Rendiconti Lincei. Scienze Fisiche e Naturali (2023) 34:703–720
https://doi.org/10.1007/s12210-023-01171-3

QUASICRYSTALS: STATE OF THE ART AND OUTLOOKS

Tomb towers and minarets: analysis of symmetries and geometries of Iranian geometrical ornaments of the Seljuq era. Pictorial requiem for the Kharraqan towers

Emil Makovicky[1,2]

Received: 28 April 2023 / Accepted: 6 June 2023 / Published online: 12 July 2023

Abstract
Ornamental adornment of the Kharraqan tomb towers, the most outstanding funeral monuments of the Seljuk era in NW Iran, and those of four best-preserved Seljuq brick minarets in northern Iran, documents the artistic canon of the pre-glaze stage of Iranian Islamic architecture. Despite some later interruptions, these monuments and their plain-brick ornaments, as well as the 'virtually interlaced' brick ornaments, stand at the beginnings of a rich development that led to the Safavid architecture of Iran. Besides documentation and study of the geometric character of early Islamic art, which was based on limited technical resources, this study offers insight into symmetry concepts developed at this stage of art and architecture development. This is the last and most complete study of the Kharraqan towers performed before their overwhelming destruction in the 2002 earthquake.

Keywords Islamic brickwork · Tomb tower · Minaret · Seljuk art · Brick patterns · Crystallographic symmetry · Kharraqan towers

1 Introduction

The Seljuq Islamic architecture of 'pre-glaze age' in Iran and immediately adjacent regions is constructed primarily of bricks, and not of worked stone which was used in Seljuq Turkey. It not only has a unique art value but also represents one of the peaks of application of symmetry to ornamental art. This study investigates ornamental adornment of the Kharraqan tomb towers that count among the most outstanding funeral monuments of their time, and of four best-preserved Seljuq brick minarets in Iran. These monuments stand at the beginnings of a rich development that resulted in such achievements as the quasi-periodic patterns of the Gunbad-e-Kabud (Blue Tower) at Maragha, W Iran (cf. Makovicky 1992, 2008, 2016; Lu and Steinhardt 2007). In parallel to the results and conclusions concerning the geometric character of early Islamic art, this study offers insight into symmetry concepts developed (or those still absent) at this stage of art and architecture development.

Concentrated attention to the Kharraqan tomb towers was directed by Stern (1966), Stronach and Young (1966), Blair (1991), Makovicky (2016), Bier (2002) and Bonner (2017). The Seljuq minarets in Iran received more attention, e.g., by Hutt and Harrow (1977), Michell (1978), Clévenot and Degeorge (2000), and Stierlin (2002).

Studies undertaken by these authors differ—they range from the analysis of inscriptions, and of the architectural details, association with metaphysical concepts, to studies of ornament geometry. Still, they left enough space for a thorough symmetry and underlying geometry analyses, which are the main topic of the current paper. I agree with Bier (2002) that the ornament designer wished to explore the mathematical properties of the two-dimensional space, doing it with exuberant expressiveness. I see, however, more of a professional vigor supported by the wishes of the

This paper belongs to the topical collection "Quasicrystals: State of the art and outlooks" originated from an international conference organized by the Accademia dei Lincei, held in Rome on November 18, 2022 in the frame of the 2022 International Year of Mineralogy.

Emil Makovicky
emilm@ign.ku.dk

[1] Institute for Geoscience and Natural Resource Management, University of Copenhagen, Østervoldgade 10, 1350 Copenhagen, Denmark

[2] Department 'SUSTAIN', Danish Technical University, 2800 Kongens Lyngby, Denmark

order-giver to represent best his clan, instead of a metaphysical approach and hidden meaning in the choice of ornaments used, which she suggested.

2 General situation

2.1 Historical frame

Arab domination of the eastern Islamic lands came to its end in 945 when Arab caliphs had to surrender their worldly authority to the military commanders of Persian origin. For about one century, this area was ruled by dynasties of Persian origin, especially by Buyids. Later, power went to assimilated Turcoman Ghaznavids (Hillebrand 1999). For the architectural history of Iran, however, the decisive moment is that in 1037, when another Turcoman group, the Seljuqs (also spelled as Saljuqs, Seljuks) conquered Merv, then the central and western Persia, occupying Esfahan in 1051, and Baghdad in 1055. In the last quarter of eleventh century, they ruled over the entire Western Asia (Rice 1991; Bonner 2017).

The Seljuq era brought about the first extensive blossoming of Iranian art (Hillebrand 1999). The Great Seljuqs of Iran were lavish building patrons—mosques, madrasahs, minarets, small mausolea (*imamzadas*), caravanserais, and the typical *gumbads* (tomb towers). The same high level of activity was present in all other branches of applied arts—ceramics, metalwork, book painting, etc.

This activity, however, came to an abrupt halt by Mongol invasions in 1219 and the following years. True renaissance of Iranian arts comes only about 90 years later, under the Ilkhanid ruler Ghazan Khan (1295–1304) (Hillebrand 1999).

2.2 Materials and their use

Owing to the level of technological development of the day, much of the Seljuq architecture is adorned by unglazed brick ornaments, in a style called 'brick style' or *hezar baf* (thousand weavings) (Clevenot and Degeorge 2000). This technique created ornamental panels or entire walls from variously laid and variously oriented bricks, which were occasionally halved, quartered, cut, or beveled to angles other than 90°. Sometimes, they were combined with terracotta stamped with designs or with stucco. The latter, however, was used amply for interiors. These techniques were mostly inherited from older periods (as witnessed, e.g., by the Tarik Khana Minaret, about 1027) but they were brought to perfection by the builders of the Seljuq period. Monumental inscriptions adorn these buildings; the entire 'brick style' requires the play of strong light and shadows produced by the slowly moving sun in the mostly clear Iranian skies.

The pale-to-azure blue glaze appears later in the Seljuq period, e.g., as an epigraphic pale-blue band on the brick-style minaret of the Jami Mosque in Damghan (about 1058), colored bands and inscriptions on Masjid-i-Jami in Qazvin (1113) or a geometrical band of glazed bricks on the minaret of the Kalyan Mosque (1127). Extensive use of glazed brickwork is found on the Blue Tomb of Maragha (1147) (Makovicky 2008; Makovicky and Ghari 2018) but its blossoming is connected with the Ilkhanid architecture. In the Timurid period (1370–1506) the glazed bricks/tile mosaics took over the role of architectural ornament (Clevenot and Degeorge 2000), leading to virtual extinction of the 'brick style' (Makovicky 1989).

For the studies of ornamental art, many authors use combination of fundamental geometry and subjective impressions; the fullest representation of such description is that by Bier (2002, part 3.1). There exists an exact language, however, which deals with a distribution of individual motifs and their combinations—it describes symmetries of zero-dimensional, as well as of (periodic) uni-and two-dimensional patterns.

The symmetry language was created and defined mostly for physical sciences, but it fits perfectly to the study and classification of periodic ornaments. Moreover, we shall see that the architect was in some cases thinking it terms of two-sided two-dimensional patterns (one visible side and one symmetry-related, usually identical, inaccessible side), which are a result of *illusionary interlace* (Bier 2002)—the language of symmetry manages this problem with ease.

The present study is based on own observations and photographic evidence collected in Iran in 2002.

Unfortunately, this contribution is a *pictorial requiem* for the beautiful ornamental towers of Kharraqan. Just three weeks after my visit in 2002, on July 22nd, a powerful earthquake (Walker et al. 2005; Wikipedia anonymous) with epicenter located 59 km SW of Abhar (SW of Qazvin), and with magnitude calculated to 6.5 or more at the epicenter, irreparably destroyed the towers (which otherwise survived centuries). It caused one of them to sink into the ruins of its lower portion and split the other into upward-raging ruins (Jean-Marc Castera, private communication); all of this destroyed many ornaments on the walls. Locally, it was the strongest earthquake in about 900 years. Thus, this contribution is the last one describing the well-preserved and professionally restored situation before the demise. Although still described in the present tense in this paper, the optimal condition of these towers, described here, is a matter of the past.

3 Data on the localities and objects described

3.1 The Kharraqan tomb towers

Perhaps the most spectacular object of early Seljuk architecture are two Seljuq tomb towers at Kharraqan in

Western Iran (Fig. 1), built by one and the same architect for Turkoman chieftains in the years 1067 and 1086, respectively. These tomb towers lie about 120 km to the southwest of Qazvin, about 30 km up a broad lonely valley stretching to the west from the village of Abgerme, which itself is situated on the Takestan—Razan Road. The locality is less than one kilometer to the west of a small, poor village, known in Azari as Hesar Armani, later Hesar Valiars.

Local stories describe these towers as mausolea for an assassinated holy man and his sister who shared his fate or as a place of living of a Seljuq governor. In spring, before harvest, people from surrounding villages used to gather for a celebration honoring the legendary assassinated imam with a sacrifice of a sheep.

The inscription on the W tomb tower proclaims Muhammad ibn Makki al-Zanjani as the architect, although the names of clients are less certain. The tower tombs are octagonal prisms, about 13 m tall, each terminated with a brick dome, which apparently had been restored. Both towers have powerful semicircular buttresses attached to all corners. They were built of buff-fired bricks, indicating calcium-enriched clay compositions.

The ornamental fields of the two towers differ in their layout. The eastern tower has large arched blind 'windows' which take up most of the flat wall space (Fig. 1). These 'windows' are surmounted by broad horizontal friezes that have an inscription at their base. Simple *cmm* brickwork fills the triangular spaces left between these elements. Buttresses are covered by ornamental brickwork as well, and flanked by diminutive, ornamented pillars; whereas, the upper, arched frames of ornamental 'blind windows' are left unadorned. Lower portions of the walls and buttresses lost their ornamental coating to humidity and very little is left of the original dome as well.

