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Volume Growth Estimates of Gradient Ricci Solitons

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Abstract

In this paper, we survey the volume growth estimates for shrinking, steady, and expanding gradient Ricci solitons. Together with the known results, we also prove some new volume growth estimates for expanding gradient Ricci solitons.

Keywords Volume estimates · Shrinking · Steady · Expanding gradient Ricci solitons

Mathematics Subject Classification 53C25

1 Introduction

Ever since Hamilton [37] invented the Ricci flow, the study of gradient Ricci solitons has been an central element of this field. Up to this point, gradient Ricci solitons have already been extensively studied. Many classic results have been proved, and some played an important role in the whole field of Ricci flow. For instance, Ricci shrinkers can be classified completely in dimension two (c.f. [6, 38, 39]) and in dimension three (c.f. [39, §26], [59, Lemma 1.2], [11, 57, 60]), and with additional conditions in higher dimensions (c.f. [13, 14, 44, 54, 55]). Steady solitons also admit a classification in dimension two (c.f. [6, 39]). In dimension three, the only noncollapsed steady soliton

Dedicated to Professor Peter Li on the occasion of his seventieth birthday.

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is the Bryant soliton [7], although there is at least one collapsed example [41]. In higher dimensions, more examples of noncollapsed steady solitons are found by Appleton [1] and Lai [41]. The case of expanding solitons is much less clear, since even in dimension 3, one may find a one-parameter family of Ricci expanders [28].

When a classification is not available, one naturally turns to consider the qualitative or quantitative properties of a general soliton, in hope that the analytic properties may have some implications on the geometric properties, or vice versa. Research results in this respect are massive. It is, therefore, impossible to include every aspect of the field within one survey paper. Thus, we shall focus on one important aspect—that of volume estimates. In this paper, we shall survey volume estimates of all three kinds of Ricci solitons. Furthermore, we prove some new volume growth estimates for Ricci expanders. In particular, our estimate in dimension three is sharp, since it shows that every three-dimensional expander with bounded scalar curvature and proper potential function has a volume growth rate no greater than r^3, and this rate is satisfied by the Gaussian expander.

2 Volume Estimates for Shrinking Gradient Ricci Solitons

In this section, we consider a complete shrinking gradient Ricci soliton (M^n, g, f) normalized in the way that

$$\begin{gathered}\mathrm{Ric} + \nabla^2 f = \tfrac{1}{2} g. \\ \int_M (4\pi)^{-\frac{n}{2}} e^{-f} \, dg = 1.\end{gathered} \tag{2.1}$$

Because of the logarithmic Sobolev inequality of Bakry-Emery [2], Carrillo-Ni [17] observed that, for a shrinker normalized as in (2.1), the constant

$$\mu_g := 2\Delta f - |\nabla f|^2 + R + f - n \tag{2.2}$$

is equal to Perelman's functional $\mu(g, 1)$ defined in (2.4).

2.1 The Triviality of the Entropy on a Ricci Shrinker

Obviously, if we consider the canonical form $(M^n, g_t)_{t\in(-\infty,0)}$ of a Ricci shrinker (M^n, g, f), where

$$g_t := -t\phi^*_{\log(-t)} g, \quad \partial_s \phi_s = -\nabla f \circ \phi_s, \quad \phi_0 = \mathrm{id},$$

then its asymptotic shrinker in the sense of Perelman [58, Proposition 11.2] is the shrinker itself.

On the other hand, the Nash entropy on an ancient solution always converges to the entropy of its asymptotic (metric) soliton (c.f. [4, 46]). So if the ancient solution in question is the canonical form (M, g_t), then the Nash entropy converges to the quantity

μ_g defined in formula (2.2). We, thus, observe that the Nash entropy on the canonical form of a shrinker is always bounded from below. In particular, for any $x \in M$, we have

$$\inf_{\tau>0} \mathcal{N}_{x,-1}(\tau) = \mu_g > -\infty. \tag{2.3}$$

Therefore, we may regard (the canonical form of) a Ricci shrinker as an ancient Ricci flow with bounded Nash entropy, and many results in [3, 4] can be applied in this case. This provides a perspective of understanding some classical results. Here, we remind the reader that although in general, a shrinker is not known to have bounded curvature, one may still apply the cut-off argument and heat kernel gaussian estimates of Li–Wang [45] to carry out many of Bamler's proofs. Let us first of all recall the uniform Sobolev inequality of Li–Wang [45]. In fact, a uniform Sobolev inequality is equivalent to a lower bound of Perelman's ν-functional defined as follows:

$$\begin{aligned} \overline{\mathcal{W}}(g, u, \tau) &:= \int_M \tau\left(\frac{|\nabla u|^2}{u} + Ru\right) dg - \int_M u \log u dg - n - \frac{n}{2}\log(4\pi\tau), \\ \mu(g,\tau) &:= \inf\left\{\overline{\mathcal{W}}(g,u,\tau) \,\middle|\, u \geq 0,\ \sqrt{u} \in C_0^\infty(M),\ \int_M u dg = 1\right\}, \\ \nu(g) &:= \inf_{\tau>0} \mu(g,\tau). \end{aligned} \tag{2.4}$$

Theorem 2.1 [45] *Let (M^n, g, f) be a shrinker normalized as in (2.1). Then we have*

$$\nu(g) := \inf_{\tau>0} \mu(g,\tau) = \mu_g,$$

where μ_g is the constant in (2.2).

We briefly describe the idea of the proof. One may consider the canonical form $(M^n, g_t)_{t\in(-\infty,0)}$ and estimate $\nu(g_{-1})$. Let $\tau > 0$ be any number and let u be any test function for $\mu(g_{-1}, \tau)$. Define

$$u_t(x) := \int_M K(x,t\,|\,\cdot,-1)u\, dg_{-1}, \quad \tau_t = \tau - 1 - t, \quad t \leq -1,$$

where K is the conjugate heat kernel. Then we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathcal{W}}(g_t, u_t, \tau_t) \geq 0, \quad \lim_{t\to-1} \overline{\mathcal{W}}(g_t,u_t,\tau_t) = \overline{\mathcal{W}}(g,u,\tau), \\ &\lim_{t\to-\infty} \overline{\mathcal{W}}(g_t,u_t,\tau_t) = \mu_g. \end{aligned} \tag{2.5}$$

To see why the last equation is true, one may refer to [20, Section 9]. In fact, when $-t \to \infty$, the difference between $-t$ and τ_t is negligible. Now, apparently, (2.5) implies Theorem 2.1. To make the argument rigorous, one may find the cut-off argument and heat kernel gaussian bounds in [45] helpful.

