Positive solutions for m -point boundary-value problems with one-dimensional p -Laplacian

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Abstract In this paper, we investigate the existence of positive solutions for a classes of m -point boundary value problems with p -Laplacian. By applying a monotone iterative technique, some sufficient conditions for the existence of twin positive solutions are established.

Keywords Multi-point boundary value problems · Monotone iterative technique · p -Laplacian operator · Iterative positive solutions

Mathematics Subject Classification (2000) 34B15 · 34B25

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1 Introduction

In this paper, we study the existence of positive solutions to the following multi-point boundary value problem (BVP for short) with the one-dimensional p -Laplacian:

$$\begin{cases} (\phi_p(x'))' + q(t)f(t, x) = 0, & 0 < t < 1, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), & x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \quad (1.1)$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and satisfying:

- (H1) $\alpha_i, \beta_i \in [0, 1)$, and $0 \leq \sum_{i=1}^{m-2} \alpha_i < 1$, $0 \leq \sum_{i=1}^{m-2} \beta_i < 1$.
- (H2) $q(t) \in L^1([0, 1])$ is nonnegative on $(0, 1)$, and $q(t)$ does not vanish identically on any subinterval of $(0, 1)$.
- (H3) $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous and nondecreasing, and there exists a constant $\mu > 0$ such that, for any $t \in [0, 1]$, $u \in [0, +\infty)$,

$$f(t, \lambda u) \geq \lambda^\mu f(t, u), \quad \forall 0 \leq \lambda \leq 1. \quad (1.2)$$

- (H4) $f(t, 0) \not\equiv 0$ and $f(t, 1) \not\equiv 0$ for $t \in [0, 1]$, and let

$$\rho = \max_{t \in [0, 1]} f(t, 1). \quad (1.3)$$

The equation with a p -Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics [1, 4], nonlinear elasticity and glaciology [3], combustion theory [5], population biology [6, 7], nonlinear flow laws [3, 7, 8], and system of Monge-Kantorovich partial differential equations [2]. There exist a very large number of papers devoted to the existence of solutions of the p -Laplacian operator. The problem (1.1) with different boundary conditions has been studied by many authors, and for details the reader is referred to [9–18] and the references therein.

In the reference [10], the authors also investigated BVP (1.1) with the boundary conditions $u(0) = 0$, $u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$ (which is the special case when $\alpha_i = 0$ in (1.1)). Unfortunately, the expressions of solutions for the BVPs in [10] are not correct, which was pointed out by the authors of [19]. In [19], the authors have given the correct conclusions for BVP (1.1). Recently, by using the well-known Krasnoselskii fixed point theorem and defining $f_{\gamma, R, R}$ and $f^{0, r}$, Feng [13] present sufficient conditions for the existence of multiple positive solutions for the boundary value problem (1.1). Motivated by the works in [11–13] mentioned above, the purpose of this paper is to show the existence of multiple positive solutions to the BVP (1.1). Different from [11–13], we study the existence of positive solutions to the BVP (1.1) for the one-dimensional p -Laplacian by applying a monotone iterative technique. In suitable growth conditions, we find that there should exist twin iterative positive solutions which are the maximal and minimal solutions of the BVP (1.1). At the same time, we also establish iterative schemes for approximating this twin solutions, which start with a known simple linear function. Moreover, the figure of iterative sequences of solutions is explicit, which is helpful to compute for us. Some examples are also worked out to illustrate our main results.

2 Preliminaries and lemmas

Lemma 2.1 *If $f(t, u)$ satisfies (H3), then for any $\lambda \geq 1$, $(t, u) \in (0, 1) \times [0, +\infty)$, we have*

$$f(t, \lambda u) \leq \lambda^\mu f(t, u).$$

Proof In fact, for any $\lambda \geq 1$, $(t, u) \in (0, 1) \times [0, +\infty)$, by (1.2) we obtain

$$f(t, u) = f\left(t, \frac{1}{\lambda} \cdot \lambda u\right) \geq \left(\frac{1}{\lambda}\right)^\mu f(t, \lambda u),$$

so

$$f(t, \lambda u) \leq \lambda^\mu f(t, u). \quad \square$$

Let $E = C[0, 1]$ be our Banach space with the maximum norm $\|x\| = \sup_{t \in [0, 1]} |x(t)|$, and

$$C^+[0, 1] = \{x \in C[0, 1] : x(t) \geq 0, t \in [0, 1]\}.$$

Let

$$K = \{x \in C^+[0, 1] : x(t) \text{ is a concave function on } [0, 1]\},$$

then K is a cone of E . By the reference [10], for any $x(t) \in C^+[0, 1]$, define an operator