The western tower (Fig. 1) has 'blind windows' divided into tall, rectangular lower portions and arched upper portions. They are separated by a row of three small ornamental panels surmounted by trefoil arches. Again, the ornamental friezes with an inscription at the base surmount the walls. Buttresses carry rich ornamentation. Small pilasters flank the flat, ornamented walls up to the horizontal dividing line. Top arches are partly ornamented by a linear ornament, partly left unadorned. Some buttresses and the bottom portions of the walls lost their ornamental coating.

Fig. 1 The Seljuq tomb towers of Kharraqan before the 2002 earthquake, seen from SE; the eastern tower in foreground

In the interior of the W tower, a mihrab wall is preserved. It is a flat panel of two-dimensional ornament, surmounted by a smaller panel under a pointed trefoil arch. Both towers underwent expertly restoration without forcing new additions upon the original brickwork.

Although coming to Western attention at a rather late date (Stronach and Young 1966), panoramatic picture of Kharraqan towers and less frequent pictures of one or two of their ornamental panels can be found in many publications on Islamic art of Iran, e.g., Ettinghausen and Grabar (2001), actually too many to quote. A more comprehensive selection of the latter is in Makovicky (2016), although it is much smaller than the true number of extant two-dimensional ornaments. The current publication attempts to remedy the situation, especially because many depicted patterns succumbed to the earthquake. Among the specialized publications devoted to the Kharraqan towers, we might mention Stern (1966) who deals with the inscriptions, Stronach and Young (1966) dealing with the tower architecture, and Bier (2002) who suggests that the geometric patterns on the towers have metaphysical meaning and combine to form a programmatic cycle of meanings.

3.2 Seljuk minarets

Towns on the old commercial route (Silk Road) along the southern limits of the Elburz Mountains, and those encircling the vast Dasht-e-Kavir desert, were dotted with mosques that boasted with exquisite minarets adorned with complex brick ornamentation. In some cases, these minarets are the last vestiges of the old building, the mosque itself having succumbed to numerous alterations.

I studied the minaret of the Jami' Mosque at Saveh, northwest Iran (built in 1110–1111 according to Hutt and Harrow1977) at first-hand, as well as those of the Jami' Mosque in Semnan (eleventh century), Jami' Mosque in Damghan (about 1058), and the Tarik Khana minaret of the Great Mosque in Damghan (probably pre-Seljuq, 1027). All three are situated in northern Iran. Books by Hutt and Harrow (1977), Michell (1978), Stierlin (2002), and Clévenot and Degeorge (2000) offer additional, although rather sparse material concerning these objects.

4 Methods and problems

As mentioned above, two-dimensional crystallography offers unambiguous alternatives to the usual classification of ornaments as ‘complicated’, ‘dynamic’, ‘static’, etc., by identifying individual symmetry elements in the ornamental patterns studied, and analyzing their combinations, known as the point groups (supplement data: Fig. S1), plane groups, layer groups and 1-dimensional frieze groups of symmetry. We can obtain overall, or partial *statistics of plane/frieze group symmetries* used by the artisans/architects from such studies. Practical experience reveals that *friezes* can be divided into (*a*) possible cut-outs of 2D patterns observed in the region and (*b*) the truly one-dimensional patterns; it was done by Makovicky and Fenoll Hach-Alí(1997b) for the Andalusian Islamic art. Especially for the Kharraqan case, the illustrations included in the text, will be supplemented by a supplementary set, indicated by ‘S’ and a running number of supplementary illustrations.

Often, several different patterns (in one region) have the same plane-group symmetry. For their differentiation and a more detailed classification, *special and general positions* of individual elements (motifs) of the pattern (so-called Wyckoff sets) placed in the unit mesh are studied (Fig. 2). This means that the motif elements (tesserae) can either be placed on rotation axes or they can straddle reflection planes of symmetry (these are *the special positions*) or they can be situated in spaces *between* these symmetry operators (in so-called *general positions*). This determines both their shapes and the multiplicity with which they occur in the pattern. There may be interpretational problems with application of this principle to those patterns which are composed of endlessly interlaced ribs: should we consider the recessed finite volumes between ribs as ‘tiles’ or should we take the protruding continuous lines as the basic elements? In such cases, we must select elements (interpretations) which appear to be closest to the understanding of the original masters, e.g., by virtue of their frequent and repeated use in the region.

Based on the types of positions occupied by the different elements of the pattern, and on the number of times individual position types are occupied by distinct elements, patterns can be divided into three categories. *Simple patterns* have only one set occupied and only once, *intermediate patterns* have 2–3 Wyckoff sets occupied or a certain set occupied more than once, and the *complex patterns,* in which many elements combine, usually display a multiple occupation of certain positional sets.

It is not the purpose of this paper to re-explain and tabulate *groups of symmetry; the purpose is to use them.* The plane groups of symmetry can be found illustrated and geometrically described in the International Tables

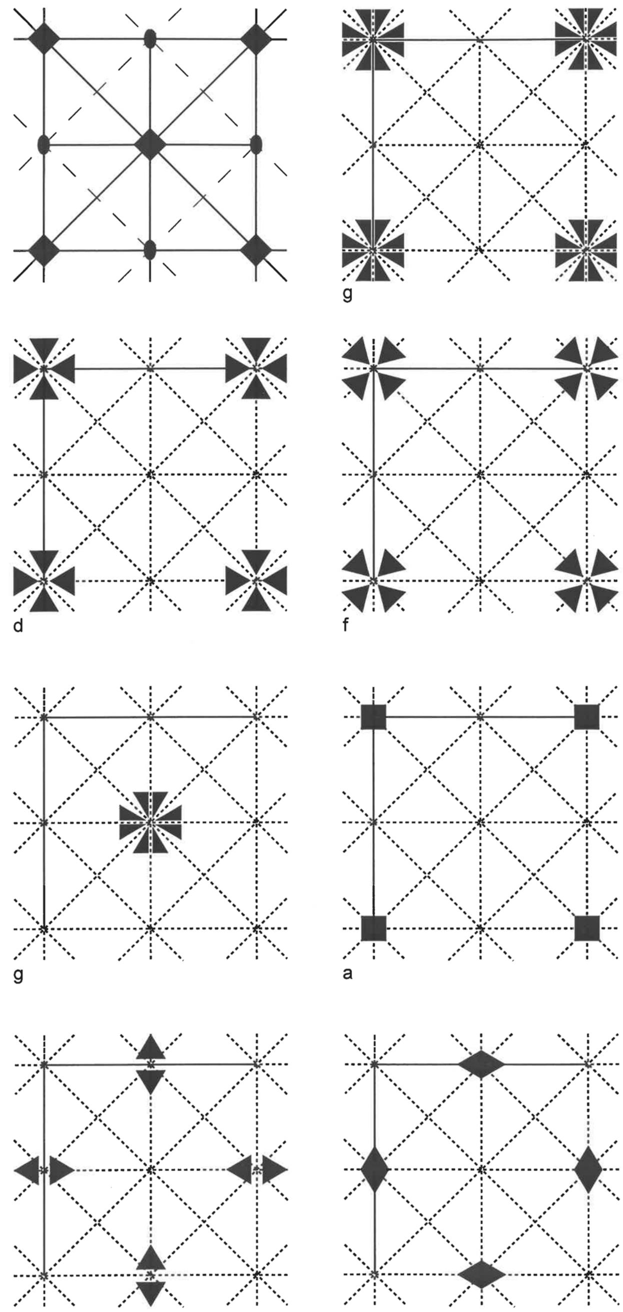

Fig. 2 Plane group *p*4*mm* and the tile positions and schematic shapes (presented as individual asymmetric triangle), either in general positions (**g**) of this plane group or in different types of special positions [either placed on the fourfold, or on twofold axes of rotation symmetry (denoted as **a**, **c**), or straddling the reflection planes (indicated as **c**–**f**)]. Their symbols suggest the local tile symmetry and shape, constructed as coalesced groups from the small right-angle triangles, which were used as *separated* only for the general position **g**. Different position types shown here can coexist in one pattern

for Crystallography published by the IUCr's Commission on International Tables, volume A (2016) (quoted as IT below, available online), as well as in the *reference books* by Washburn and Crowe (1998 and re-prints), Abbas and Salman (2007) and Makovicky (2016). A copy of a table/illustration of these groups in your hand is recommendable for fully appreciating the following chapters of the present paper. The purpose of our contribution is to use the symmetry groups in a critical way, as it was done by, e.g., Makovicky and Fenoll (1997a). The frequently occurring question of pseudosymmetry (when metrics/symmetry of the *underlying* geometric scheme is higher than that of the resulting pattern), or of the excess local symmetry, will be addressed as well.

Islamic artists were the greatest masters of *two-sided layer symmetry,* and the *interlaced patterns,* which are based on this kind of symmetry, are a frequent adornment of the important localities of Islamic art. Ignoring the true *layer symmetry* of these patterns, and replacing them automatically by plane-group symmetry, is rather frequent in literature, but it distorts the understanding of given art style.

Very interesting is the profusion of interlaced patterns already in early Seljuk art, including the studied localities. With the technical limitations of the patterns created by brick arrangement, the Seljuks artists mimicked (mostly in an error-free fashion) the interlacing by carefully arranging the component bricks in such a way that the pattern creates an impression of interlacing with each string alternatively 'going over' and 'going under' the strings it meets. The best source of 'layer groups of symmetry' (Fig. 3), needed for this portion of study, are the International Tables for Crystallography, published by the IUCr's Commission on International Tables, volume E (2010), whereas instructive examples are in, e.g., Makovicky (2016).

