Generalized (α, β) -derivations on Jordan ideals in $\ast$ -prime rings

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Abstract Let R will be a 2-torsion free $\ast$ -prime ring and $\alpha, \beta \in \text{Aut } R$. F be a nonzero generalized (α, β) -derivation of R with associated nonzero (α, β) -derivation d which commutes with $\ast$ and J be a nonzero $\ast$ -Jordan ideal and a subring of R . In the present paper, we shall prove that R is commutative if any one of the following holds: (i) $[F(u), u]_{\alpha, \beta} = 0$, (ii) $F(u)\alpha(u) = \beta(u)d(u)$, (iii) $F(u^2) = \pm\alpha(u^2)$, (iv) $F(u^2) = 2d(u)\alpha(u)$, (v) $d(u^2) = 2F(u)\alpha(u)$, for all $u \in U$.

Keywords $\ast$ -Prime rings · $\ast$ -Jordan ideal · Generalized (α, β) -derivation

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1 Introduction

Let R will be an associative ring with center Z . For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and the Jordan product $xoy = xy + yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $\ast : R \rightarrow R$ is called an involution if $(xy)^\ast = y^\ast x^\ast$ and $(x^\ast)^\ast = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $\ast$ -ring. A ring with an involution is said to $\ast$ -prime if $xRy = xRy^\ast = 0$ or $xRy = x^\ast Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $\ast$ -prime but the converse need not hold general. An example due to Oukhtite [7] justifies the above statement that is, R be a prime ring, $S = R \times R^o$ where R^o is the opposite ring of R . Define involution $\ast$ on S as $\ast(x, y) = (y, x)$. S is $\ast$ -prime, but not prime. This example shows that $\ast$ -prime rings constitute a more general class of prime rings. In all that follows the symbol $S_{a_\ast}(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of R , i.e. $S_{a_\ast}(R) = \{x \in R \mid x^\ast = \pm x\}$.

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An additive subgroup J of R is said to be a Jordan ideal of R if $uor \in J$, for all $u \in J, r \in R$. A Jordan ideal is said to be a $*$ -Jordan ideal if $J^* = J$. Let I be the ring of integers. Set

$$R = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, c, d \in I \right\}$$

and

$$J = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \in I \right\}$$

We define the following maps on $R : * \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$. Then it is easy to see that J is a nonzero $*$ -Jordan ideal of R .

An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \rightarrow R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. An additive mapping $f : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d : R \rightarrow R$ such that

$$f(xy) = f(x)y + xd(y), \quad \text{for all } x, y \in R.$$

This definition was given by Bresar in [3]. Let α and β be any two automorphisms of R . An additive mapping $d : R \rightarrow R$ is called a (α, β) -derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Inspired by the definition (α, β) -derivation, the notion of generalized derivation was extended as follows: Let α and β be any two automorphisms of R . An additive mapping $f : R \rightarrow R$ is called a generalized (α, β) -derivation on R if there exists a (α, β) -derivation $d : R \rightarrow R$ such that

$$f(xy) = f(x)\alpha(y) + \beta(x)d(y), \quad \text{for all } x, y \in R.$$

Of course a generalized (1,1)-derivation is a generalized derivation on R , where 1 is the identity mapping on R .

Let S be a nonempty subset of R . A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. The study of centralizing and commuting mappings was initiated by Posner in [12] (Posner's second theorem). Several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations which are centralizing and commuting on appropriate subsets of R (see, e.g., [1, 4, 2, 6] and references therein). Recently, Oukhtite et al. [8, Theorem 1] proved Posner's second theorem to rings with involution in the case of characteristic not 2: Let R be a 2-torsion free $*$ -prime ring and U a square closed $*$ -Lie ideal of R . If R admits a nonzero derivation d centralizing on U , then $U \subseteq Z$. They generalized this theorem to generalized derivations centralizing on Jordan ideals in rings with involution [11, Theorem 1]. Also, in [5], the authors investigated some commutativity theorems for Jordan ideals in rings with generalized derivation. In the present paper, we shall study these results Jordan ideals of $*$ -prime rings with generalized (α, β) -derivation of R .