2.2 Upper Volume Growth Estimate

Cao–Zhou [16] proved the following nice growth estimates for the shrinker potential f (see also [40]). These estimates show that, to estimate the volume growth rate, one may consider the sub-level sets of f instead of geodesic disks; the former proves to be much more analytically tractable than the latter.

Theorem 2.2 [16] *Let* (M^n, g, f) *be a Ricci shrinker normalized as in* (2.1)*, then we have*

$$\frac{1}{4}\left(\mathrm{dist}(x,p)-5n\right)_+^2 \le f(x)-\mu_g \le \frac{1}{4}\left(\mathrm{dist}(x,p)+\sqrt{2n}\right)^2,$$

where p *is the point where* f *attains its minimum and* μ_g *is the quantity in* (2.2).

Following the idea mentioned at the beginning of this subsection, let $\mathcal{V}(s)$ and $\mathcal{R}(s)$ be the functions

$$\mathcal{V}(s)=\int_{\{f<\mu_g+s\}} dg \quad \text{and} \quad \mathcal{R}(s)=\int_{\{f<\mu_g+s\}} R\, dg. \tag{2.6}$$

By integrating the trace of (2.1), Cao–Zhou [16] proved the following differential equation for $\mathcal{V}$ and $\mathcal{R}$:

$$\tfrac{n}{2}\mathcal{V}(s)-\mathcal{R}(s)=s\mathcal{V}'(s)-\mathcal{R}'(s). \tag{2.7}$$

Using the above equation, Cao–Zhou [16] and Munteanu [48] showed that a complete gradient shrinker has at most Euclidean volume growth (see also [40]). The volume estimate is sharp on the Gaussian shrinker and on the conical shrinker constructed by Feldman–Ilmanen–Knopf [36]. See also the generalization by Munteanu–Wang [50] to smooth metric measure space with $\mathrm{Ric}_f \ge 1/2$.

Theorem 2.3 [16, 48, 50] *Let* (M^n, g, f) *be a complete shrinking gradient Ricci soliton normalized as in* (2.1)*. Then there is a dimensional constant* $c = c(n)$ *such that for any* $p \in M$ *and* $r > 0$,

$$|B_r(p)| \le c e^{f(p)-\mu_g} r^n. \tag{2.8}$$

As pointed out in [40], if the center p is chosen to be the minimum point of f, then it follows from (2.2) that $f(p)-\mu_g \le n/2$ and the constant on the R.H.S. of (2.8) can be made dimensional. However, the recent work of Bamler [3] implies a stronger result than Theorem 2.3. Since shrinkers all have nonnegative scalar curvature (c.f. [23]), one may apply the result of [3, Theorem 8.1] to Ricci shrinkers and show that, based at any point, the volume growth rate has an Euclidean upper bound. One may use the techniques in [45] to make this argument rigorous. Here, we only present the following conclusion.

Theorem 2.4 [3, Theorem 8.1] *Let (M^n, g, f) be a (not necessarily normalized) Ricci shrinker. Then for any point $x \in M$ and any $r > 0$, we have*

$$|B_r(x)| \leq C(n)r^n,$$

where $C(n)$ is a dimensional constant.

A common class of shrinkers which model the nondegenerate neck pinch of the Ricci flow is that of the cylindrical shrinkers. Their volumes grow much slower than the Euclidean space. Under the assumption of positive lower bound of the scalar curvature, Zhang [67] established an upper volume bound for gradient shrinkers (see also [17] for upper estimates under the bounded and nonnegative Ricci curvature conditions, and [26] under the assumption of integral upper bound of R). This estimate is optimal because of the cylindrical examples $\mathbb{R}^{n-k} \times \mathbb{S}^k$ for $k \geq 2$.

Theorem 2.5 [67] *Suppose that (M^n, g, f) is a complete shrinking gradient Ricci soliton normalized as in (2.1) with scalar curvature bounded from below by a positive constant ε, i.e., $R \geq \varepsilon$ on M. Then for all $p \in M$, there exists a constant C such that for all $r \geq 0$*

$$|B_r(p)| \leq Cr^{n-2\varepsilon}.$$

Ni [56] showed that any nontrivial complete gradient Ricci shrinker with nonnegative Ricci curvature has scalar curvature uniformly bounded from below by a positive constant. In particular, it follows from Theorem 2.5 that such a shrinker has zero asymptotic volume ratio. This was also proved by Carrillo–Ni [17, Proposition 2.1].

2.3 Lower Volume Growth Estimate

Munteanu–Wang [49] first proved that the volume growth rate of a shrinking gradient Ricci soliton is at least linear. This rate is optimal since it is fulfilled by the standard round cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. However, their estimation constant $C(n)e^{c_1 \mu_g}$, where $c_1 = c_1(n) > 1$, is not optimal. Later, by applying their uniform Sobolev inequality on shrinking solitons, Li–Wang [45] improved the estimation constant of [49].

Theorem 2.6 (Lower volume growth estimate for shrinkers) *Let (M^n, g, f) be a complete noncompact shrinking gradient Ricci soliton normalized as in* (2.1). *There exist dimensional constants $c_0 = c_0(n)$ and $r_0 = r_0(n)$ such that for all $r \geq r_0$,*

$$|B_r(p)| \geq c_0 e^{\mu_g} r.$$

We briefly summarize the proof of Theorem 2.6. Consider the set $D(s) := \{2\sqrt{f - \mu_g} < s\}$ instead and its volume $V(s) := |D(s)|$. Indeed, we have $V(s) = \mathcal{V}(s^2/4)$. If, by contradiction, the conclusion is not true, then one may integrate (2.7) to obtain that the volume of the annulus $|D(s+1)\backslash D(s)|$ is very small in comparison to e^{μ_g} whenever s is large enough, and thus, $|D(s+1)\backslash D(s)|$ is much smaller than $e^{\frac{2\mu_g}{n}}|D(s+1)\backslash D(s)|^{\frac{n-2}{n}}$. On the other hand, the Sobolev inequality in Theorem 2.1, when applied to a proper cut-off function, implies that $|D(s+1)\backslash D(s)|^{\frac{n-2}{n}}$ is smaller than $Ce^{-\frac{2\mu_g}{n}}\left(|D(s+2)\backslash D(s-1)| + \int_{D(s+2)\backslash D(s-1)} R dg\right)$. When everything is, thus, added together, one may conclude that the shrinker has finite volume. However, this cannot happen because of [49, Lemma 6.2].