There exists another classification scheme for patterns, which is the so-called *structural classification.* This classification separates them into families, which contain the same or very similar elements and element-combination principles, but their members differ by well defined, incremental changes. For example, certain segments of the pattern grow incrementally, and the consecutive members of the family differ by a number of brick courses that build these portions. At the same time, the other portions, situated between the former ones, remain unchanged. These are so-called *homologous series* constructed by means of *element accretion.* Alternatively, one can modify the pattern by intercalation of additional ornamental elements into the original pattern *('baroquization'* of the pattern or creating *pattern-complication series* of gradually developing patterns). Further structural types of patterns and series are the expansion–reduction series, omission derivatives, element-substitution series, partial-overlap derivatives, vortex series, etc., which were sometimes developed by the artists (Makovicky 1989; Makovicky and Fenoll 1997a); for similar pattern categories, see Roe (1980). Although the present study offers only a limited set of examples for these categories, we shall mention them whenever applicable.

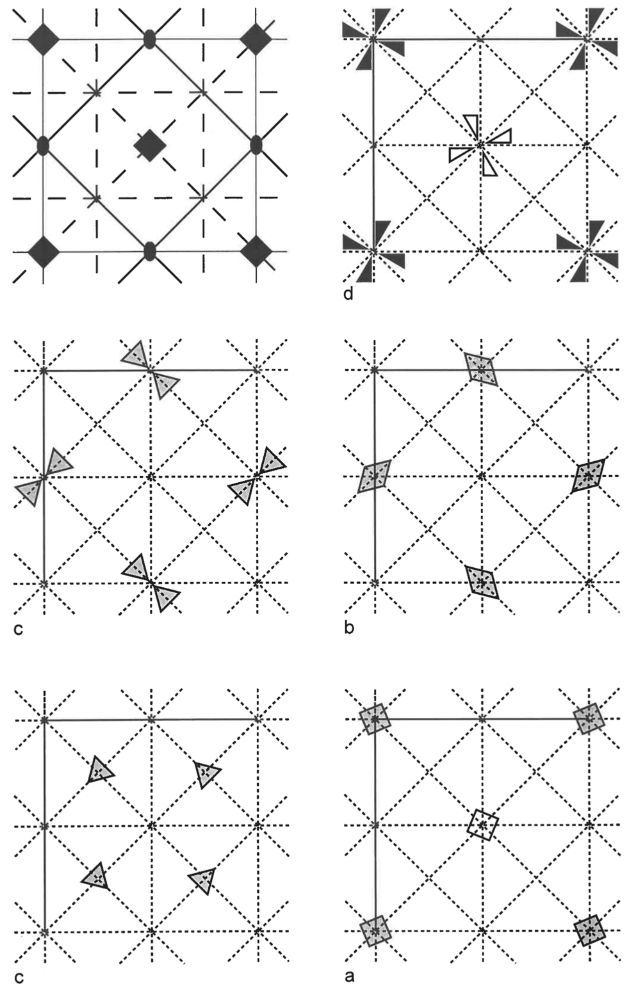

Fig. 3 Layer group *p*422, a frequent layer group in Islamic *illusionary-interlaced patterns*. Vertical ('perpendicular') rotation axes are indicated by symbols, horizontal twofold axes run in diagonal directions of the projection plane, and horizontal screw axes are indicated by dash-lines. Point materializations on the upper side of projection plane are black, and on the lower side as void symbols. General positions are illustrated by slim triangles surrounding the fourfold axes. Special positions are on, or straddling, different sets of rotation axes or reflection planes. Only **a** and **d** do not lie in the projection plane itself (which would be expressed by stippling of the symbols). Numerous examples are in Makovicky (2016)

5 The patterns

The one-dimensional and two-dimensional *brick patterns* of the Kharraqan tomb towers and of the Seljuq minarets can be divided into three categories:

(a) Flat-surface patterns created by orientation of brick-stacking ('flat-brick patterns'),
(b) Patterns created by protruding ribs and recessed background, both materialized as brickwork ('rib patterns'),
(c) Complex and very rich, 'loaded' rib patterns.

I shall describe the diverse ornamental patterns observed on the Kharraqan tomb towers as first. Patterns used on the walls of minarets display specific common features, and we shall handle them separately in the subsequent text.

The flat-brick-stacking patterns of the two towers (Fig. 4) consist of standard 1:2:3 bricks, which were inserted in the body of the wall as a combination of 'horizontal' and 'vertical' orientations, i.e., as two orientations 90° apart. Sometimes the patterns were completed by adding the 1:1 square bricks to the standard ones. There are two practical interpretation problems connected with these patterns:

(1) The large scale. They often exceed or nearly exceed the size of the panel, making determination of their periodicity and plane-group symmetry problematic.
(2) The vertical *versus* horizontal orientation of the bricks, used to create a visually distinct pattern of shadows, was the principal tool of artistic expression. From a distance it results in a visual *'damask effect'*. This arrangement invites dual interpretation of pattern symmetry—either an approach, which ignores the brick orientation and considers only the resulting *shapes and disposition of the fields outlined by them* (this approach will be used here), or an approach, which includes *brick orientations* into final interpretation, and results in reduced symmetry.
(3) The size of certain portions of these patterns can sometimes be varied by increments, adding another brick or a brick course (stripe) to them. In this way, we obtain the already mentioned *homologous series* of patterns. All members of such a series are constructed using the same pattern principle.

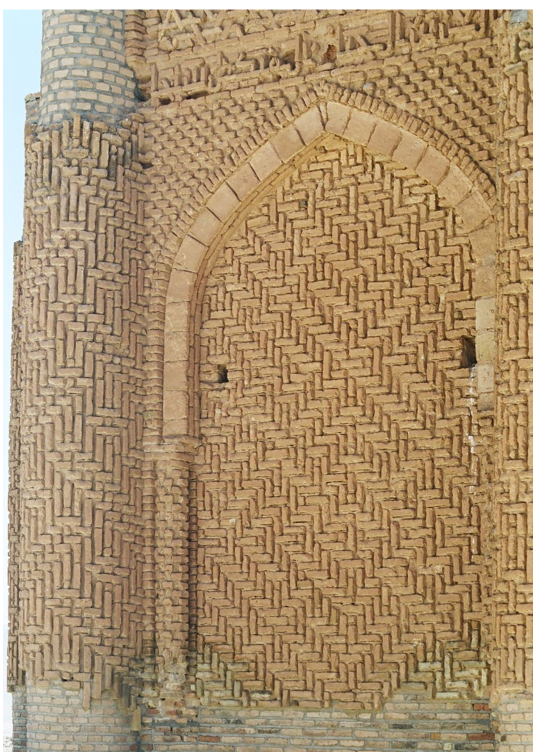

Fig. 4 Pattern of brick-composed squares, framed by an isometric framework. The flat-brick pattern is based on two brick orientations, resulting in the *damask* effect and the p4*mm* plane group of symmetry (symmetry expresses the final shape, not the orientation of individual bricks in it). A flat-brick *p*4*gm* pattern coats the adjacent buttress. SW panel, E tower

Two principal observed types of flat-brick patterns are (a) patterns of (sometimes framed) squares with or without a small central cross which is formed by a standing brick and two square bricks which flank it (Figs. 4, 5), and (b) patterns with larger framed crosses accompanied by smaller squares or frames (Figs. S2, S3, S5).

Figure 5 shows two members of a homologous *p*4*mm* series built from ornamental squares, and situated on two adjacent buttresses. The higher member has two additional brick courses added to the edge of the diamond-oriented squares, which characterize both members of the series. Centers of these squares define the vertices of the square unit mesh and house fourfold axes of rotation symmetry (in the interpretation sub (a) above; the other fourfold rotation axis lies in the intersection of single-brick courses

Fig. 5 Two members of the homologous series formed by a square-based *p*4*mm* pattern, with 7 and 11 tiers counted along the vertical diagonal of the unit square, covering two separate buttresses. A framed *p*4*mm* pattern of squares and crosses covers the intervening flat panel. Buttresses flanking the S panel of the E tower

separating the squares. Because of the brick orientation, a strong visual impression of *cmm* is present (this ambiguity was mentioned sub (2) above). The square diagonals in the two homologues are, respectively, 7 and 11 brick thicknesses long. The Damghan minaret has the same pattern with square diagonals 9 brick thicknesses in diameter, and with recessed central portions. Fig. S4 (in the supplementary set) and Fig. 4 display the squares framed by different frames, with symmetry *p*4*mm* (Fig. 4) but the patterns are too large to be resolved unambiguously.

The extant brick patterns with crosses have between six and nine bricks per entire arm of the cross. A well-preserved linear pattern *p*2*mm* is on a buttress shown in Fig. S3. The pattern with smallest crosses contains squares of two sizes (*p*4*mm,* Fig. 6) whereas those with larger crosses have double-crosses framed with a swastika-like configuration (*p*4*gm,* Fig. S5). Double crosses were also used in another large *p*4*gm* design. Finally, some large panels were interpreted only as point-group designs, with symmetries 4 and 4*mm* (Fig. S1).

The niches of the W tower are small and only rarely contain brick-stacking patterns. In Fig. 7a, they accommodate a simple *p*4*gm* pattern, composed of a combination of bricks with square bricks and an indefinite fragment of another, larger pattern, perhaps *p*2*mg*.

