{4} + \frac{P'}{2} \right) \right]$$

where, i is the imaginary unit

$$G = \frac{1 - \cos 2\alpha_c + i \sin 2\alpha_c}{12}$$

$$E = E_c + R \cdot \text{imag}(H + GH^3)$$

$$N = N_c + R \cdot \text{real}(H + GH^3)$$

Inverse case

To compute the ellipsoidal coordinates (φ , λ) given the cartographic coordinates (E, N), it is required that the complex parameter G is already computed in the direct case and a re-iterative solution for the other complex parameter H whose starting values are:

$$H_0 = \frac{N - FN}{R} + i \frac{E - FE}{R} \quad k = 0$$

$$H_1 = \frac{H_0}{H_0 + GH_0^3} \quad k = 1$$

$$H_{k+1} = \frac{H_0 + 2GH_k^3}{1 + 3GH_k^2}$$

Re-iterate until $|\text{real}(H_0 - H_k - GH_k^3)| \leq 1E - 11$.

$$L' = -1 \cdot \text{real}(H_k)$$

$$P' = 2 \cdot \arctan\left(e^{\text{imag}(H_k)}\right) - \frac{\pi}{2}$$

where, e is the base of natural logarithms

$$U' = \cos P' \cdot \cos L' \cdot \cos \varphi_s + \cos P' \cdot \sin L' \cdot \sin \varphi_s$$

$$V' = \sin P'$$

$$W' = \cos P' \cdot \cos L' \cdot \sin \varphi_s - \cos P' \cdot \sin L' \cdot \cos \varphi_s$$

$$d = \sqrt{U'^2 + V'^2}$$

$$\begin{aligned} \text{if } d \neq 0, \text{ then } L &= 2 \cdot \arctan\left(\frac{V'}{U'+d}\right) \text{ and } P = \arctan\left(\frac{W'}{d}\right) \\ \text{if } d = 0, \text{ then } L &= 0 \text{ and } P = \text{sign}(W') \cdot \frac{\pi}{2} \end{aligned}$$

We can now compute the longitude by referring to the Paris meridian:

$$\lambda = \lambda_c + \frac{L}{B}$$

$$q' = \frac{\ln\left[\tan\left(\frac{\pi}{4} + \frac{P}{2}\right)\right] - C}{B}$$

The final solution for latitude requires a second re-iterative process, where the first element is

$$\phi'_0 = 2 \arctan\left(e^{q'}\right) - \frac{\pi}{2}$$

where, e is the base of natural logarithms

$$\phi'_k = 2 \arctan\left[\left(\frac{1 + e \sin \phi'_{k-1}}{1 - e \sin \phi'_{k-1}}\right)^{e/2} \cdot e^{q'}\right] - \frac{\pi}{2}$$

Iterate until $|\phi'_k - \phi'_{k-1}| \leq 1\text{E} - 11$, and finally we obtain

$$\phi = \phi'_k.$$

Approximation with the oblique Mercator projection

Within 450 km from the center of projection, in Antananarivo, the Laborde formulas can be approximated by Hotine rectified skew orthomorphic oblique Mercator projection with deviations of less than 2 cm (Hotine 1946–47). But beyond this limit, the two projections diverge rapidly and the differences are of the order of 1 m already at a distance of 600 km, and exceed 1.5 m to the limits of the Madagascar territory. The approximation of the Laborde projection by the Hotine can, therefore, be useful for medium-scale cartographic application, but not for geodetic calculations or for large-scale engineering mapping although it is implemented in several commercial softwares.

Table 2 Laborde cartographic coordinates and geodetic coordinates, datum Tananarive 1925. Orthometric heights are reported

	E	N	H	φ		λ			
Orangea	717,742.200	1,532,648.160	116.556	−12	15	19.725	47	1	0.198
Mahajanga	390,871.180	1,152,710.110	48.561	−15	42	42.063	44	0	51.447
Maintirano	144,319.180	891,125.190	29.032	−18	3	39.709	41	41	4.397
Morondava	174,896.770	644,434.630	31.421	−20	17	34.191	41	56	38.745
Teadro	129,292.030	293,987.700	207.476	−23	27	7.159	41	27	3.900
Ambohibe	520,006.790	801,977.670	1458.372	−18	52	43.155	45	14	21.842
Lakaria	718,010.410	964,103.520	46.593	−17	23	38.591	47	5	36.782
Manakara	563,292.180	440,734.460	2.376	−22	8	17.587	45	41	0.330
Resampy	439,502.040	122,810.214	54.547	−25	0	58.563	44	29	42.513

Transformation to WGS84 datum

This work stems from the need to recover geodetic and cartographic heritage of Madagascar in the context of modern satellite positioning, and then moving to the WGS84 datum. The Foiben-Taosarintanin'i Madagasikara (Institut Géographique et Hydrographique National de Madagascar) does not publish official transformation parameters at present, but only the monographs of nine geodetic control points of the zero-order geodetic network recently calculated in the WGS84 datum.