$$(Tx)(t) = \begin{cases} \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^{\xi_i} \phi_p^{-1} \left(\int_s^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau \right) ds \\ \quad + \int_0^t \phi_p^{-1} \left(\int_s^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau \right) ds, & 0 \leq t \leq \sigma_x, \\ \frac{\sum_{i=1}^{m-2} \beta_i}{1 - \sum_{i=1}^{m-2} \beta_i} \int_{\xi_i}^1 \phi_p^{-1} \left(\int_{\sigma_x}^s q(\tau) f(\tau, x(\tau)) d\tau \right) ds \\ \quad + \int_t^1 \phi_p^{-1} \left(\int_{\sigma_x}^s q(\tau) f(\tau, x(\tau)) d\tau \right) ds, & \sigma_x \leq t \leq 1, \end{cases} \quad (2.1)$$

where σ_x is a constant, which satisfies

$$\begin{aligned} & \frac{1 - \sum_{i=1}^{m-2} \beta_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_p^{-1} \left(\int_0^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau - \int_0^t q(s) f(s, x(s)) ds \right) dt \\ & + \left(1 - \sum_{i=1}^{m-2} \beta_i \right) \int_0^1 \phi_p^{-1} \left(\int_0^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau - \int_0^t q(s) f(s, x(s)) ds \right) dt \\ & + \sum_{i=1}^{m-2} \beta_i \int_{\xi_i}^1 \phi_p^{-1} \left(\int_0^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau - \int_0^t q(s) f(s, x(s)) ds \right) dt = 0. \end{aligned}$$

Thus, it follows from [10] that $\|Tx\| = (Tx)(\sigma_x)$ and the BVP (1.1) is equivalent to the fixed point problem of the operator T .

Lemma 2.2 $T : K \rightarrow K$ is a completely continuous and nondecreasing operator.

Proof At first, since for any $x \in K$,

$$(\phi_p(Tx)')'(t) = -q(t)f(t, x) \leq 0, \quad t \in (0, 1),$$

which implies that $\phi_p((Tx)'(t))$ is nonincreasing, then $(Tx)'(t)$ is nonincreasing, i.e., $(Tx)''(t) \leq 0$, which implies that $(Tx)(t)$ is concave on $[0, 1]$. Thus we have $T(K) \subset K$.

By Lemma 2.4 in [10], we know $T : K \rightarrow K$ is completely continuous. Further, noticing the monotonicity of f on x and the definition of T , we also have that the operator T is nondecreasing. $\square$

3 Main results

Define a constant

$$C = \frac{\phi_p^{-1}(\int_0^1 q(s)ds)}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

Theorem 3.1 Suppose conditions (H1)–(H4) hold. If there exists a positive constant $c > 1$ such that

$$\phi_p(C) \leq \frac{\phi_p(c)}{\rho c^\mu}, \quad (3.1)$$

where μ, ρ are defined by (1.2) and (1.3), respectively, then the BVP (1.1) has the maximal and minimal solutions x^* and y^* , which are positive and concave such that

$$0 < \|x^*\| \leq c, \quad 0 < \|y^*\| \leq c.$$

Moreover for initial values $x_0 = c, y_0 = 0$, define the iterative sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_n = Tx_{n-1} = T^n x_0, \quad y_n = Ty_{n-1} = T^n y_0, \quad (3.2)$$

then

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} T^n x_0 = x^* \quad \text{in } E, \quad \lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} T^n y_0 = y^* \quad \text{in } E$$

where T is defined by (2.1).

Proof Let $\overline{K}_c = \{x \in K : 0 \leq \|x\| \leq c\}$, we firstly prove $T\overline{K}_c \subset \overline{K}_c$.

In fact, for any $x \in \overline{K}_c$, we have

$$0 \leq x(t) \leq \max_{t \in [0,1]} x(t) = \|x\| \leq c.$$

By the assumption **(H3)** and Lemma 2.1, we have

$$0 \leq f(t, x(t)) \leq f(t, c) \leq c^\mu f(t, 1) \leq c^\mu \sup_{t \in [0, 1]} f(t, 1) = \rho c^\mu. \quad (3.3)$$