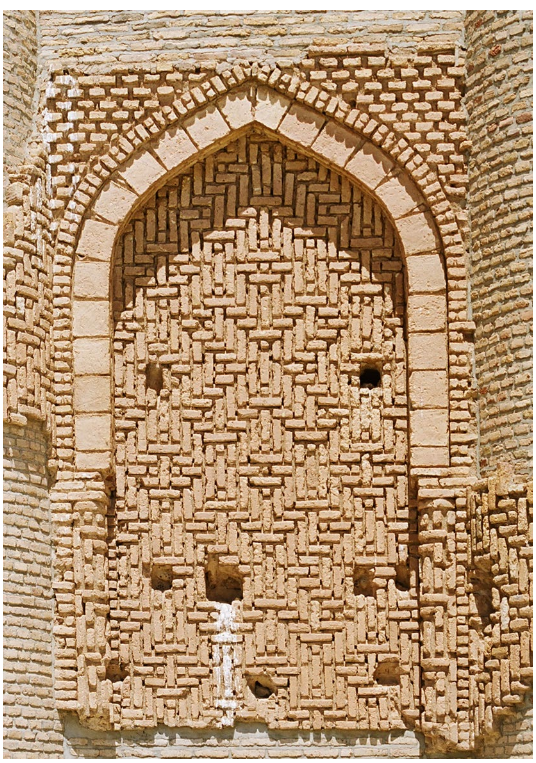

Fig. 6 Flat-brickwork; *p*4*mm* pattern of small crosses surrounded by squares of two sizes. E panel of the E tower

A rich spectrum of rib patterns with a contrast of raised bricks (viewed edge-on) and recessed fields, was used for tympana, panels, niches, and friezes of the tomb towers. Architects used it extensively also for the upper portions of minarets to create patterns visible from afar; whereas, the lower portions of the same minarets were usually covered by flat-brick-stacking patterns instead. In many patterns on the Kharraqan towers, the intricate brick layout, with one strand of (beveled) bricks crossing another, has been purposely designed as an *illusionary-interlaced pattern* and has to be interpreted as such, i.e., as a ‘virtual’ two-sided pattern and not as a plane pattern. This may not be true, however, for large-scale patterns on minarets, where the mutual positioning of bricks appears to be a technical matter.

The bricks employed in these patterns are flat bricks of different (exposed) length. When necessary, they were beveled on the ‘wall-inserted’ edges from the original 90° to (ideally) 67.5°, 60°, or 45°, to accommodate the orientation changes of adjacent bricks in the illustrated patterns.

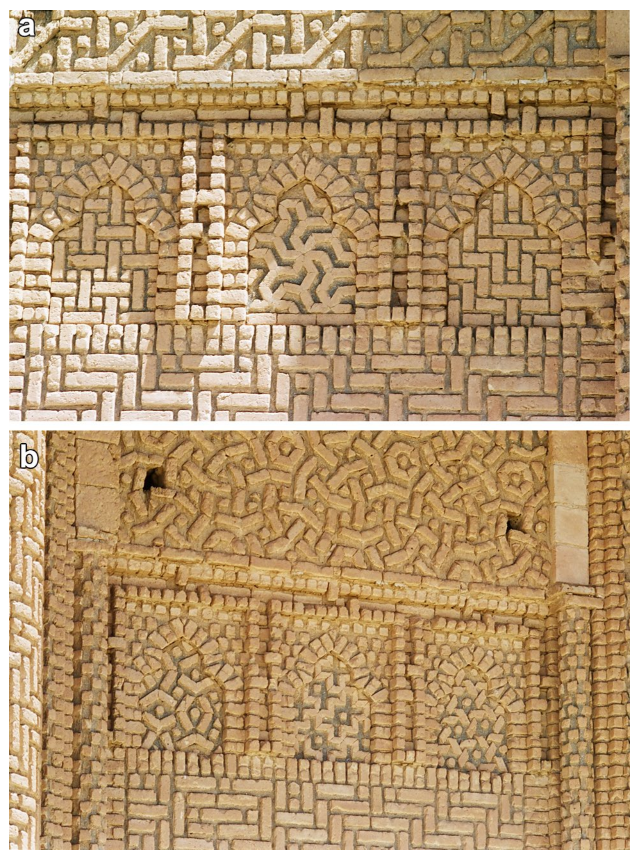

Fig. 7 **a** Flat-brickwork patterns in the niches of the western tower. Lateral niches: *p*4*gm* and *p*2*mg*; central niche: rib pattern *p*6. SSE panel of the W tower. **b** Niches of the western tower: two geometric aspects of *p*4*gm* as *rib patterns*, and a *p*6 diaper pattern composed of lozenge bricks and round buttons (right-hand side). Pilasters have a *cmm* pattern. SSW panel of the W tower

Ornaments of this category, constructed using lower-symmetry *plane groups*, are largely absent. Only two 'diaper ornaments' were observed, adorning the lateral pilasters, which flank some panels, and were composed either of raised lozenges or of shaped mirror-symmetrical elements on a recessed background. They display plane groups *cmm* and *cm,* respectively (pilasters flanking several panel photographs). The *cmm* pattern of disjoint cubes is on pilasters in Fig. 7b.

The plane group *p*4 is present in two patterns. A 'curvilinear pattern' on a buttress (Fig. 8) has fourfold 'flowers' and squares, respectively, positioned on the two sets of fourfold axes. An impressive flat panel has swastikas alternating with small squares (Fig. 9); the latter have local symmetry in excess of that of the *p*4 plane group. The swastika pattern can also be interpreted using layer group *p*-4, when we examine the central bars of swastikas in detail: we find that the vertical segment 'virtually overlaps' the horizontal arm. This, however, might be a technical rather than esthetic factor: in this way, bricks can be better fixed to the brickwork.

Several interesting rib patterns obey *plane group p*4*gm*. A 'double-fork' ornament in a large tympanum (Fig. 10), and a related 'X'-based ornament on a panel (Fig. 11), lead to a 'pattern of pointed parentheses' (Fig. S6). The latter pattern, in all important features, is a *dual* of the X-based ornament. The forked elements in Figs. 10 and 11 are positioned in the points with local symmetry 2*mm*, with dot-like elements in two different positions. Finally, the fourfold axes in the *p*4*gm* pattern can be occupied by swastikas, in agreement with their point symmetry, and together they form small, indented rectangles on 2*mm* (Fig. 7b, central niche). This pattern is known from large panels elsewhere (Makovicky 1989, Fig. 34) as also is the 'maple leaf' *p*4*gm* pattern in Fig. S7. We can see that the architect used the *p*4*gm* symmetry to construct several pattern types.

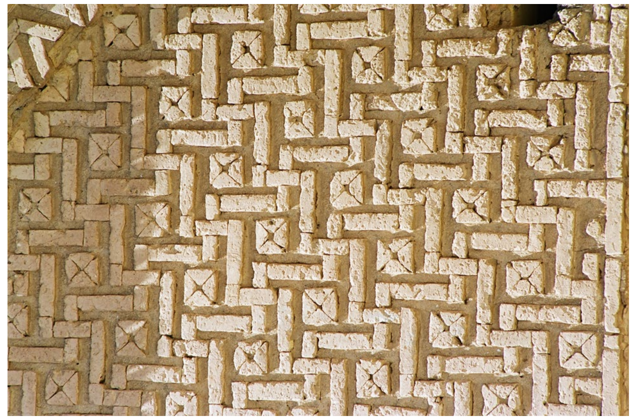

Fig. 9 The *p*4 pattern of raised bricks, combining interconnected swastikas of identical orientation and small squares. They are situated on alternative sets of fourfold axes of the plane group. NNE tympanum panel, W tower

Although it is not immediately obvious, the apparent '*p*4*mm* pattern' of interwoven brick ribs (Fig. 12) is composed of overlapping octagons, part of which comes from the outside of the pattern field. In spite of panel limitations, it is a 2D pattern, not a frieze pattern. The true, two-sided symmetry group is a layer group *p*422, and not *p*4*mm;* the virtual interweaving is flawless and intentional, with an ornamental purpose.

Bewildering by its complexity is a large-scale rib pattern in one of the tympana of the W tomb tower, composed of squares, pentagons and triangles with *a dual net* drawn over all of these elements (Fig. 13, see also Makovicky and Ghari 2018). The visual prominence of partitioned pentagons and rhombs obscures its true *p*4*gm* symmetry. The play of tilted

Fig. 8 A deceptively curvilinear pattern composed of straight brick segments. Symmetry *p*4 with fourfold rotation axes on the 'flowers' and on squares. A buttress of the W tower

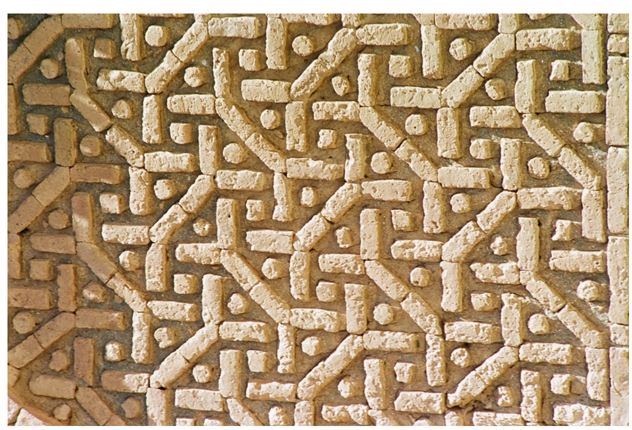

Fig. 10 An expressive *p*4*gm* panel with a double-Y (double-fork) pattern in the tympanum of the W tower. Twin forks are positioned on twofold axes and intersection of reflection planes, i.e., in 2*mm* positions. The loose square configurations are situated on fourfold axes. SSE panel of the W tower

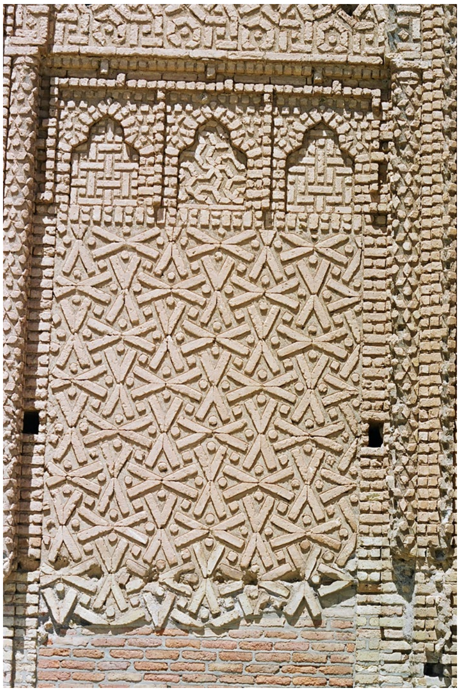

Fig. 11 An X-based 'chopstick' version of the *p*4*gm* pattern. The ENE panel of the W tower

Fig. 12 A pattern of interwoven ribs with a layer group *p*422. E tower. This pattern of conspicuous fourfold rosettes actually is a pattern of overlapping octagons, which are present as 4 + 4 octagons for each central octagon. See also Fig. S14, the top frieze

squares stands in interesting contrast to the strict symmetry of the surrounding patterns.