To date, the only parameters of transformation characterized by a degree of formality have been published by the U.S. National Imagery and Mapping Agency (NIMA) for the transition from the Tananarive 1925 datum to WGS84 datum by the Molodensky formulas. Unfortunately, we do not know anything, neither the accuracy of these parameters nor the way they have been calculated. Even in Mugnier J. C. (2000), an unofficial solution that is slightly different is reported and is always compatible with the Molodensky transformation. A seven-parameter Helmert datum transformation has never been estimated before the present work.

To solve this problem, we know nine points with cartographic coordinates in Laborde projection, referred to the Tananarive 1925 datum, and their orthometric heights. We also know the WGS84 geodetic coordinates in the ITRF94 realization and their ellipsoidal heights.

The well-known rigorous approach requires:

1. In the starting datum RF₁:
 - a. to transform the map coordinates (E , N) into geographic coordinates (ϕ_I , λ_I);
 - b. to transform orthometric heights (H_I) to ellipsoidal (h_I) using geoid undulation N_1 ;
 - c. to transform geodetic (ϕ_I , λ_I , and h_I) associated to the ellipsoid Σ_I into geocentric coordinates (X_I , Y_I , and Z_I);

Table 3 Geodetic coordinates, datum ITRF94. Ellipsoidal and orthometric heights are reported, where N is the geoid undulation

	φ			λ			h	H	N
Orangea	-12	15	24.153	49	21	14.026	96.023	118.307	-22.284
Mahajanga	-15	42	46.590	46	21	4.581	31.588	49.834	-18.246
Maintirano	-18	3	44.476	44	1	17.137	19.328	30.275	-10.947
Morondava	-20	17	39.079	44	16	51.516	27.771	32.977	-5.206
Teadro	-23	27	12.324	43	47	16.715	211.530	208.732	2.798
Ambohibe	-18	52	47.898	47	34	35.226	1457.941	1458.872	-0.931
Lakaria	-17	23	43.249	49	25	50.497	36.749	47.218	-10.469
Manakara	-22	8	22.440	48	1	14.180	4.131	4.276	-0.145
Resampy	-25	1	3.742	46	49	56.013	61.811	54.692	7.119

2. To estimate the datum transformation Π using the geocentric coordinates of points known in both datum, $(X_1, Y_1, \text{ and } Z_1)$ and $(X_2, Y_2, \text{ and } Z_2)$, and compute the transformed coordinates (X'_2, Y'_2, Z'_2) .
3. In the final datum RF_2 :
 - a. to transform geocentric (X'_2, Y'_2, Z'_2) into the geodetic coordinates $(\phi'_2, \lambda'_2, h'_2)$ associated to the ellipsoid Σ_2 ;
 - b. to transform ellipsoidal heights (h'_2) to orthometric (H'_2) using geoid undulation N_2 .

The disadvantage of the 3-D rigorous approach is that for the common points, ellipsoidal heights are required, while often, only orthometric heights are available. Also, in the present case, unfortunately, it is not possible to rigorously perform step 1a because we do not know N_1 or a geoid model associated to the Tananarive 1924 datum.

In the evolution process towards a 3-D geodesy, the problem is not new. In a local bi-dimensional geodetic network, ellipsoidal latitude and longitude are available in a local reference system. From a global 3-D geodetic network, ellipsoidal latitude, longitude, and height are known in a global reference system. It has to be emphasized that due to the older separation of horizontal and vertical reference surface, the ellipsoidal height h_1 in the local coordinate system is not available. The differential transformation between curvilinear geodetic coordinates have been treated by Molodensky et al. (1960), Soler (1976), and Grafarend et al. (1995). While Molodensky gave differential equations equivalent to the conventional similarity transformation and Soler developed the differential transformations between Cartesian and curvilinear orthogonal coordinates, Grafarend

formulated the curvilinear datum transformation for ellipsoidal latitude and longitude (ϕ_1, λ_1) in the local reference system RF_1 as a function of the ellipsoidal coordinates $(\phi'_2, \lambda'_2, h'_2)$ in the global reference frame RF_2 .