In view of Theorem 2.3, a shrinker has at most Euclidean volume growth. Therefore, it is interesting to see when it has maximal volume growth. Note that a shrinker has maximum volume growth if and only if it has positive asymptotic volume ratio (AVR)

$$\mathrm{AVR}(g) := \lim_{r\to\infty} \frac{|B_r(o)|}{r^n}.$$

Chow–Lu–Yang [31] established an equivalent condition for a shrinker to have positive asymptotic volume ratio.

Theorem 2.7 [31] *Let (M^n, g, f) be a complete noncompact shrinking gradient Ricci soliton normalized as in (2.1). M has positive asymptotic volume ratio if and only if*

$$\int_{n+2}^{\infty} \frac{\mathcal{R}(s)}{s\mathcal{V}(s)}\, ds < \infty,$$

where $\mathcal{R}(s)$ and $\mathcal{V}(s)$ are functions defined in (2.6).

3 Volume Estimates for Steady Gradient Ricci Solitons

In this section, we investigate the volume bounds for steady gradient Ricci solitons, i.e., a complete Riemannian manifold (M, g) with a smooth potential function $f : M \to \mathbb{R}$ satisfying

$$\mathrm{Ric} + \nabla^2 f = 0.$$

Steady Ricci solitons are natural generalizations of Ricci-flat manifolds. Therefore, it is expected that they share similar nice volume bounds. However, the volume growth of a steady soliton in full generality is less clear than the shrinker case.

It was shown by Hamilton [39] that $\nabla\left(|\nabla f|^2 + R\right) = 0$. Hence, a non-Ricci-flat steady soliton, by scaling the metric if necessary, can be normalized in the way that

$$|\nabla f|^2 + R = 1. \tag{3.1}$$

As in the case of Ricci shrinkers, by [23], a steady gradient Ricci soliton has nonnegative scalar curvature.

Munteanu–Sesum [53] showed the following volume growth estimates of steady gradient Ricci solitons.

Theorem 3.1 [53] *Suppose that (M^n, g, f) is a complete noncompact steady gradient Ricci soliton. Then for all $p \in M$, there exist positive constants a and c such that for all large $r \gg 1$, it holds that*

$$c^{-1}r \leq |B_r(p)| \leq ce^{a\sqrt{r}}.$$

As an application of the above estimates, in the non-Ricci-flat case, Munteanu–Sesum [53, Corollary 5.2] established bounds for

$$F := -f.$$

Namely, for any $p \in M$, there exists a large positive c such that for all large positive r,

$$r - c\sqrt{r} \leq \sup_{\partial B_r(p)} F \leq r + c. \tag{3.2}$$

Using the weighted volume comparison theorem for smooth metric measure space, Wu [65] proved a similar estimate and showed the weak decay of scalar curvature on complete noncompact steady gradient Ricci solitons, i.e., $\liminf_{x\to\infty} R = 0$. Under some comparison condition on the function $f + r$, Wei–Wu [64] showed that the $\sqrt{r}$ term in (3.2) can be improved to $\ln r$, and that M^n has at most Euclidean volume growth, i.e., for all large r

$$\begin{aligned} r - c\ln r &\leq F \leq r + c; \\ |B_r(p)| &\leq Cr^n. \end{aligned} \tag{3.3}$$

There are two generic types of scalar curvature decay for steady Ricci solitons, namely the linear decay $R \leq Cr^{-1}$ and the exponential decay $R \leq Ce^{-r}$. We shall prove that, under the conditions of proper potential function and linear scalar curvature decay, the estimate for F in (3.3) holds and consequently, M has at most polynomial volume growth. The linear scalar curvature decay and proper potential function conditions are satisfied by a number of recent examples of steady soliton [1, 8–10, 32, 34, 35, 63, 66].

Theorem 3.2 *Let (M^n, g, f), where $n \geq 4$, be a complete noncompact non-Ricci-flat steady gradient Ricci soliton normalized as in (3.1). If the scalar curvature R of M decays at least linearly, i.e., $R \leq C_1/(r+1)$ and f is proper, then there is a positive constant C such that*

$$r - C\ln(r+1) - C \leq F \leq r + C \text{ on } M,$$

where $F := -f$ and $r(\cdot) := \mathrm{dist}(\cdot, p)$ is the distance function based at p.

Remark 3.3 The upper and lower estimates of $-f$ are sharp as, upon scaling, $|r+f| = O(1)$ on the Hamilton's cigar soliton, and $|r+f| \sim \ln(r+1)$ on the Cao's soliton on $\mathbb{C}^n$ with $n \geq 2$ (see [10, 32, 64]). Estimates on F under Ricci curvature conditions were also obtained in [13, 17, 30].

Using the volume estimate of Wei–Wu [64, Remark 3.2], we have

Corollary 3.4 *Under the same assumption of the previous theorem, M has at most polynomial volume growth.*

Proof of Theorem 3.2 Let $F := -f$. Rewriting (3.1) as

$$|\nabla F|^2 + R = 1 \tag{3.4}$$

and applying the fact that $R \geq 0$ on a complete steady gradient Ricci soliton, we obtain the upper bound for F. By our assumptions and the upper estimate for F, there are large positive constants ρ_0 and C_0 such that on $\{F \geq \rho_0\}$

$$|\nabla F|^2 = 1 - R \geq 1/2 \quad \text{and} \quad R \leq C_0/F. \tag{3.5}$$

Let Ψ_s be the flow of the vector field $\nabla F/|\nabla F|^2 = -\nabla f/|\nabla f|^2$ with $\Psi_{\rho_0} = \text{id}$. Moreover for $s \geq \rho_0$, Ψ_s maps $\{F \geq \rho_0\}$ to $\{F \geq s\}$ and, hence, $\Psi_s(x)$ exists for all $s \geq \rho_0$ for any $x \in \{F \geq \rho_0\}$. Let $\Gamma_s := \{F = s\}$. Then Γ_s is a smooth compact hypersurface in M and when Ψ_s is restricted on Γ_{ρ_0}, it becomes a diffeomorphism from Γ_{ρ_0} to Γ_s. For all $z \in \{F \geq \rho_0\}$, we denote $F(z)$ by s. Then there is a $z_0 \in \Gamma_{\rho_0}$ such that $\Psi_s(z_0) = z$ and for all $t \in [\rho_0, s]$,

$$F\left(\Psi_t(z_0)\right) - F(z_0) = \int_{\rho_0}^{t} \langle \nabla F, \dot{\Psi}_\tau(z_0)\rangle \, \mathrm{d}\tau = \int_{\rho_0}^{t} \langle \nabla F, \nabla F\rangle/|\nabla F|^2 \, \mathrm{d}\tau = t - \rho_0.$$