The tomb towers of Kharraqan display a profusion of beautiful hexagonal and trigonal rib patterns. The simplest hexagonal patterns are 'diaper' ornaments (e.g., Fig. 13, in the triangular side panels). Raised lozenges outline the recessed hexagonal stars in these patterns. Depending on the orientation of lozenges, both *p*6 and *p*6*mm* patterns were

Fig. 13 The complex *p*4*gm* rib pattern of squares, lozenges, triangles, and pentagons, with superimposed dual lines. Surmounted by a diaper *p*6 pattern and a linear *p*2*mm* pattern. The NNW tympanum panel of the W tower

Fig. 14 A pattern formed by intermeshed hexagons positioned on sixfold axes, resulting in the layer group *p*622. It is a pattern with two different alternating line spacings. Present on ENE and WSW panels of the W tower

created. Several rib patterns with hexagonal symmetry were generated either by large overlapping hexagons centered on sixfold axes (Fig. 14) or, alternatively, by smaller overlapping hexagons centered on threefold axes (Fig. 15, top).

Nested hexagons create near-continuous parallel double-lines, which are combined with sixfold stars in the tympanum shown in the upper portion of Fig. 16 and in Fig. 14. A complex rib pattern of hexagons and zig-zag lines in Fig. 16 contains local 12.2.2 symmetries of the *two-sided layer type*, surrounded by 622 hexagons. It 'predicts', in a

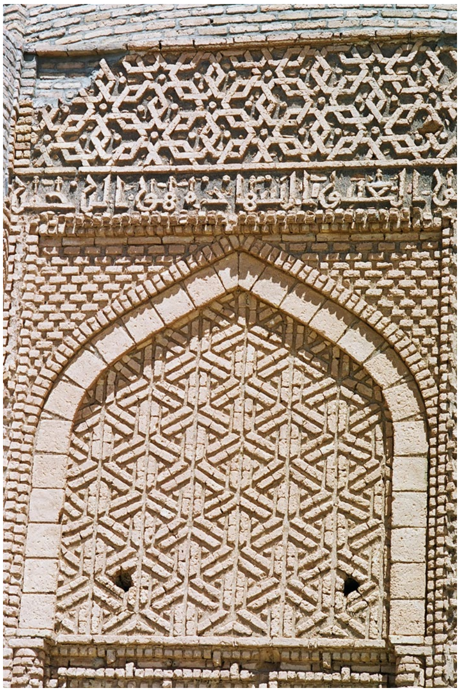

Fig. 15 An interwoven two-sided *p*-312/*m* pattern consisting of two kinds of beveled bricks. Surmounted by a frieze of small hexagons positioned on threefold axes (layer group *p*622). The ESE panel of the W tower

Fig. 16 Frieze: a pattern of nested hexagons and hexagonal stars with a (partly obscured) layer group *p*622; a cut-out of a 2D pattern. Tympanum below: A complex hexagonal pattern (layer group *p*622) with 'Allah' written in the 12-fold star-shaped recesses. Intervening wall: a brick layer course *cmm* with plugs between bricks. N panel of the E tower

way, the frequent Islamic pattern types used by later generations of artists. The frieze in Fig. 16 and the unique, hardly recognizable type of frieze in Fig. S8 differ by 30° rotation of hexagons against the double lines. *In all these patterns,* the plane group *p*6*mm* has been altered into a *layer group p*622. Only very rare interlacing errors were located, showing a high-quality work.

In another category of patterns, the pattern can be understood as constructed from, or completed by, zig-zag lines alone, with symmetry *p*6. Recessed spaces are based on threefold rotation axes, as triangular fields (Fig. 17), or they occur as S-shaped fields, which are based on twofold axes (Fig. 7a, the central niche). These two pattern types are duals of one another. A simple zig-zag line creates the *p*622 pattern in Fig. S9 whereas the construction of the double-hexagon version (Fig. S10) is more complex. These two versions differ also in the spacing of the underlying parallel construction lines: 1:1 spacings for Fig. S9 and approximate 3:1 for the S10 a pattern. Figure S10 b contains three different spacings in approximate ratio 4:2:1. Pattern in Fig.14 has alternating narrow and broader spacings as well. All this suggests that these simple underlying construction principles were already known and in use at the time of tower tomb construction.

The *mihrab* in the interior of the W tower hosts a repetition of the above-mentioned *p*6 pattern of S-shaped recessed fields in its upper portion. The principal panel of the mihrab wall, however, has a complicated *p*6 pattern composed of 'arrows' with side attachments. The arrows point outwards from the sites of sixfold axes. It is a two-sided *layer pattern* with intertwined ribs but its sixfold axes are *polar* and

Fig. 17 A *p*6 pattern of shaped bricks in the frieze of the NNE panel; W tower. A simple pattern with one type of recessed fields, positioned on threefold rotation axes

there are no layer-reversing elements of symmetry present (Fig. 18).

There are two distinct trigonal patterns present: a 'palmate' $p31m$ pattern with 'threefold snowflakes' on one set of threefold axes, and threefold swastikas on the other set of threefold axes (Fig. 19), and a pattern with a virtual 'interweaving' of double-strands that meet in one set of threefold axes of the layer group p-312 (Fig. 15, a large panel). The same pattern occurs on buttresses and, in a narrow form, as a frame of the adjacent tympanum (the 1D group is pg, now mostly preserved as $p1$ due to weathering). It is transitional to the flat-brick stacking.

The loaded rib patterns, each of them rich in a number of ornamental elements, form some of the most spectacular panels on the Kharraqan towers. The other patterns of this category are those applied to buttresses, in which the verticality of the motif becomes enhanced.

Most of these patterns were constructed with swastikas or recurved swastikas. However, nowadays they mostly contain 'lost swastikas', in which some of the short arm portions, probably originally fashioned out of stucco, disintegrated with time and in wet weather (Fig. S11). In two of these cases, remnants of the lost parts are recognizable

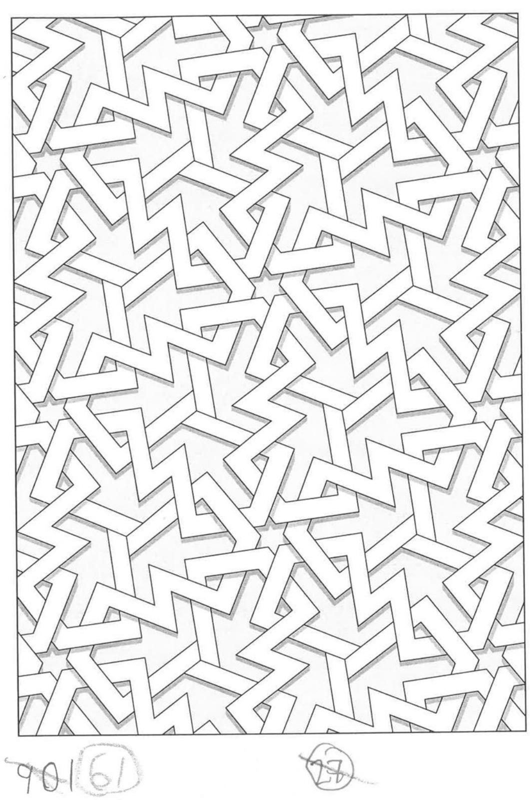

Fig. 18 The two-sided interlaced panel with a polar, non-reversing layer group $p6$. The recessed motif (with traces of blue paint) consists of arrows pointing outwards from the sixfold axes, adorned with side attachments. Mihrab inside the W tower; redrawn from a photograph

Fig. 19 A $p31m$ pattern of 'threefold dendrites' ('snowflakes') positioned on one set of threefold axes; the alternative set is occupied by triskelions (threefold swastikas). SSW panel of the W tower

in those panel portions, which remained protected by the arch of the tympanum. Thus, in the Fig. S11 pattern, the original $p4gm$ symmetry was reduced to pgg by the loss of swastika arms and, in the weathered pattern, further reduced to pg because the small bricks that extend the diagonally oriented pair of bricks and are always situated at their upper end of the diagonal, were made bare and conspicuous. The pattern in Fig. 20 is more complicated because of the *recurving* swastikas, now reduced to sets of parallel, vertically oriented bricks. The original plane group $p4gm$ has been reduced in the same way as in Fig. S11. Figures 20 and S11 are two versions of one pattern type; mostly they differ only in details. For example, in the modification from Fig. S11, dots replace the small triangles, which are present in the version from Fig. 20.

Degraded swastikas characterize also the large-scale, originally hexagonal pattern of squares based on recurving swastikas, and arranged around sixfold stars and around a common vertex of three squares situated on threefold axes. The adjacent wreaths of swastikas share some of the swastikas. The underlying star-and-square pattern $p6mm$ became reduced by insertion of swastikas to $p6$, but by further degradation of them by weathering, to final $p2$ (Fig. 21). The weathered version of virtually 'overlapping ribbons' acquired its own beauty. An expanded version of this pattern, in which the wreaths of swastikas around sixfold axes do not overlap, contains most of the stucco arms which were lost elsewhere (although they are present in a damaged form), and forms a nice $p6$ pattern (Fig. 13). We observed no two-sided reversal in these planar patterns.