2 Preliminaries

Throughout the paper, R will be a 2-torsion free $*$ -prime ring and $\alpha, \beta \in \text{Aut } R$. F be a nonzero generalized (α, β) -derivation of R with associated nonzero (α, β) -derivation d which commutes with $*$ and J be a nonzero $*$ -Jordan ideal and a subring of R . Also, we will make some extensive use of the basic commutator identities:

$$\begin{aligned}
[x, yz] &= y[x, z] + [x, y]z \\
[xy, z] &= [x, z]y + x[y, z] \\
[xy, z]_{\alpha, \beta} &= x[y, z]_{\alpha, \beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha, \beta}y \\
[x, yz]_{\alpha, \beta} &= \beta(y)[x, z]_{\alpha, \beta} + [x, y]_{\alpha, \beta}\alpha(z) \\
xo(yz) &= (xoy)z - y[x, z] = y(xoz) + [x, y]z \\
(xy)oz &= x(yoz) - [x, z]y = (xoz)y + x[y, z] \\
(xo(yz))_{\alpha, \beta} &= (xoy)_{\alpha, \beta}\alpha(z) - \beta(y)[x, z]_{\alpha, \beta} = \beta(y)(xoz)_{\alpha, \beta} + [x, y]_{\alpha, \beta}\alpha(z) \\
((xy)oz)_{\alpha, \beta} &= x(yoz)_{\alpha, \beta} - [x, \beta(z)]y = (xoz)_{\alpha, \beta}y + x[y, \alpha(z)]
\end{aligned}$$

Lemma 1 [9, Lemma 2] *Let R be a 2-torsion free $\ast$ -prime ring, J a nonzero $\ast$ -Jordan ideal of R and $a, b \in R$. If $aJb = a^\ast Jb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2 [9, Lemma 3] *Let R be a 2-torsion free $\ast$ -prime ring and J a nonzero $\ast$ -Jordan ideal of R . If $[J, J] = 0$, then $J \subseteq Z$.*

Lemma 3 [10, Lemma 3] *Let R be a 2-torsion free $\ast$ -prime ring and J a nonzero $\ast$ -Jordan ideal of R . If $J \subseteq Z$, then R is commutative.*

Lemma 4 *Let R be a 2-torsion free $\ast$ -prime ring, J a nonzero $\ast$ -Jordan ideal of R and d a nonzero (α, β) -derivation of R . If d commutes with $\ast$ and $d(J) = 0$, then R is commutative.*

Proof By the hypothesis, we obtain that

$$d(ur + ru) = 0, \quad \text{for all } u \in J, r \in R.$$

Expanding this term and using the hypothesis, we get

$$(d(r), u)_{\alpha, \beta} = 0, \quad \text{for all } u \in J, r \in R. \quad (2.1)$$

Replacing r by $2rv$, $v \in J$ in (2.1) and using (2.1), $d(J) = 0$, we have

$$d(r)\alpha([v, u]) = 0, \quad \text{for all } u, v \in J, r \in R.$$

Substituting rs , $s \in R$ for r in this equation and using this, we find that $d(r)\alpha(s)\alpha([v, u]) = 0$, and so

$$d(r)R\alpha([v, u]) = 0, \quad \text{for all } u, v \in J, r \in R. \quad (2.2)$$

Writing $r^\ast$ by r in the last equation, we get

$$d(r^\ast)R\alpha([v, u]) = 0, \quad \text{for all } u, v \in J, r \in R.$$

Since d commutes with $\ast$, the last equation follows

$$d(r)^\ast R\alpha([v, u]) = 0, \quad \text{for all } u, v \in J, r \in R. \quad (2.3)$$

Applying the $\ast$ -primeness of R , because of (2.2) and (2.3), we conclude that $d(r) = 0$ or $[v, u] = 0$, for all $u, v \in J, r \in R$. Since d is a nonzero (α, β) -derivation of R , we arrive at $[J, J] = 0$, and so R is commutative by Lemmas 2 and 3. $\square$

3 Results

Theorem 1 *Let R be a 2-torsion free $*$ -prime ring and J a nonzero $*$ -Jordan ideal and a subring of R . If R admits a nonzero generalized (α, β) -derivation F associated with nonzero (α, β) -derivation d which commutes with $*$ such that $[F(u), u]_{\alpha, \beta} = 0$, for all $u \in J$, then R is commutative.*

Proof Suppose that

$$[F(u), u]_{\alpha, \beta} = 0, \quad \text{for all } u \in J. \quad (3.1)$$

Linearizing (3.1) and using this, we obtain that

$$[F(u), v]_{\alpha, \beta} + [F(v), u]_{\alpha, \beta} = 0, \quad \text{for all } u, v \in J. \quad (3.2)$$

Replacing v by vu in (3.1), we get

$$[F(u), vu]_{\alpha, \beta} + [F(v)\alpha(u) + \beta(v)d(u), u]_{\alpha, \beta} = 0.$$

That is

$$\begin{aligned} & [F(u), v]_{\alpha, \beta}\alpha(u) + \beta(v)[F(u), u]_{\alpha, \beta} + [F(v), u]_{\alpha, \beta}\alpha(u) \\ & + F(v)[\alpha(u), \alpha(u)] + \beta(v)[d(u), u]_{\alpha, \beta} + [\beta(v), \beta(u)]d(u) = 0, \\ & \text{for all } u, v \in J. \end{aligned}$$