In the present work, three possible algorithms have been carried out and tested:

- the proposed adaptation of the rigorous seven-parameter Helmert 3-D transformation,
- the standard five-parameter Molodensky formulas, and
- the nine-parameter Grafarend formulas.

Adapted Helmert algorithm

As reported by Schmitt et al. (1991), incorrect heights of the common points often have a negligible effect on the plane coordinates estimated by Helmert transformation. However, to obtain small residuals for the height component, it is necessary to give an approximate value for the geoid undulations. In step 3b, we can use the geoid undulations extrapolated by the EGM96 model that refer to the WGS84 (G873) ellipsoid. The accuracy of WGS84 (G873), with respect to ITRF94, is about 5 cm which is enough for our cartographic purposes. ITRF94 orthometric heights are reported in Table 3. Comparing heights in Tables 2 and 3, it is clear that Laborde cartographic coordinates are associated with orthometric heights, which is obvious even if not declared.

Even if we cannot know N_1 , we can approximate it with N_2 or refer the EGM96 geoid model to the Tananarive 1924 datum using the inverse transformation Π^{-1} . Because Π is unknown a priori, it is necessary to modify the rigorous approach as follows:

1. In the starting datum, RF_1 transform the orthometric heights (H_1) to ellipsoidal (h_1) using as first approximation of N_1 the geoid undulation N_2 ;
2. Estimate the datum transformation Π .
3. Apply the inverse transformation Π^{-1} to the EGM96 model described in RF_2 by the points (ϕ_2, λ_2, N_2) , obtaining a second approximation of N_1 .

Table 4 Height RMS in meters using different geoid corrections

	σ_E	σ_N	σ_h
No geoid correction	0.48	1.04	2.81
Correction at step 2	0.56	1.01	0.24
Correction at step 5	0.57	1.03	0.20

Table 5 The estimated Helmert parameters

	Without outlier rejection (nine parameter)			With outlier rejection (six parameter)		
TX	−173.24011	±22.9779	m	−164.48510	±11.4646	m
TY	−194.63775	±19.4813	m	−211.62672	±11.0365	m
TZ	41.88801	±14.3030	m	46.92458	±6.4894	m
Rotation X	−3.06419	±0.39956	"	−2.48367	±0.22212	"
Rotation Y	4.19445	±0.52337	"	4.90223	±0.23754	"
Rotation Z	−0.77665	±0.85702	"	−0.53688	±0.45177	"
scale	3.13400	±1.8307	ppm	4.27741	±0.9354	ppm

4. Transform the orthometric heights (H_1) to ellipsoidal (h_1) using the new approximation of N_1
5. Re-estimate the datum transformation Π .

It is not possible to use this procedure recursively on local or regional networks because of the very high correlation between the geoid undulation and the translation of the ellipsoid origin that results in a slow divergence of the estimated translation. In the present case, the height root-mean-square (RMS) without geoid undulation correction is ~ 2.81 m, with undulation correction at step 2 is ~ 0.24 m, and at step 5 is ~ 0.20 m. Note that in Table 4, as previously emphasized, height correction of the common points has a very small effect on horizontal residuals.

The inverse datum transformation Π^{-1} must be used to refer the EGM96 geoid model to the Tananarive 1924 datum in order to obtain the ellipsoidal heights of cartographic points before applying the direct transformation Π (Table 5).

We have estimated the seven parameters of Helmert transformation by Procrustean analysis (Awange and Grafarend 2005). At the beginning, all the nine common points have been used, then, applying outlier rejection, only those that had deviations lower than 2 m have been used. Finally, the residuals in both cases are reported (Table 6).

Standard five-parameter Molodensky

Molodensky parameters for the transformation from Tananarive 1925 to WGS84 are published by (NIMA 2000), Mugnier (2000), and by Institut Géographique National (IGN). All these parameter sets have been estimated to transform orthometric heights directly to ellipsoidal, neglecting the geoid correction. For comparison purposes, I used the nine common points to compute a standard Molodensky transformation, obtaining similar results (Fig 1). This solution is reported as 3-D Molodensky in Table 7. The transformation parameters are t_x , t_y , t_z , $\delta\alpha$, and δf , where t_x , t_y , and t_z are the coordinate of the origin of RF₂ with respect to RF₁, $\delta\alpha$ is the change in the semi-major axes of the reference ellipsoid and δf is the change of its flattening. Only t_x , t_y , and t_z have to be estimated, while $\delta\alpha$ and δf are given.