Poorly preserved swastika arms occur also in a difficult, large-scale $p4$ pattern, which is the 3x3 superstructure of a simple swastika stacking which contains 2 x 2 blocks of swastikas (Fig. 22). The horizontal twofold axes, suggested by the framing (diagonally oriented) elements in the inter-block

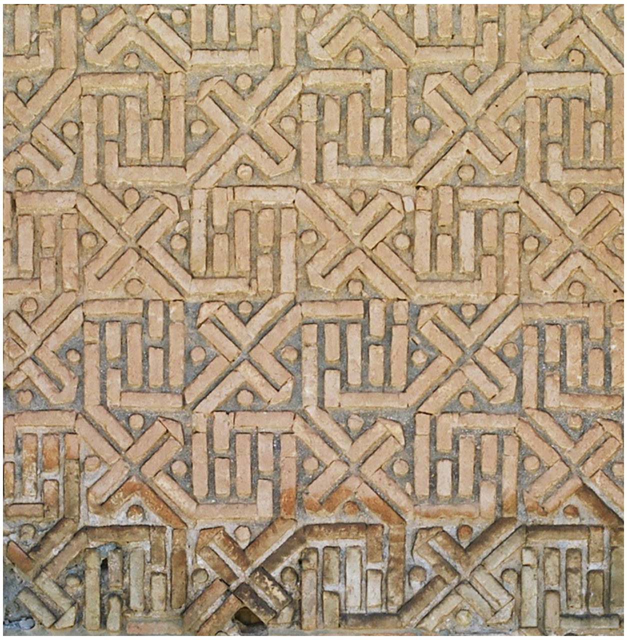

Fig. 20 The WSW panel of the W tower. In the original *p*4*gm* pattern, swastikas were on two sets of fourfold axes, whereas lying crosses were situated on twofold axes. Weathered to *pgg;* for details see text

Fig. 21 WNW panel of the W tower. A *p*6*mm* tile scheme altered into the *p*6 pattern by presence of swastikas, which became reduced to blocks consisting of parallel bricks by a loss of plaster cross-arms during weathering. The apparent symmetry today is *p*2

Fig. 22 A large-scale superstructure based on swastikas and eightfold stars. Rudiments of swastika cross-arms are locally visible; two-sidedness of the motif is rudimentary. NE panel of the E tower

strips, are not active for the swastikas and swastika blocks, leaving only the *p*4 symmetry for the entire pattern.

The hexagonal pattern in Fig. 23, constructed from winding *double-strands* that create a pattern of hexagons, lozenges, squares and triangles, has a layer group *p*622. These elements occupy positions with local symmetries 62, 222, the general positions, and the 32 position of the *layer group*. No closed line loops are present and both the open-V configurations and the hexagons are composed of flat-brick segments. Equally loaded is the structurally related pattern of overlapping dodecagons that enclose nested hexagons in Fig. 24, in which the plane group *p*6*mm* has been altered quite purposely into a layer group *p*622. Its 'empty', not loaded version is the cartwheel pattern adorning a frieze of the E tower, with the same symmetry modification. It is shown in Fig. S13.

The *'arched' patterns* are found mostly on corner buttresses; in their design, they follow the verticality of the buttress. Several *cm* patterns (Figs. 25, 26) and one *pm* buttress ornament (Figs. 26 and S14) were constructed with prefabricated floral embellishment. The swastikas mostly suffered the same weathering damage as was observed in the patterns already mentioned. The only *cm* pattern of this type, spread over a panel (Fig. 14) is very closely related to the *cm* pattern on buttresses and shares with it a number of motifs and overall design.

6 Pattern statistics

With their profusion of patterns, the Kharraqan towers invite a statistical treatment of pattern types and symmetry groups used. The eastern Kharraqan tower (Fig. S14) has

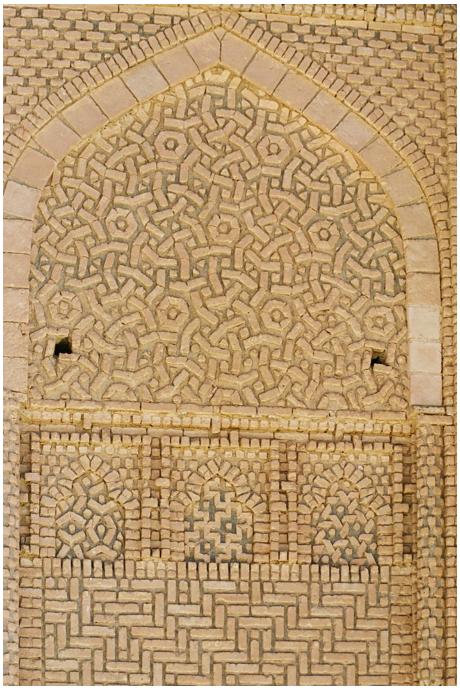

Fig. 23 An interwoven *p*622 pattern with five types of recessed 'tiles': nested hexagons, lozenges, squares and triangles. Constructed as interweaving of double-strands of sinuous lines. SSW panel of the W tower

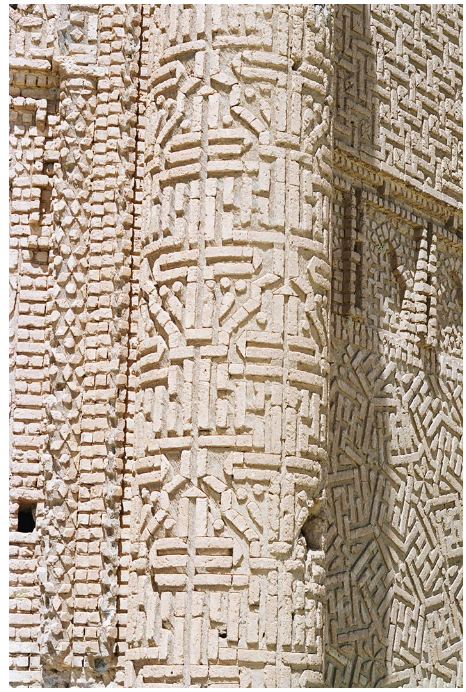

Fig. 25 A heavily loaded vertically oriented large-scale *cm* pattern with damaged swastikas and extensive superficial similarity between non-equivalent triangular fields. Local symmetry of swastika-containing horizontal segments does not extend beyond them into the segments of triangular fields. N buttress, W tower

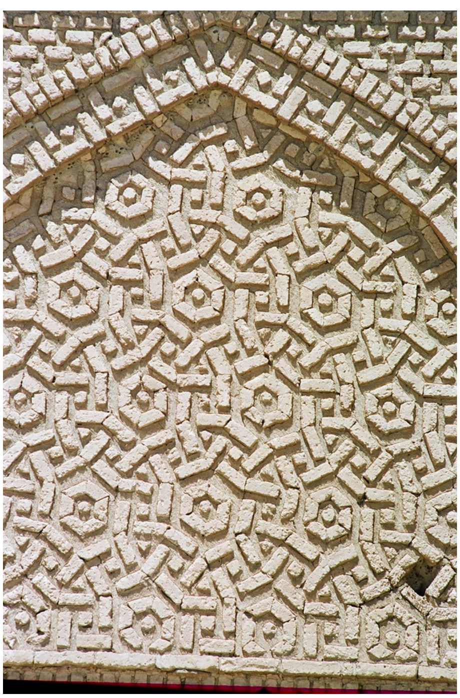

Fig. 24 Pattern of interwoven 12-gons centered on sixfold axes, which are marked by hexagons. A complex pattern, which is a slight geometric modification of the pattern in Fig. S 13, obtained by reducing the size of dodecagons. All 'recessed' fields are centered by dots. ENE panel, W tower

two patterns on each prism face; whereas, the western tower has six patterns on each face, except for the entrance panel. Corner buttresses and miniature pilasters flanking the panels carry additional patterns. For the western tower it ideally makes 45 panel patterns plus 16 buttress/pilaster patterns, i.e., 61 patterns. With two additional patterns in the interior, it makes ideally 63 patterns in all (without subtraction of the destroyed ones) on flat and curved surfaces. To them, we must add several one-dimensional patterns positioned on the tympanum arches. In this way, we reach potentially almost 70 patterns displayed on the W tower. The eastern tower has 32 patterns, what makes up to 36 patterns when the unidimensional patterns are included; from this we have to subtract some patterns which were destroyed. All top friezes on both towers are broad enough to allow interpretation as unmodified 2D patterns and not as 1D patterns. Overview of pattern distribution has been summarized in Table 1a and b, as well as in Figs. S15 a–d.

The visual role of patterns on flat surfaces and flat panels differs from that of the patterns enveloping cylindrical buttresses and pilasters. We shall treat them separately because the ornaments in these locations also differ in the choice of symmetry groups.

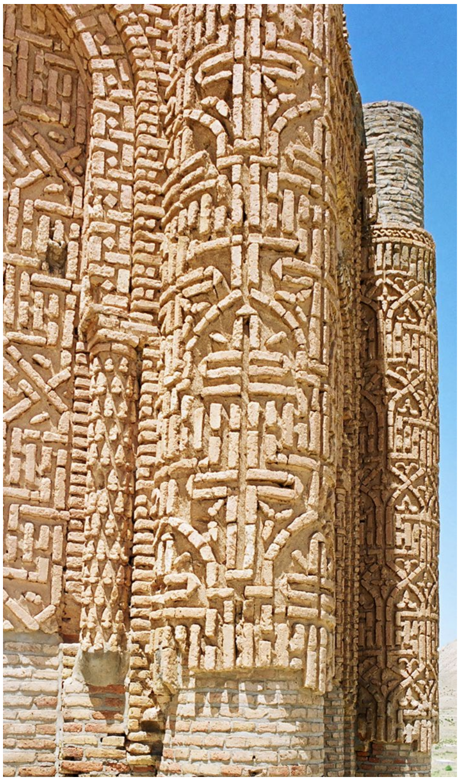

Fig. 26 Buttresses at the NW side of the E tower: arched patterns *cm* (foreground) and *pm* (rear). Left-hand side: a pilaster with a *cm* diaper pattern of pointed tiles surmounted by a *p*2 rim of a panel

6.1 Panels on prism faces

When we exclude an occasional repetition of some patterns on a tower, the E tower has 5 distinct *flat-brick* (*i.e.,* brick-stacking) patterns *versus* 11 *rib* patterns whereas the W tower has 6 flat-brick patterns and 21 rib patterns. Hexagonal/trigonal patterns are absent among the flat-brick patterns; by their nature, they are primarily rectangular and quadratic patterns. Brick stacking patterns are absent in tympana.