Now combining (3.1) and (3.2) in the last equation, we find that

$$\beta(v)[d(u), u]_{\alpha, \beta} + [\beta(v), \beta(u)]d(u) = 0, \quad \text{for all } u, v \in J. \quad (3.3)$$

Again replace v by vw in (3.3) and use (3.3), to get

$$[\beta(v), \beta(u)]\beta(w)d(u) = 0, \quad \text{for all } u, v, w \in J.$$

Since β is an automorphism of R , we see that

$$[v, u]J\beta^{-1}(d(u)) = 0, \quad \text{for all } u, v \in J.$$

Since J is a nonzero $*$ -Jordan ideal of R yields that

$$[v, u]^*J\beta^{-1}(d(u)) = 0, \quad \text{for all } v \in J, u \in J \cap S_{a_*}(R).$$

Hence we get

$$[v, u]J\beta^{-1}(d(u)) = [v, u]^*J\beta^{-1}(d(u)) = 0, \quad \text{for all } v \in J, u \in J \cap S_{a_*}(R). \quad (3.4)$$

By Lemma 1, we get either $[v, u] = 0$, for all $v \in J$ or $d(u) = 0$ for each $u \in J \cap S_{a_*}(R)$. Let $u \in J$, as $u + u^*, u - u^* \in J \cap S_{a_*}(R)$ and $[v, u \pm u^*] = 0$, for all $v \in J$ or $d(u \pm u^*) = 0$. Hence we have $[v, u] = 0$ or $d(u) = 0$, for all $u, v \in J$. We obtain that J is union of two additive subgroups of U such that

$$K = \{u \in J \mid d(u) = 0\}$$

and

$$L = \{u \in J \mid [v, u] = 0, \quad \text{for all } v \in J\}.$$

Moreover, J is the set-theoretic union of K and L . But a group can not be the set-theoretic union of two proper subgroups, hence $K = J$ or $L = J$. In the former case, we get R is commutative by Lemma 4. In the latter case, $[J, J] = (0)$. That is $J \subseteq Z$ by Lemma 2 and R is commutative by Lemma 3. This completes the proof. $\square$

In the view of Theorem 1, we get the following result:

Corollary 1 *Let R be a 2-torsion free $\ast$ -prime ring and J a nonzero $\ast$ -Jordan ideal and a subring of R . If R admits a nonzero (α, β) -derivation d which commutes with $\ast$ such that $[d(u), u]_{\alpha, \beta} = 0$, for all $u \in J$, then R is commutative.*

Theorem 2 *Let R be a 2-torsion free $\ast$ -prime ring and J a nonzero $\ast$ -Jordan ideal and a subring of R . If R admits a nonzero generalized (α, β) -derivation F associated with nonzero (α, β) -derivation d which commutes with $\ast$ such that $F(u)\alpha(u) = \beta(u)d(u)$, for all $u \in J$, then R is commutative.*

Proof We have

$$F(u)\alpha(u) = \beta(u)d(u), \quad \text{for all } u \in J \quad (3.5)$$

Replacing u by $u + v$ in (3.5) and using this, we get

$$F(v)\alpha(u) + F(u)\alpha(v) = \beta(v)d(u) + \beta(u)d(v), \quad \text{for all } u, v \in J. \quad (3.6)$$

Writting vu for v in (3.6) and using (3.6), we obtain that

$$2\beta(v)d(u)\alpha(u) = \beta(uov)d(u), \quad \text{for all } u, v \in J.$$

Taking wv instead of v in the above equation and using this, we have

$$\beta([w, u])\beta(v)d(u) = 0, \quad \text{for all } u, v, w \in J.$$

Hence we arrive at

$$[w, u]J\beta^{-1}(d(u)) = 0, \quad \text{for all } u, v, w \in J.$$

Since J is a nonzero $\ast$ -Jordan ideal of R yields that

$$[w, u]^{\ast}J\beta^{-1}(d(u)) = 0, \quad \text{for all } u, v, w \in J.$$

Therefore, we get

$$[w, u]J\beta^{-1}(d(u)) = [w, u]^{\ast}J\beta^{-1}(d(u)) = 0, \quad \text{for all } v \in J, u \in J \cap S_{a_{\ast}}(R).$$

The similar arguments as used after equation (3.4), we get the required result. $\square$

Theorem 3 *Let R be a 2-torsion free $\ast$ -prime ring and J a nonzero $\ast$ -Jordan ideal and a subring of R . If R admits a nonzero generalized (α, β) -derivation F associated with nonzero (α, β) -derivation d which commutes with $\ast$ such that $F(u^2) = \pm\alpha(u^2)$, for all $u \in J$, then R is commutative.*

Proof Linearizing the hypothesis, we get

$$F(u)\alpha(v) + \beta(u)d(v) + F(v)\alpha(u) + \beta(v)d(u) = \alpha(uv + vu), \quad \text{for all } u, v \in J. \quad (3.7)$$