Hence, $F\left(\Psi_t(z_0)\right) = t$. Using (3.4)

$$\begin{aligned}
\operatorname{dist}\left(\Psi_s(z_0), z_0\right) &\leq \int_{\rho_0}^{s} \frac{1}{|\nabla F|} \, \mathrm{d}\tau = \int_{\rho_0}^{s} \frac{1}{|\nabla f|} \, \mathrm{d}\tau \\
&= s - \rho_0 + \int_{\rho_0}^{s} \frac{1 - \sqrt{1-R}}{\sqrt{1-R}} \, \mathrm{d}\tau \\
&\leq s - \rho_0 + \int_{\rho_0}^{s} 2 - 2\sqrt{1-R} \, \mathrm{d}\tau \\
&\leq s - \rho_0 + \int_{\rho_0}^{s} 2R \, \mathrm{d}\tau \\
&\leq F(z) - \rho_0 + \int_{\rho_0}^{s} 2C_0/F \, \mathrm{d}\tau \\
&= F(z) - \rho_0 + \int_{\rho_0}^{s} 2C_0/\tau \, \mathrm{d}\tau \\
&= F(z) - \rho_0 + 2C_0 \ln s - 2C_0 \ln \rho_0,
\end{aligned} \tag{3.6}$$

where we have applied (3.5). Hence,

$$\begin{aligned}-f(z) = F(z) &\geq \mathrm{dist}(z, p) - \sup_K \mathrm{dist}(\cdot, p) - 2C_0 \ln s \\ &\geq \mathrm{dist}(z, p) - \sup_K \mathrm{dist}(\cdot, p) - 2C_0 \ln\left(\mathrm{dist}(z, p) + F(p)\right),\end{aligned} \tag{3.7}$$

where K is the compact set $\{F \leq \rho_0\}$ and we have used that fact that $F(z) = s \leq \mathrm{dist}(z, p) + F(p)$. This completes the proof of the theorem. □

For general complete noncompact Kähler manifold M^m of complex dimension m with nonnegative holomorphic bisectional curvature everywhere and positive bisectional curvature at some point, Chen–Zhu [24] showed that for all large r

$$|B_r(p)| \geq c r^m. \tag{3.8}$$

An integration by parts argument was used by Cui [33] to show that (3.8) also holds on any complete Kähler steady gradient Ricci soliton with positive Ricci curvature and with scalar curvature attaining its maximum. The lower bound is sharp on the Cao soliton on $\mathbb{C}^m$ with positive bisectional curvature, where $m \geq 2$ [10]. In real dimension three, the Hamilton–Ivey estimate implies that any complete ancient solution to the Ricci flow has nonnegative sectional curvature (see [23] and the references therein). In particular, any three-dimensional complete gradient shrinking or steady soliton has nonnegative sectional curvature. When the curvature is positive and the scalar curvature attains its maximum on M, the authors [21] established a sharp quadratic volume estimate on the steady soliton.

Theorem 3.5 [21] *Suppose that (M^3, g, f) is a three-dimensional complete steady gradient Ricci soliton with positive sectional curvature. Assume that the scalar curvature attains its maximum somewhere. Then M has quadratic volume growth, i.e., $|B_r(p)| \sim r^2$ for all $r \gg 1$.*

Remark 3.6 Very recently, Lai [42, Theorem 1.4] has shown that on any three-dimensional complete steady gradient Ricci soliton with positive sectional curvature, the scalar curvature R must attain its maximum. Hence, the condition on R in Theorem 3.5 is superfluous.

For a complete Riemannian manifold (M^n, g), if $\mathrm{Ric} \geq 0$ outside a compact set, the asymptotic volume ratio (AVR)

$$\mathrm{AVR}(g) := \lim_{r\to\infty} \frac{|B_r(o)|}{r^n}$$

is well defined and does not depend on the choice of the basepoint $o \in M$. By a dimension reduction argument, Chow, Deng and the second named author proved the following result in dimension 4.

Theorem 3.7 (Theorem 1.10 in [30]) *Suppose that (M^4, g, f) is a four-dimensional non-Ricci-flat steady Ricci soliton with nonnegative Ricci curvature outside a compact set. If the scalar curvature decays uniformly, then* $\mathrm{AVR}(g) = 0$.

In fact, if we assume that $\mathrm{Ric} \geq 0$ everywhere on M and do not assume uniform scalar curvature decay, by Theorem 1.8 in [30], we still have that $\mathrm{AVR}(g) = 0$.

In higher dimensions, under some conditions on the curvature tensor, the second and third named authors showed that the asymptotic volume ratio of a noncollapsed steady soliton with $\mathrm{Ric} \geq 0$ is zero [46]. The proof relied on the existence of Perelman's asymptotic shrinkers [58, Section 11] and the authors followed the arguments of Ni [56].

Theorem 3.8 (Corollary 5.2 in [46]) *Let (M^n, g, f) be a complete noncompact and noncollapsed steady gradient Ricci soliton with nonnegative Ricci curvature. Furthermore, suppose that M is non-flat, and there exists a constant $C > 0$ such that*

$$|\mathrm{Rm}| \leq CR \quad on \quad M. \tag{3.9}$$

Then M has zero asymptotic volume ratio, i.e., $\mathrm{AVR}(g) = 0$.

Remark 3.9 Recently, the authors [22] have showed that (3.9) holds on any four-dimensional complete non-Ricci-flat steady soliton singularity model (see also [12, 15, 18, 51]).

The authors and Bamler have found an optimal volume growth estimate for noncollapsed steady gradient Ricci solitons. Their notion of noncollapsing is defined in terms of the boundedness of Nash entropy. More precisely, a steady soliton (M^n, g, f) is called noncollapsed, if there exists a point $p \in M$ and a nonpositive number μ_∞, such that the canonical form $(M, g_t)_{t\in(-\infty,\infty)}$ satisfies

$$\inf_{\tau>0} \mathcal{N}_{p,0}(\tau) = \mu_\infty \quad \text{for all} \quad \tau > 0. \tag{3.10}$$

Note that this notion of noncollapsing is different from (indeed, stronger than) the classical definition of Perelman. Nevertheless, it is more natural for application and is proved to be equivalent to Perelman's definition in several special cases [46, 68].