Two kinds of statistical tabulations will be performed for the material from Kharraqan:

(a) *Frequency of plane groups of symmetry* on all panels of a tower, including all repetitions of any pattern type with its *plane group of symmetry* as well as all repetitions of the plane group by means of different pattern types. This is the principle behind Table 1a and b.
(b) *Occurrence of distinct pattern types*, discounting potential repetitions of some of the patterns on a tower, again expressed using their plane groups of symmetry. The results given in a graphical form are in Figs. S15a–d.

Both towers have a clear preponderance of the plane group *p*6*mm*, which, in a more detailed analysis, always is a *layer group p*622 *in all cases*. The western tower (Fig. S16) is distinguished by important occurrence of *p*4*gm* and *p*6 patterns, whereas for E tower *p*4*mm* (partly *p*422) is important, instead. The generally low frequency of *p*2, *p*4, *cm, cmm,* primary *pgg,* and *pmg,* and absence of *p*1, *p*3, (also of *p*6 on the eastern tower), *pm, pmm,* and *p*3*m*1 are conspicuous for panel ornaments in the Table 1.

Care must be exercised in the interpretation, as the *p*4*gm* pattern of swastikas has almost always been *reduced* to *pgg* in most of its area by weathering, except for the portions protected from the elements by the ornamental arch ledge. As already mentioned, the weather-unstable portions of swastikas ‘withered away’ during centuries of exposure. The long arms of the swastikas are composed of a long and a short brick in succession, reducing in some cases the fully weathered pattern to a plane group *pg*. Only the *original plane groups were included in the count.*

Repetition of patterns on towers is minimal, except for the niches where it is considerable, being one of the main reasons of differences between Table 1 and Fig. S15a–d. Search for correlations between plane groups of patterns on the same face/wall of the octagonal W tower gave *no preferred combinations* for panel-tympanum pattern pairs. Similar analysis of correlations between top friezes and large panels on both towers revealed that its results are determined by the absolute preponderance of *p*6*mm* (in fact, the layer group *p*622) in top friezes (Table 1a, b). The combination of any plane group, out of the entire spectrum of plane groups found on large panels or tympana, with a hexagonal ‘*p*6*mm’* pattern on the frieze, is the usual scheme employed. Combination *p*6*mm*–*p*6*mm* between the panel and the frieze is rare on both towers, as also is occurrence of two quadratic groups together on the E tower. These statistics do not confirm the conclusions of Bier (2002) about pattern combinations.

6.2 Patterns on pilasters and corner buttresses

In the majority of cases, patterns on the curved surfaces of pilasters and corner buttresses can be unambiguously interpreted as cut-outs of 2D patterns; exceptions are several flat-brick, large-scale patterns, for which the extension itself is unclear. Only rarely, we deal with obvious 1D patterns, an example being the prominent brick-stacking pattern *p*2*mm* on two adjacent buttresses of the W tower (Fig. S16).

Frequency diagrams for this category (last two columns in Table 1, Figs. 15c, d) differ in several substantial features from those for the flat patterns which are on the faces of the octagonal prism, i.e., of the tomb tower. Principal plane groups of small pilasters are *cm* and *cmm*, representing primarily small-scale diaper-type patterns. For buttresses, especially the plane group *p*4*mm* occurs as flat-brick patterns. The rest of plane groups are present as one or two patterns each. *Plane* groups *p*1, *p*3, *p*6, *p*2*mm*, *p*3*m*1, *pg*, *p*2*mg*

Table 1 Distribution of ornamental patterns over the panels and buttresses of the western (a) and eastern (b) Kharraqan tower (situation before earthquake)

Ornamental patterns of the W tower[a]								
Octagon face	Lower panel	a	Niches b	c	Tympanum	Frieze	Small pilasters	Corner buttresses
ENE	p4gm	p4gm (F)	p6	cmm? (F)	p6mm	p6mm	p6mm	ENE/ESE p4(F, damaged)
ESE	p2? (large, F)	p6	p4gm	p4gm	p31m*	p6mm	p2	ESE/SSE Linear pgm (F)
SSE	p4mm?	p4gm(F)	p6	pmg (F)	p4gm	p6mm	cmm	SSE/SSW Linear pgm (F)
SSW	n.d.	p4gm	p4gm	p6	p6mm	p31m	cmm	SSW/WSW p4mm (F)
WSW	pgg	p4gm (F)	p6	p4gm	p4gm	p6mm	cmm	WSW/WNW p4
WNW	p2	p6	p6	p4gm	cm	p6mm	cm	WNW/NNW p31m*
NNW	Taken by	Quran	texts		p4gm; p6	p4gm	cm	NNW/NNE P31m*
NNE	p6	p4gm	p4gm	p6	p4	p6	cm	NNE/ENE cm
mihrab	p6*	–	–	–	p6	–	–	–

Ornamental patterns of the E-tower[b]				
Octagon face	Principal panel	Frieze	Small pilasters	Corner buttresses
E	p4mm (F)	destroyed	p2	E/SE p4mm? (F, mostly destroyed)
SE	cmm (F)	p6mm	p2	SE/S p4mm (F)
S	p4mm (F)	p4mm*	cmm	S/SW p4mm (F)
SW	p4mm (F)	p6mm	cmm	SW/W p4gm (F)
W	p4gm	p6mm	cmm (F)	W/NW p4mm? (F)
NW	pgg	p6mm	cmm	NW/N pm
N	p6mm+point group 4	p6mm	pgg (F)	N/NE cm
NE	p4	p4mm*	cm	NE/E p4mm? (F, partly destroyed)

[a]All instances of the *p6mm* plane group of symmetry are *p*622 on very close inspection. Those plane groups in the 'tympanum' and 'buttress' columns, which are indicated by asterisk, are layer groups on close inspection of their brick laying, as well. F indicates flat-brick patterns; the rest are rib patterns. Question marks indicate very large patterns, which exceed the panel area in size.

[b]All cases of the plane group of symmetry quoted as *p6mm* are those of layer group *p*622 under more detailed inspection. The cases of *p4mm** are those of layer group *p*422 in a similar way (details in the text). F = flat-brick patterns, all the rest are rib patterns

are missing in this position on both towers. Hexagonal and trigonal patterns are extremely rare: we registered two *p*31*m* rib patterns on buttresses on the W tower, where they occur also on its tympanum, and found a diaper *p6mm* pattern on one of its pilasters.

The *cmm* patterns from pilasters were constructed mostly from bricks that are close to a square shape, with recessed plaster between them. Although they may approximate a square system, their geometry remains rectangular, i.e., *cmm* patterns. An impressive diaper *cmm* pattern composed of 'standing' slim lozenges is present on pilasters framing the NW side of the E tower.

6.3 Patterns used on frames of the panels

Several narrow frieze patterns form the frames of selected planar panels. Certain of them are fashioned of bricks, e.g., the *p*2 frieze framing the arch of the NE panel of the E tower, and surmounted by a *p2mm* frame consisting of bricks as well (Fig. 22). The complete combination occurs also on other faces of this tower and as an arch on the NEE panel of the W tower. Bead-like strings *p2mm* of shaped bricks and a damaged *pg* frieze derived from a trigonal pattern, framed by a *p2mm* stripe, are present on the W tower (Fig. 27).

6.4 Seljuq minarets

Except for the minaret in Saveh that has lower portions devoid of ornamentation because this fell victim to the elements, all the studied minarets have lower portions, two–four ornament tiers high, covered by flat-brick ornaments (Fig. 28). These are followed by upper portions, with between two and five tiers of rib ornaments (Figs. 29, S17, S18). Number of different positions on, or between, the

Fig. 27 The WNW panel of the W tower. Tympanum: a *cm* pattern of complex upward pointing elements and dotted infill. Note similarity of design to the *cm* buttress pattern in Figs. 25 and 26. Niches: two types of *p*6 patterns and the *p*4*gm* pattern of 'pointed parentheses'

Fig. 28 The bottom portion of the Damghan minaret is conical and the (possibly intended) *p*4*mm* flat-brick pattern becomes split and modified in its lower half. At the top of this portion, a repetition of nine 'standing' bricks and 14 lying bricks results in the *cmm* pattern with a separation of horizontal zig-zag strips. In the conical bottom portions, we observe a growing misfit of strips. In the cylindrical top portion, a 16×16 brick pattern creates large squares, spanning seven bricks, and is modulated by empty squares and braided fragments; plane-group symmetry *p*4*mm*. Graphic illustration of this pattern is in Makovicky (1989), (his Fig. 23)

symmetry operators of the flat-brick ornaments is limited. Thus, the flat-brick ornaments are *intermediate-complexity patterns* at maximum, with frames altered by intercalated elements. To the contrary, the very impressive rib ornaments on these buildings are (*very*) *complex patterns*, with many recessed fields (some of them nested) which are positioned on a spectrum of general and special positions (Fig. 29).

What makes the minaret ornaments distinct from the Kharraqan tomb towers is the decidedly limited choice of plane groups of symmetry. The bulk of cases are patterns in the plane group *p*4*mm*, with very rare occurrences of *p*4*gm* and of clear *cmm* (Figs. 28, 29, S17, S18). On the Semnan and Damghan minarets, several *p*4*mm* patterns are slightly compressed or extended along the vertical axis, yielding *cmm* when exact geometry is considered. For the cases, when such distortion is caused by brick/mortar thickness and dimensions, we shall ignore these obviously pure technical modifications. No persuasive cases of interweaving were seen; the technical aspects of bricklaying and attachment seem to have determined the relations between adjacent bricks on these patterns. *These patterns were designed to be contemplated from afar,* from distances at which individual bricks coalesce into continuous ribs. Shadows created by the rib *versus* recess interplay play a decisive role.