Nevertheless, taking into account orthometric heights to estimate the transformation parameters is incorrect. It is possible to apply Molodensky formulas to latitude and longitude (maybe including also the height equation with a low weight in the least squares adjustment) and estimate the three translation of the origin to be used in transforming latitude and longitude only. This solution is reported as 2D+1D Molodensky in Table 7. Ellipsoidal heights can be

Table 6 Residuals of the transformations without outlier rejection and with outlier rejection. The residuals are in meters

	Without outlier rejection (nine parameter)			With outlier rejection (six parameter)		
	ΔE	ΔN	Δh	ΔE	ΔN	Δh
Orangea	−3.16	5.13	0.24	−5.93	9.63	1.04
Mahajanga	1.57	−2.07	0.03	0.52	1.25	0.12
Maintirano	−0.53	−2.18	0.23	−0.36	0.41	−0.26
Morondava	0.15	−3.31	0.60	0.49	−1.69	0.29
Teadro	−1.46	−0.11	0.45	−0.56	0.29	0.14
Ambohibe	0.38	−2.84	−0.74	−0.75	−1.03	−0.13
Lakaria	−1.03	−2.18	−0.77	−3.10	0.03	0.32
Manakara	3.56	6.27	0.75	2.73	6.68	1.63
Resampy	0.66	1.36	−0.80	0.66	0.77	−0.17
RMS	1.80	3.34	0.58	0.57	1.03	0.20

Fig. 1 Differences ΔE , ΔN , and ΔP between Laborde and oblique Mercator projection for Madagascar area. The nine common points used to estimate the datum transformation are represented

derived roughly using orthometric heights and the geoid undulations related to the RF_2 :

$$h'_2 = H_1 + N_2 + \delta h$$

The parameter δh have to be estimated in order to have zero mean residuals ($h'_2 - h_2$).

Grafarend nine-parameter approach

The Grafarend curvilinear datum transformation for ellipsoidal latitude and longitude (ϕ_1, λ_1) in the local reference

system, RF_1 is formulated as a function of the ellipsoidal coordinates (ϕ_2, λ_2 , and h_2) in the global reference frame, RF_2 . The curvilinear coordinate transformation of conformal type (datum transformation) extended by ellipsoidal parameters is given by

$$y = \begin{bmatrix} \lambda_1 - \lambda_2 \\ \phi_1 - \phi_2 \end{bmatrix} = A \cdot [t_x \quad t_y \quad t_z \quad \alpha \quad \beta \quad \gamma \quad s \quad \delta a_2 \quad \delta f_2]^T$$

where, A is the Jacobi matrix, which contains the partial derivative of ellipsoidal coordinates with respect to the datum parameters as well as ellipsoidal form parameters.

Table 7 Transformation parameters reported by literature compared and statistics on residuals. The residuals of the Molodensky transformation are computed on all the nine common points. The

3-D Molodensky have been estimated by least square adjusting of standard Molodensky formulas. The 2D+1D Molodensky by least square adjustment of latitude and longitude formulas only

			NIMA 2000	Mugnier 2000	IGN 2008	Roggero 2008	
			Molodensky	Molodensky	Molodensky	3-D Molodensky	2D+1D Molodensky
Par.	Translation [m]	x	-189	-191.745	-179	-191.145	-198.383
		y	-242	-226.365	-229	-232.756	-240.517
		z	-91	-115.609	-109	-111.533	-107.909
Residuals	Mean [m]	φ	6.32	-13.84	-5.30	0.08	-0.01
		λ	-4.42	4.90	-6.18	0.06	0.00
		h	-9.70	8.00	11.59	-0.06	0.00
		3-D	12.39	16.72	14.16	0.03	0.00
	RMS [m]	φ	3.10	2.43	2.37	2.24	2.22
		λ	11.60	4.18	4.28	4.19	4.11
		h	3.01	4.38	3.27	2.97	0.56
		3-D	12.37	6.53	5.88	3.23	2.72

Table 8 Residual RMS of the Grafarend and Helmert transformations compared

			Grafarend	Helmert
Residuals	RMS [m]	φ	0.70	0.57
		λ	0.72	1.03
		h	0.56	0.20
		3-D	0.66	1.19

The incremental scale s and the incremental semi-major axis δa_2 cannot be determined independently, but δa_2 and δf_2 can be fixed and removed from the analysis.