Theorem 3.10 [5] *Suppose that (M^n, g, f) is a complete steady gradient Ricci soliton, normalized as in (3.1) if g is not Ricci flat. Assume that the canonical form $(M^n, g_t)_{t\in\mathbb{R}}$ satisfies (3.10). Additionally, assume that* ***either one*** *of the following conditions is true:*

1. *$(M^n, g_t)_{t\in\mathbb{R}}$ arises as a singularity model; or*
2. *(M^n, g) has bounded curvature.*

Then

$$c(n, \mu_\infty) r^{\frac{n+1}{2}} \leq |B_r(o)| \leq C(n) r^n \quad \textit{for all} \quad r > \bar{r}(n, \mu_\infty),$$

where $\mu_\infty := \inf_{\tau>0} \mathcal{N}_{o,0}(\tau) = \lim_{\tau\to\infty} \mathcal{N}_{o,0}(\tau) > -\infty$ and $c(n, \mu_\infty)$ a positive constant of the form

$$c(n, \mu_\infty) = \frac{c(n)}{\sqrt{1-\mu_\infty}} e^{\mu_\infty}.$$

Furthermore, the upper bound is also true for all $r > 0$ (instead of $r \geq \bar{r}(n, \mu_\infty)$).

4 Volume Estimates for Expanding Gradient Ricci Solitons

In this section, we study the volume growth of expanding gradient Ricci solitons, i.e., a triple (M^n, g, f) consisting of a Riemannian manifold (M, g) and $f \in C^\infty(M)$ satisfying

$$\mathrm{Ric} + \nabla^2 f = -\tfrac{1}{2} g.$$

They are analogs of Einstein manifolds with negative Ricci curvature. Hamilton [39] showed that $\nabla\left(|\nabla f|^2 + R + f\right) = 0$. By adding a constant to f, we may normalized the expander in the way that

$$|\nabla f|^2 + R = -f. \tag{4.1}$$

It was proved by Pigola–Rimoldi–Setti [61] and Zhang [67] that the scalar curvature of any complete Ricci expander satisfies

$$R \geq -\frac{n}{2}, \tag{4.2}$$

with equality holds somewhere if and only if M is Einstein. This inequality also follows by applying the lower estimate for the scalar curvature $R \geq -n/2t$ to the canonical form of the expander (see [29]).

4.1 Upper Volume Growth Estimate

For general Ricci expanders, because of the trivial examples of hyperbolic spaces, one cannot expect a volume growth estimate with a growth rate slower than exponential. In this respect, Munteanu–Wang [50] proved a sharp exponential volume bound for smooth metric measure spaces with a lower bound of the Bakry–Emery Ricci curvature $\mathrm{Ric}_f \geq \lambda$ and a linear upper bound of $|\nabla f|$; these assumptions are satisfied by expanding gradient solitons.

Theorem 4.1 [50] *Let (M^n, g, f) be a complete expanding gradient Ricci soliton. Then there is a constant $C > 0$ such that for all $r > 0$*

$$|B_r(p)| \leq C e^{\sqrt{(n-1)}r}. \tag{4.3}$$

Without assumption on the potential function, the bound cannot be improved to a polynomial one as the hyberbolic space form $\mathbb{H}^n$ has exponential volume growth.

Chen–Deruelle [25] showed the existence of a conical structure on a complete noncompact Ricci expander with finite Riemann curvature ratio $\limsup_{x\to\infty} r^2|\mathrm{Rm}| < \infty$. In particular, the conical expander has Euclidean volume growth (see also [26] for estimates under $\lim_{x\to\infty} r^2|\mathrm{Rm}| = 0$ or integral decay condition on R).

Theorem 4.2 [25] *Any complete noncompact expanding gradient Ricci soliton (M^n, g, f) with $\limsup_{x\to\infty} r^2|\mathrm{Rm}| < \infty$ must be asymptotically conical in the*

Gromov–Hausdorff sense. In particular, it has Euclidean volume growth, namely, $|B_r(p)| \sim r^n$ *for all* $r \gg 1$.

4.2 Lower Volume Growth Estimate

It was proven by Hamilton that any complete gradient Ricci expander with positive Ricci curvature has positive asymptotic volume ratio (c.f. [27, Proposition 9.26]). Under some lower bound of the scalar curvature, Ni–Carrillo [17] proved the following monotonicity formula for the volume of expanding gradient Ricci soliton.

Theorem 4.3 [17] *Let* (M^n, g, f) *be a complete noncompact expanding gradient Ricci soliton with scalar curvature bounded from below by a constant, i.e.,* $R \geq -\beta$, *where* $\beta \geq 0$. *Then for any* $p \in M$ *and* $r \geq r_0$, *it holds that*

$$|B_r(p)| \geq |B_{r_0}(p)| \left(\tfrac{r+\alpha}{r_0+\alpha}\right)^{n-2\beta},$$

where $\alpha := \sqrt{\beta - f(p)}$.

Remark 4.4 The lower estimate is sharp on the hyperbolic cylinder.

4.3 Estimate Under Scalar Curvature Upper Bound

Munteanu–Wang [49] showed that if $R \geq -(n-1)/2$, then the gradient Ricci expander either is connected at infinity or splits isometrically like $\mathbb{R} \times N$, where N is a compact Einstein manifold. Munteanu–Wang [52] also proved that, a Kähler expander with proper potential function f must be connected at infinity.

E is an end of M relative to a compact subset K if it is an unbounded component of $M \backslash K$ [43, Definition 20.2]. In general, an expander may have more than one ends, even if it does not split. The ends of an expander can behave very differently from each other. Indeed, there is a three-dimensional expanding gradient Ricci soliton with two ends, one being hyperbolic like, another being asymptotically conical [62] (see also [6, 47] for other examples). The potential function f of the soliton is proper on the conical end and remains bounded on the hyperbolic end. Therefore, it is natural to localize the volume estimates on those ends with proper potential f. We first establish the following volume estimate for three-dimensional expanding solitons:

Theorem 4.5 *Let* (M^3, g, f) *be a three-dimensional complete noncompact expanding gradient Ricci soliton, and let* E *be an end of* M *relative to some compact subset. Suppose the following conditions are satisfied:*

- $\lim_{x\in E, x\to\infty} f(x) = -\infty$;
- *scalar curvature* R *is bounded on* E.