Replacing v by vu , $u \in J$ in (3.7) and applying this equation, we arrive at

$$\beta(uov)d(u) = 0, \quad \text{for all } u, v \in J. \quad (3.8)$$

Writting vw for v in (3.8) and using (3.8), we obtain that

$$\beta([u, v])\beta(w)d(u) = 0, \quad \text{for all } u, v, w \in J,$$

and so

$$[u, v]J\beta^{-1}(d(u)) = 0, \quad \text{for all } u, v \in J.$$

Since J is a nonzero $*$ -Jordan ideal of R yields that

$$[u, v]^*J\beta^{-1}(d(u)) = 0, \quad \text{for all } v \in J, u \in J \cap S_{a_*}(R).$$

Therefore, we get

$$[u, v]J\beta^{-1}(d(u)) = [u, v]^*J\beta^{-1}(d(u)) = 0, \quad \text{for all } v \in J, u \in J \cap S_{a_*}(R).$$

The similar arguments as used after equation (3.4), we get the required result. $\square$

Theorem 4 *Let R be a 2-torsion free $*$ -prime ring and J a nonzero $*$ -Jordan ideal and a subring of R . If R admits a nonzero generalized (α, β) -derivation F associated with nonzero (α, β) -derivation d which commutes with $*$ such that $F(u^2) = 2d(u)\alpha(u)$, for all $u \in J$, then R is commutative.*

Proof We get

$$F(u^2) = 2d(u)\alpha(u), \quad \text{for all } u \in J.$$

That is

$$F(u)\alpha(u) + \beta(u)d(u) = 2d(u)\alpha(u), \quad \text{for all } u \in J. \quad (3.9)$$

Linearizing (3.9) and using this, we obtain

$$F(u)\alpha(v) + F(v)\alpha(u) + \beta(u)d(v) + \beta(v)d(u) = 2d(u)\alpha(v) + 2d(v)\alpha(u), \\ \text{for all } u, v \in J. \quad (3.10)$$

Taking vu instead of v in (3.10) and using this equation, we have

$$\beta(uov)d(u) = 2\beta(v)d(u)\alpha(u), \quad \text{for all } u, v \in J. \quad (3.11)$$

Letting v by wv in (3.11) and using (3.11), we arrive at

$$\beta([u, w])\beta(v)d(u) = 0, \quad \text{for all } u, v, w \in J. \quad (3.12)$$

That is

$$[u, w]J\beta^{-1}(d(u)) = 0, \quad \text{for all } u, w \in J.$$

Since J is a nonzero $*$ -Jordan ideal of R yields that

$$[u, w]^*J\beta^{-1}(d(u)) = 0, \quad \text{for all } w \in J, u \in J \cap S_{a_*}(R).$$

Therefore, we get

$$[u, w]J\beta^{-1}(d(u)) = [u, w]^*J\beta^{-1}(d(u)) = 0, \quad \text{for all } w \in J, u \in J \cap S_{a_*}(R).$$

Further application of similar arguments as used after (3.4), we get the required result. $\square$

Theorem 5 *Let R be a 2-torsion free $*$ -prime ring and J a nonzero $*$ -Jordan ideal and a subring of R . If R admits a nonzero generalized (α, β) -derivation F associated with nonzero (α, β) -derivation d which commutes with $*$ such that $d(u^2) = 2F(u)\alpha(u)$, for all $u \in J$, then R is commutative.*

Proof Expanding $d(u^2) = 2F(u)\alpha(u)$, we have

$$d(u)\alpha(u) + \beta(u)d(u) = 2F(u)\alpha(u), \quad \text{for all } u \in J. \quad (3.13)$$

Linearizing (3.13) and using this, we obtain

$$d(u)\alpha(v) + d(v)\alpha(u) + \beta(u)d(v) + \beta(v)d(u) = 2F(u)\alpha(v) + 2F(v)\alpha(u), \\ \text{for all } u, v \in J. \quad (3.14)$$

Writing vu instead of v in (3.14) and using this equation, we have

$$\beta(uov)d(u) = 2\beta(v)d(u)\alpha(u), \quad \text{for all } u, v \in J. \quad (3.15)$$

Replacing v by wv in the last equation and using this, we arrive at

$$\beta([u, w])\beta(v)d(u) = 0, \quad \text{for all } u, v, w \in J.$$

The similar arguments as used after equation (3.12) in the proof of Theorem 4, we get the required result. $\square$

We can give the following corollary by Theorem 4 (or 5):

Corollary 2 *Let R be a 2-torsion free $\ast$ -prime ring and J a nonzero $\ast$ -Jordan ideal and a subring of R . If R admits a nonzero (α, β) -derivation d which commutes with $\ast$ such that $d(u^2) = 2d(u)\alpha(u)$, for all $u \in J$, then R is commutative.*

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