Technically interesting is the limitation of ornamental ribs on the minaret walls to the vertical and horizontal ones, with rare cases of diagonal ribs, at 45° to the bulk of ribs present in the pattern. We did not observe any analogs to the rich choice of symmetries and geometries seen at Kharraqan. Details of brick arrangement reflect the technical solutions. In general, the superstructures are not simple multiples of small squares; sometimes half-squares were approximated.

Conspicuous is the repetition of these patterns over different locations. The flat-brick patterns with either elongate elements or full crosses inserted into framing occur on both minarets in Damghan, the latter one also at Semnan. Identical patterns of simple framed squares occur on the Tariq Khana minaret and the Semnan minaret. Two *p*4*mm* rib patterns on the two minarets built in Damghan are identical and one of them is repeated in Semnan. A pattern of large octagons 'interconnected' by elliptical links (Fig. 29) is common to Saveh ('upper' pattern, Fig. S18) and Masjid-e-Jameh, Damghan; its simplified versions are found on later minbars, etc. elsewhere in Iran.

Fig. 29 The top portion of the Damghan minaret with four tiers of tetragonal rib patterns, all with the plane group *p*4*mm*. Patterns from the bottom: (1) Diagonally oriented unit cell with small squares and void crosses, respectively, on two sets of fourfold axes. (2) Eight-fold stars and octagons define unit cell vertices and regular crosses are in the center of the cell. Oval polygons interconnect the octagons in this Islamic pattern repeatedly used at different monuments. (3) A very large diagonally poised unit cell with empty small squares in the corners and a complex cross-like arrangement of small squares in the center. (4) Pattern with a small unit cell and tighter line overlap; small squares and empty crosses mark the fourfold rotation axes

The illustrated 'lower' Saveh pattern (Fig. S17) can be interpreted in two ways:

(*a*) an intercalation derivative of a *p*4*mm* pattern in which the double-Y elements were first split and then doubled and separated from one another by a strip of short braided elements, or by the adjacent broader patch of fourfold groups, or (*b*) as a tetragonal primitive arrangement of octagons (which contain infill of 4 crosses) separated by 'collapsed' strips in which two octagon fragments overlap and enclose a braided remnant of the previous infills (Fig. S17). Both interpretations are possible, and both represent truly outstanding geometric creativity.

The just described pattern distribution can be extended to other Iranian minarets. The Barsiyan Mosque Minaret (1097) has a flat-brick pattern with squares 7 brick courses in diameter and crosses inserted in the framing, the Malik Mosque Minaret, Kerman (11 century), has squares 9 crosses broad, and simple, elongate inserts in framing, whereas the Shrine-of-Bayazid Minaret at Bistam (probably 1120) is topped by two *p*4*mm* rib patterns, one of them a simplified version of the Damghan rib patterns (Hutt and Harrow 1977).

7 Conclusions

The planar ornaments on the faces of the octagonal *tomb towers* at Kharraqan show about equal numbers of tetragonal (square) patterns and hexagonal patterns, a phenomenon a bit unusual for ornaments constructed from bricks. For example, such ornaments from the times of Ummayad Emirate of Cordoba, Spain, contain substantial proportion of tetragonal patterns, besides the rectangular symmetry groups with twofold axes (Makovicky and Fenol Hach-Alí 1997a) but no hexagonal patterns. The statistic of plane groups from Kharraqan resembles rather closely that of the collection of Islamic ornaments by Bourgoin (1973), i.e., it looks like the average distribution of plane groups of symmetry, independent of the medium used. All '*p*6*mm*' patterns belong to a two-sided layer group *p*622 in 'virtual' presentation. Buttresses of the younger, W tower have similar distribution, except for a high frequency of the *cm* designs. Buttresses of the E tower appear more influenced by the construction material. They lack three- and sixfold axes and resemble the Cordoban statistics.

Erosion of certain small elements of ornaments may have altered the overall plane-group symmetry and careful analysis of the entire pattern may be required to re-establish its original character (e.g., from *pgg* back to *p*4*gm*). Slight affine distortions, especially those of complex tetragonal patterns obviously were unintentional, a construction problem, and should be ignored. There exist exceptions, however, e.g., the minaret of the Jami'Mosque in Gurgan, Iran (12th century) on which the original *p*4 group has been affinely distorted to *p*2, with the original diagonals of a square altered from the ratio 1:1 to 1:2 (Makovicky 1989, his Fig. 18).

Except for flat-brick patterns of framed squares, among which cases with 5, 7, and 9 brick courses across a square occur, structural homologous series are rare in the sampled material, and limited to a few pattern pairs. Baroquisation of patterns by intercalation of additional elements is occasionally present.

The very narrow choice of plane groups of symmetry applied to the *minarets* studied, and the repetition, on different minarets in the studied region, of complex rib patterns that require complicated schemes of bricklaying, points towards one and the same workshop or to a tightly followed tradition with a sequence of pupils turned masters.

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s12210-023-01171-3.

Acknowledgements The present study in 2002 was made possible by a research travel grant from C.L. Davids Fond og Samling, Copenhagen, Denmark. Mrs. Britta Munch's professional help in graphic works during her employment by the University of Copenhagen, is gratefully acknowledged. To stress again: *this is a pictorial requiem* for the two magnificent Kharraqan tomb towers, which succumbed to the powers of Nature, and not to a human hand (as it unfortunately is usual). I express my admiration for master builder Muhammad ibn Makki al-Zanjani who left behind such architectural jewels, and *I direct urgent appeals to the Ministry of Culture of the Islamic Republic of Iran to save at least something from these damaged monuments before it is too late.* My thanks to the anonymous reviewers for the careful inspection of the manuscript. Last but not least, I thank my wife, Dr. Milota Makovicky for her infinite patience and understanding.

Author contributions One sole author (including photography); Mrs. Britta Munch made the drawings when she was employed as a draftswoman by the Geological Institute of the University of Copenhagen.

Funding Open access funding provided by Royal Library, Copenhagen University Library. Part of expenses for the year 2002 were covered by a travel grant of C.L. Davids Fond og Samling (Copenhagen).

Data availability All buildings/objects studied or examined are publicly accessible without prohibitions on photography.

Declarations

Conflict of interest No competing interests of financial or personal nature were found.

Ethical approval Not applicable.

References

Abbas SJ, Salman AS (2007) Symmetries of islamic geometrical patterns, 2nd edn. World Scientific Publishing Co., Singapore

Bier C (2002) Geometric patterns and the interpretation of meaning: two monuments in Iran. Bridges 2002 Conference Proceedings, pp 67–78

Blair S (1991) The monumental inscriptions from early Islamic Iran and Transoxiana. Brill, Leiden

Bonner J (2017) Islamic Geometric Patterns. Springer, New York

Bourgoin J (1973) Arabic geometrical pattern & design. Dover, New York

Clévenot D, Degeorge G (2000) Ornament and Decoration in Islamic architecture. Thames and Hudson, London, p 224

Ettinghausen R, Grabar O (2001) Islamic art and architecture 650–1250. Yale University Press, New Haven

Hillebrand R (1999) Islamic art and architecture. Thames and Hudson, London

Hutt A, Harrow L (1977) Islamic Architecture. Iran 1. Scorpion Publ. Ltd, London, p 32

IUCr's Commission on International Tables (2010) International Tables for Crystallography. Volume E: Subperiodic Groups (online)

IUCr's Commission on International Tables (2016) International Tables for Crystallography. Volume A: Space-group Symmetry (online)

Lu P, Steinhardt P (2007) Decagonal and quasicrystalline tilings in medieval Islamic architecture. Science 315:1106–1110

Makovicky E (1989) Ornamental brickwork. Theoretical and applied symmetrology and classification of patterns. Comput Math Appl 17:955–999

Makovicky E (1992) 800-year-old pentagonal tiling from Maragha, Iran, and the new varieties of aperiodic tiling it inspired. In: Hargittai I (ed) Fivefold symmetry. World Scientific, Singapore, pp 67–86

Makovicky E (2008) Another look at the blue tomb of Maragha, a site of the first quasicrystalline Islamic pattern. Symmetry 19:127–151

Makovicky E (2016) Symmetry through the eyes of old masters. De Gruyter, Berlin, p 240

Makovicky E, Fenoll Hach-Alí P (1997a) Brick-and-marble ornamental patterns from the Great Mosque and the Madinat al-Zahra palace in Cordoba, Spain. I. Symmetry, structural analysis and classification of two-dimensional ornaments. Boletín De La Sociedad Española De Mineralogía 20:1–40

Makovicky E, Fenoll Hach-Alí P (1997b) Brick-and-marble ornamental patterns from the Great Mosque and the Madinat al-Zahra palace in Cordoba, Spain. II. Symmetry and classification of one-dimensional patterns. Boletín De La Sociedad Española De Mineralogía 20:115–134

Makovicky E, Ghari M (2018) Neither simple nor perfect: From defect symmetries to conscious pattern variations in Islamic ornamental art. Symmetry 29:279–301

Michell G (ed) (1978) Architecture of the Islamic world. Thames and Hudson, London, p 288

Rice DT (1991) Islamic art. Thames and Hudson, London

Roe PG (1980) Art and residence among the Shipibo Indians of Peru—study in micro-acculturation. Am Anthropol 82(1):42–71

Stern SM (1966) The inscriptions in the Kharraqan mausoleums. Iran 4:21–22

Stierlin H (2002) Islamic art and architecture. Thames and Hudson, London, p 319

Stronach D, Young TC (1966) Three Seljuk tomb towers. Iran 4:1–20

Walker RT, Bergman E, Jackson J, Ghorashi M, Talebain M (2005) The 2002 June 22 Changureh (Avaj) earthquake in Qazvin province, northwest Iran: epicentral relocation, source parameters, surface deformation and geomorphology. Geophys J Int 160:707–720

Washburn DK, Crowe DW (1998) Symmetries of culture: theory and practice in plane pattern analysis. University of Washington Press, Seattle