Then there is a large positive constant C *such that for all* $r \gg 1$

$$|B_r(p) \cap E| \leq Cr^3.$$

Moreover, either one of the following holds for all large r:

1. $|B_r(p) \cap E| \geq C^{-1}r^3$;
2. $|B_r(p) \cap E| \leq Cr$.

Remark 4.6 The two cases hold on conical and cylindrical ends, respectively. Under the assumptions of Theorem 4.5, one may wonder if the end E must either be conical or cylindrical, or neither.

Using the result by Ni–Carrillo [17], we also have

Corollary 4.7 *Let (M^3, g, f) be a three-dimensional complete noncompact expanding gradient Ricci soliton with bounded scalar curvature R and proper potential function. Then it has at most Euclidean volume growth. Specifically, fixing any $p \in M$, there exists a large constant C such that for all large $r > 0$*

$$|B_r(p)| \leq Cr^3.$$

In particular, any 3-dimensional complete noncompact gradient Ricci expander with $\lim_{x\to\infty} R = 0$ (f is automatically proper in this case) has Euclidean volume growth, namely, $|B_r(p)| \sim r^3$.

Remark 4.8 The estimate is sharp on Gaussian soliton and the positively/negatively curved Bryant expanding soliton. In view of the hyperbolic space $\mathbb{H}^3$, we see that the properness of f is essential and cannot be removed.

In higher dimensions, we also have

Theorem 4.9 *Let (M^n, g, f) be an n-dimensional complete noncompact expanding gradient Ricci soliton and let E be an end of M, where $n \geq 4$. Suppose the following conditions are satisfied:*

- $\lim_{x\in E, x\to\infty} f(x) = -\infty$;
- *There is a constant L_0 (not necessarily nonnegative) such that*

$$R \leq L_0 \text{ on } E. \tag{4.4}$$

Then for any $p \in M$, there exists a large constant C such that for all large $r > 0$

$$|B_r(p) \cap E| \leq Cr^{n+2L_0}.$$

Remark 4.10 The estimate is sharp on the hyperbolic cylinder. It can be viewed as the expanding analog of the volume estimate of shrinker by Zhang [67], as well as the reverse estimate of [17] (see also Theorems 2.5 and 4.3).

Corollary 4.11 *Any complete noncompact expanding gradient Ricci soliton with bounded scalar curvature and proper potential function f has at most polynomial volume growth.*

When f is proper and the scalar curvature R is bounded on an end E of the expander M, then by [29, Theorem 27.4] and by the same proof as in [19, lemma 7], the potential function f satisfies the following estimates near the infinity of E,

$$\tfrac{1}{5}\,(\mathrm{dist}(x,p))^2 \le v(x) \le \tfrac{1}{4}\,(\mathrm{dist}(x,p))^2 + \mathrm{dist}(x,p)\sqrt{v(p)} + v(p), \tag{4.5}$$

where p is a fixed point on M, $\mathrm{dist}(\cdot,p)$ is the distance function based at p and $v := \frac{n}{2} - f$. The above estimates will be used in the proofs of Theorems 4.5 and 4.9, since it relates the geodesic balls and sub-level sets of v.

Proof of Theorem 4.5 ($n = 3$ case) Since the scalar curvature is bounded, by the same argument as in [19, Theorem 11], $|\mathrm{Rm}|$ is bounded on the end E. We may invoke the Shi's estimate to see that, $|\nabla^k \mathrm{Rm}|$ is also bounded on E for all k. Let $v = \frac{n}{2} - f$. Then by the properness of f, the boundedness of R, (4.1), and (4.5), we have $\lim_{x\in E, x\to\infty} v = \infty$ and $|\nabla v|^2 = v - n/2 - R > 0$ outside a compact subset of E. Hence, one can find large s_0 such that on $\{x \in E : v(x) \ge s_0\}$, it holds that $\nabla v \ne 0$. Moreover, the level sets $\Gamma_s := \{x \in E : v(x) = s\}$ are compact smooth hypersurfaces with N components for all $s \ge s_0$, where N is a constant integer, and $\{x \in E : v(x) \ge s_0\}$ is diffeomorphic to $[0,\infty) \times \Gamma_{s_0}$. By abuse of notation, we denote the induced metric on Γ_s by g and consider the flow ψ_s of the vector field $\frac{\nabla v}{|\nabla v|^2}$ with $\psi_{s_0} = \mathrm{id}$. When restricted on Γ_{s_0}, $\psi_s : \Gamma_{s_0} \longrightarrow \Gamma_s$ are diffeomorphisms for all $s \ge s_0$. Let h_s be the pull back metric $\psi_s^* g$ on Γ_{s_0}. Then we have

$$\frac{\partial}{\partial s} h_s = \psi_s^* \mathcal{L}_{\frac{\nabla v}{|\nabla v|^2}} g = \psi_s^* \left(\frac{2\nabla^2 v}{|\nabla v|^2} \right) = \psi_s^* \left(\frac{2\,\mathrm{Ric} + g}{|\nabla v|^2} \right),$$

where $\mathcal{L}_{\frac{\nabla v}{|\nabla v|^2}}$ is the Lie derivative with respect to $\frac{\nabla v}{|\nabla v|^2}$. We denote by dh_s the volume form induced by the metric h_s, then by (4.1) and choosing s_0 sufficiently large

$$\begin{aligned} \frac{\partial}{\partial s} dh_s &= \tfrac{1}{2}\,\mathrm{tr}_{h_s} \frac{\partial}{\partial s} h_s\, dh_s \\ &= \psi_s^* \left(\frac{R - \mathrm{Ric}(\mathbf{n},\mathbf{n}) + 1}{|\nabla v|^2} \right) dh_s \\ &= \psi_s^* \left(\frac{R - \mathrm{Ric}(\mathbf{n},\mathbf{n}) + 1}{s - 3/2 - R} \right) dh_s \\ &\le \psi_s^* \left(\frac{R - \mathrm{Ric}(\mathbf{n},\mathbf{n}) + 1}{s} \right) dh_s + \frac{C}{s^2}\, dh_s, \end{aligned} \tag{4.6}$$

where $\mathbf{n}$ is the normal vector $\frac{\nabla v}{|\nabla v|}$. Note that the second fundamental form of Γ_s is $\frac{\nabla^2 v}{|\nabla v|} = \frac{\mathrm{Ric} + g/2}{\sqrt{s - 3/2 - R}} = O(s^{-1/2})$. Let $\sigma_1 \le \sigma_2$ be the principal curvatures of Γ_s. By the