The inverse curvilinear datum transformation is given by

$$\begin{bmatrix} \lambda'_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \varphi_1 \end{bmatrix} + B \cdot [t_x \ t_y \ t_z \ \alpha \ \beta \ \gamma \ s \ \delta a_1 \ \delta f_1]^T = \begin{bmatrix} \lambda_1 \\ \varphi_1 \end{bmatrix} + B \cdot x$$

The matrix B contains the partial derivative of the ellipsoidal coordinates in RF_1 . Since in the local reference system the ellipsoidal heights h_1 are in general not available, the matrix B can be decomposed in $B = B_0 + h_1 B_1$. It can be numerically shown that the impact of the term $h_1 B_1$ is of second order and it can be neglected, obtaining a formula that does not depend on h_1 . See Grafarend et al. (1995) for the details on the method. For geodetic networks of regional extension, Grafarend also experienced configurations close to a collinearity (near linear dependence), and discussed the methods to overcoming the problem.

Orthometric heights have been derived roughly as in the previous 2D+1D approach. Kotsakis (2008) discussed how to transform the geoid undulation from one geodetic

Table 9 Transformation residuals

	Grafarend			Helmert		
	ΔE	ΔN	Δh	ΔE	ΔN	Δh
Orangea	-0.67	0.49	-0.61	-5.93	9.63	1.04
Mahajanga	1.28	-0.54	-0.13	0.52	1.25	0.12
Maintirano	-1.07	1.41	-0.10	-0.36	0.41	-0.26
Morondava	0.57	-0.51	-0.42	0.49	-1.69	0.29
Teadro	-0.35	-0.55	-0.12	-0.56	0.29	0.14
Ambohibe	0.44	-0.29	0.64	-0.75	-1.03	-0.13
Lakaria	-0.13	-0.21	0.51	-3.10	0.03	0.32
Manakara	-8.20	-2.46	-0.76	2.73	6.68	1.63
Resampy	-0.08	0.99	0.99	0.66	0.77	-0.17
RMS	0.72	0.70	0.66	0.57	1.03	0.20

The discarded points are in gray

reference frame to another, however, assuming the Helmert transformation parameters known a priori. In Perozzi and Surace (2000), it is also emphasized that the three Molodensky formulas are largely uncorrelated, allowing to separate the horizontal and vertical components in the adjustment. So, the Molodensky model is also coherent with the separation of the two components in the classical geodetic approach, in which the horizontal and vertical positions are measured and adjusted independently.

The results are shown in Table 8 and compared with the adapted Helmert results. Note that because of the use of the ellipsoidal latitude and longitude equation only, the datum transformation parameters are not comparable with the Helmert parameters and are not reported. On the contrary, using the 3-D curvilinear coordinate transformation equations in latitude, longitude, and height, the estimated parameters are equal to the Helmert parameters when the ellipsoidal heights are available in both reference systems. Such an approach cannot be used in the present work, but it is possible to include in the adjustment the height equation with a low weight in the adjustment.

Outlier rejection has been performed on the residuals, discarding the point Manakara. The final residuals are reported in Table 9. Note that the points Orangea and Lakaria, discarded in the Helmert approach, are now accepted and show small residuals.

Implementation

The algorithms discussed in the present work are implemented in FORTRAN 90 programming language and included in the software TOPO.⁴ Helmert transformation parameters are estimated by Procrustes, while Molodensky and Grafarend approach transformation parameters are estimated by least squares. All the algorithms have been tested estimating known transformations between International Terrestrial Reference Frames (ITRFs), then applied on the Madagascar case study.

Conclusions

To convert Laborde cartographic coordinates to geodetic coordinates, the Laborde projection equations have been implemented for the direct and inverse case, avoiding the use of the Hotine approximation, usually implemented in the commercial software. Orthometric heights have been converted to ellipsoidal by means of an empirical localization of a global geoid model. Then, the Helmert transfor-

⁴ TOPO 1.10 software is developed by the author and freely available on <http://antartica60.spaces.live.com/>

mation parameters have been estimated by Procrustean analysis, with horizontal RMS of 1.2 m and vertical RMS of 20 cm; the rejected outlier horizontal residuals are of the order of some meters. For comparison with previously published results, the three origin translation parameters of a standard Molodensky transformation have been estimated, but a more rigorous solution is possible applying the Grafarend curvilinear datum transformation. This approach has given horizontal RMS of 1 m and vertical RMS of 50 cm. The modified Helmert and the Grafarend solutions seem to produce comparable results, providing metric accuracy, and are suitable for cartography recovering at the reduction scales up to 1:5,000.

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