Gauss equation, we have

$$\begin{aligned} K_s &= R_{1221} + \sigma_1\sigma_2; \\ 2K_s &= R - 2\,\mathrm{Ric}(\mathbf{n},\mathbf{n}) + 2\sigma_1\sigma_2 \\ &= R - 2\,\mathrm{Ric}(\mathbf{n},\mathbf{n}) + O(s^{-1}), \end{aligned} \tag{4.7}$$

where K_s denotes the Gauss curvature of Γ_s. We have

$$\begin{aligned} \frac{\partial}{\partial s} dh_s &\le \psi_s^*\left(\frac{R - \mathrm{Ric}(\mathbf{n},\mathbf{n}) + 1}{s}\right) dh_s + \frac{C}{s^2}\, dh_s \\ &\le \psi_s^*\left(\frac{2K_s + \mathrm{Ric}(\mathbf{n},\mathbf{n}) + 1}{s}\right) dh_s + \frac{C'}{s^2}\, dh_s. \end{aligned} \tag{4.8}$$

By virtue of $\nabla R = 2\,\mathrm{Ric}(\nabla f)$, we have $\psi_s^*(\mathrm{Ric}(\mathbf{n},\mathbf{n})) = -\frac{1}{2}\langle \nabla R, \nabla v\rangle / |\nabla v|^2 = O(|\nabla R|/|\nabla v|) = O(s^{-1/2})$. It follows from (4.8) that

$$\frac{\partial}{\partial s} dh_s \le \psi_s^*\left(\frac{2K_s + 1}{s}\right) dh_s + \frac{C''}{s^{3/2}}\, dh_s.$$

By applying the Gauss Bonnet Theorem to each components of Γ_s, we see that

$$A'(s) \le A(s)/s + C_0/s + C'' A(s) s^{-3/2},$$

where $A(s) :=$ area of Γ_s. Let $Q(s) := s^{-1} e^{2C'' s^{-1/2}}(A(s) + C_0)$. Then Q satisfies $Q'(s) \le 0$. By integrating the differential inequality, we see that for all $s \ge s_0$,

$$A(s) \le Q(s_0) s.$$

By the Coarea formula,

$$\begin{aligned} |\{s_0 \le v \le s\} \cap E| &= \int_{s_0}^{s} \int_{\Gamma_\rho} \frac{1}{|\nabla v|}\, \mathrm{d}\rho \\ &\le C \int_{s_0}^{s} \rho^{-1/2} A(\rho)\, \mathrm{d}\rho \\ &\le 2C\, Q(s_0) s^{3/2}/3. \end{aligned}$$

Using (4.5), we see that $v \sim \mathrm{dist}^2(\cdot, p)$ for some point p. Hence, we can find constants C_1 and C_2 such that for all large r

$$|B_r(p) \cap E| \le |\{v \le C_1 r^2\} \cap E| \le C_2 r^3 + |\{v \le s_0\} \cap E|.$$

This shows the Euclidean volume upper bound for the expander. Let $\Lambda := \int_{\Gamma_s} 2K_s$ which is a constant for all large s by the Gauss Bonnet theorem. We have the following two cases.

1. If there is a large $s_1 \geq s_0$ such that for all $s \geq s_1$, we have $A(s) + \Lambda \leq 1$.
2. There is a sequence of $s_i \to \infty$ such that $A(s_i) + \Lambda > 1$ for all i.

In Case 1, by virtue of Coarea formula,

$$\begin{aligned}|\{s_1 \leq v \leq s\} \cap E| &= \int_{s_1}^{s} \int_{\Gamma_\rho} \frac{1}{|\nabla v|} \leq C \int_{s_1}^{s} \rho^{-1/2} A(\rho)\, \mathrm{d}\rho \\ &\leq C \int_{s_1}^{s} \rho^{-1/2}(1 + |\Lambda|)\, \mathrm{d}\rho \\ &\leq 2C(1 + |\Lambda|)\sqrt{s}.\end{aligned}$$

$|B_r(p) \cap E| \leq Cr$ then follows from the fact that $v \sim \mathrm{dist}_g^2(\cdot, p)$ (see (4.5)). We shall show that $|B_r(p) \cap E| \geq Cr^3$ if Case 2 holds. Let i be a large integer to be specified. By the similar computation as before, we have

$$\begin{aligned}\frac{\partial}{\partial s} dh_s &\geq \psi_s^* \left(\frac{R - \mathrm{Ric}(\mathbf{n}, \mathbf{n}) + 1}{s}\right) dh_s - \frac{C}{s^2}\, dh_s \\ &\geq \psi_s^* \left(\frac{2K_s + \mathrm{Ric}(\mathbf{n}, \mathbf{n}) + 1}{s}\right) dh_s - \frac{C'}{s^2}\, dh_s \\ &\geq \psi_s^* \left(\frac{2K_s + 1}{s}\right) dh_s - \frac{C''}{s^{3/2}}\, dh_s.\end{aligned}$$

By integrating both sides over Γ_{s_0} and applying the Gauss Bonnet Theorem,

$$A'(s) \geq (A(s) + \Lambda)/s - C'' A(s) s^{-3/2}.$$

Let $W := s^{-1} e^{-2C"s^{-1/2}}(A(s) + \Lambda)$. The derivative of W satisfies

$$W'(s) \geq -C''|\Lambda| s^{-5/2} e^{-2C"s^{-1/2}} \geq -C''|\Lambda| s^{-5/2}.$$

By the definition of s_i, $A(s_i) + \Lambda > 1$. Since $\lim_{i\to\infty} s_i = \infty$, there exists a large i such that

$$e^{-2C'' s_i^{-1/2}} - 2C''|\Lambda| s_i^{-1/2} > 1/2.$$

Fixing this i, we may integrate the above inequality on $[s_i, s]$ and get

$$W(s) \geq W(s_i) - 2C''|\Lambda| s_i^{-3/2} \geq s_i^{-1} e^{-2C"s_i^{-1/2}} - 2C''|\Lambda| s_i^{-3/2} \geq (2s_i)^{-1}.$$

Hence, for all $s \geq s_i(4|\Lambda| + 1)$,

$$A(s) \geq (2s_i)^{-1} s - \Lambda \geq (4s_i)^{-1} s.$$

$|B_r(p) \cap E| \geq Cr^3$ is a consequence of a similar argument using Coarea formula and (4.5). This completes the proof of the theorem.

Proof of Corollary 4.7 The first statement is a consequence of Theorem 4.5 if we let $E = M$. The second assertion follows essentially from the argument in [17, Proposition 5.1] and Theorem 4.5. We include the argument for the sake of completeness. Since $R \to 0$ as $x \to \infty$, f is proper [19, Corollary 3] and there is a large $r_0 > 0$ such that on $M \backslash B_{r_0}(p)$

$$R \geq -\tfrac{1}{2}.$$

We integrate the trace of the expanding soliton equation over $B_r(p)$ to get

$$\begin{aligned} \left|B_r(p)\backslash B_{r_0}(p)\right| &\leq \int_{B_r(p)\backslash B_{r_0}(p)} (R + \tfrac{3}{2})\, dg \\ &\leq \int_{B_r(p)\backslash B_{r_0}(p)} (R + \tfrac{3}{2})\, dg + \int_{B_{r_0}(p)} (R + \tfrac{3}{2})\, dg \\ &= -\int_{B_r(p)} \Delta f\, dg = -\int_{\partial B_r(p)} \partial_r f \\ &\leq \left(\tfrac{r}{2} + \sqrt{\tfrac{3}{2} - f(p)}\right) A\left(\partial B_r(p)\right), \end{aligned} \tag{4.9}$$

we have used the facts that $|\nabla f| \leq \text{dist}(x, p)/2 + \sqrt{3/2 - f(p)}$ [29, Theorem 27.4] and $R \geq -\frac{3}{2}$ on M [61, 67] (see also (4.2)). Integrating the above inequality, we have that, for all $r \geq r_0 + 1$

$$\left|B_{r_0+1}(p)\backslash B_{r_0}(p)\right| \left(\frac{r+2\sqrt{3/2-f(p)}}{r_0+1+2\sqrt{3/2-f(p)}}\right)^2 \leq \left|B_r(p)\backslash B_{r_0}(p)\right|.$$

Hence, $|B_r(p)|$ grows at least quadratically in r and the cubic volume growth now follows from the dichotomy in Theorem 4.5. This completes the proof of the corollary. □

In higher dimensions, in particular when $n \geq 4$, the Gauss Bonnet Theorem cannot be applied to the $n-1$ dimensional level sets. A different argument is needed to handle the integral of the scalar curvature over the level set.

Proof of Theorem 4.9 ($n \geq 4$ case) We retain the same notation as in the proof of Theorem 4.5. We may choose s_0 large enough such that on $\{x \in E : v(x) \geq s_0\}$, it holds that $v - n/2 - R \geq s_0 - n/2 - L_0 > 0$. Moreover, using the facts that R is bounded on E and $\psi_s^*(\text{Ric}(\mathbf{n}, \mathbf{n})) = -\frac{1}{2}\langle \nabla R, \nabla v\rangle/|\nabla v|^2 = -\partial_s\left(\psi_s^* R/2\right)$, we may proceed as in the proof of Theorem 4.5 to see that

$$
\begin{aligned}
\frac{\partial}{\partial s} dh_s &= \psi_s^* \left(\frac{R - \mathrm{Ric}(\mathbf{n}, \mathbf{n}) + (n-1)/2}{s - n/2 - R} \right) dh_s \\
&= \psi_s^* \left(\frac{R + (n-1)/2}{s - n/2 - R} \right) dh_s + \psi_s^* \left(\frac{1}{2s - n - 2R} \right) \partial_s \psi_s^* R \, dh_s \\
&= \psi_s^* \left(\frac{2R + (n-1)}{2s - n - 2R} \right) dh_s + \psi_s^* \left(\frac{1}{2s - n - 2R} \right) dh_s \\
&\quad - \psi_s^* \left(\frac{1}{2s - n - 2R} \right) \partial_s \psi_s^* (s - n/2 - R) \, dh_s \\
&\le \psi_s^* \left(\frac{2L_0 + n}{2s - n - 2R} \right) dh_s - \tfrac{1}{2} \partial_s \psi_s^* \ln(2s - n - 2R) \, dh_s .
\end{aligned}
\tag{4.10}
$$

Let $J(s) := \int_{\Gamma_{s_0}} \psi_s^* \left(\sqrt{2s - n - 2R} \right) dh_s \ge 0$. Thanks to (4.2) and (4.4), it holds that $L_0 + n/2 \ge R + n/2 \ge 0$. Hence,

$$
\begin{aligned}
J'(s) &\le \int_{\Gamma_{s_0}} \psi_s^* \left(\frac{2L_0 + n}{\sqrt{2s - n - 2R}} \right) dh_s \\
&\le \frac{L_0 + n/2}{s} J(s) + \frac{C}{s^2} J(s).
\end{aligned}
\tag{4.11}
$$

By integrating the above inequality, we have for large $s \gg s_0$,

$$
\sqrt{s} A(s) \le J(s) \le s^{(L_0 + n/2)} s_0^{-(L_0 + n/2)} e^{C/s_0 - C/s} J(s_0).
$$

Hence, $A(s) = O(s^{(L_0 + (n-1)/2)})$. In view of the rigidity case of (4.2), we see that $L_0 + n/2 > 0$. Otherwise, $R = n/2$ somewhere on M and consequently, $\mathrm{Ric} \equiv -g/2$ and $\nabla^2 f \equiv 0$ on M (see [29, 61, 67]). Hence, either f is a constant or M splits off a factor of $\mathbb{R}$. The former case is impossible as f is proper along the end E. Contradiction also arises in the latter case as $\mathrm{Ric} \equiv -g/2$. Hence, there is a large s_1 such that for all $s \ge s_1$

$$
\begin{aligned}
|\{s_1 \le v \le s\} \cap E| &= \int_{s_1}^{s} \int_{\Gamma_\rho} \frac{1}{|\nabla v|} \le C \int_{s_1}^{s} \rho^{-1/2} A(\rho) \, \mathrm{d}\rho \\
&\le C' \int_{s_1}^{s} \rho^{(L_0 + (n-2)/2)} \, \mathrm{d}\rho \\
&\le C'' s^{(L_0 + n/2)}.
\end{aligned}
$$

We used the fact that $L_0 + n/2 > 0$ in the last inequality. $|B_r(p) \cap E| \le C r^{2L_0 + n}$ then follows similarly from (4.5) as in the proof of Theorem 4.5. This finishes the the proof of Theorem 4.9. □

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