It follows from Lemma 2.2 that $Tx \in K$, and by (3.3), (3.1), for any $x \in \overline{K}_c$, we have

$$\begin{aligned} \|Tx\| &= (Tx)(\sigma_x) \\ &= \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^{\xi_i} \phi_p^{-1} \left(\int_s^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\quad + \int_0^{\sigma_x} \phi_p^{-1} \left(\int_s^{\sigma_x} q(\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\leq \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 \phi_p^{-1} \left(\int_0^1 q(\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\quad + \int_0^1 \phi_p^{-1} \left(\int_0^1 q(\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\leq \frac{\phi_p^{-1}(\int_0^1 q(\tau) f(\tau, x(\tau)) d\tau)}{1 - \sum_{i=1}^{m-2} \alpha_i} \\ &\leq \frac{\phi_p^{-1}(\int_0^1 q(\tau) d\tau) \phi_p^{-1}(\rho c^\mu)}{1 - \sum_{i=1}^{m-2} \alpha_i} \\ &= C \phi_p^{-1}(\rho c^\mu) \leq c, \end{aligned}$$

which implies that $T\overline{K}_c \subset \overline{K}_c$. Let $y_0(t) = 0$, $t \in [0, 1]$, then $y_0(t) \in \overline{K}_c$. Let $y_1(t) = (Ty_0)(t)$, we have $y_1 \in \overline{K}_c$. Denote

$$y_{n+1} = Ty_n = T^{n+1}y_0, \quad n = 1, 2, \dots$$

It follows from $T\overline{K}_c \subset \overline{K}_c$ that $y_n \in \overline{K}_c$. Since T is compact, we obtain that $\{y_n\}$ is a sequentially compact set.

Since $y_1 = Ty_0 = T0 \in \overline{K}_c$, we have

$$y_1(t) = (Ty_0)(t) = (T0)(t) \geq 0 = y_0(t), \quad t \in [0, 1].$$

By induction, we get

$$y_{n+1} \geq y_n, \quad n = 0, 1, 2, \dots$$

Consequently, there exists $y^* \in \overline{K}_c$ such that $y_n \rightarrow y^*$ in E . Letting $n \rightarrow +\infty$, from the continuity of T and $Ty_n = y_{n-1}$, we obtain $Ty^* = y^*$ which implies that y^* is a nonnegative solution of the boundary value problem (1.1). From the condition **(H4)**, we know that the zero function is not the solution of the boundary value problem (1.1), thus $\max_{0 \leq t \leq 1} |y^*(t)| > 0$, and by the concavity of y^* we have

$$y^*(t) \geq \|y^*\| \min\{t, 1-t\} > 0, \quad t \in (0, 1).$$

On the other hand, let $x_0(t) = c$, $t \in [0, 1]$, then $x_0(t) \in \overline{K}_c$. Let $x_1 = Tx_0$, then from the above expressions, we have $x_1 \in \overline{K}_c$. Thus let us denote

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad n = 1, 2, \dots$$

It follows from $T\overline{K}_c \subset \overline{K}_c$ that $x_n \in \overline{K}_c$, $n = 0, 1, 2, \dots$. Since T is compact by Lemma 2.2, we can assert that $\{x_n\}$ is a sequentially compact set.

Now, since $x_1 \in \overline{K}_c$, we have

$$0 \leq x_1(t) \leq \|x_1\| \leq b = x_0(t).$$

It follows from Lemma 2.1 that $T : K \rightarrow K$ is nondecreasing, and so

$$x_2 = Tx_1 \leq Tx_0 = x_1.$$

By induction, we have

$$x_{n+1} \leq x_n, \quad n = 0, 1, 2, \dots$$

Consequently, there exists $x^* \in \overline{K}_c$ such that $x_n \rightarrow x^*$ in E . Letting $n \rightarrow +\infty$, from the continuity of T and $Tx_n = x_{n-1}$, we obtain $Tx^* = x^*$ which implies that x^* is a nonnegative concave solution of boundary value problem (1.1).

Next, noting $x_0(t) = c \geq y_0(t) = 0$, $t \in [0, 1]$, it follows from the monotonicity of T that $Tx_0 \geq Ty_0$, and by induction, we have $x_n \geq y_n$, $n = 0, 1, 2, \dots$, which implies that $x^* \geq y^*$. Thus we have

$$x^*(t) \geq y^*(t) \geq \|y\| \min\{t, 1-t\} > 0, \quad t \in (0, 1).$$

This means that x^* is a positive concave solution of the boundary value problem (1.1). Thus, the BVP (1.1) has two positive and concave solutions x^* , y^* such that $0 < \|x^*\| \leq c$, $0 < \|y^*\| \leq c$, and from the above proof, we know that the iterative sequences hold. The proof is completed. $\square$

Remark 3.1 We notice that x^* and y^* in Theorem 3.1 are the maximal and minimal solutions of the BVP (1.1) in $\overline{K}_c$. These two solutions may coincide, and in this case the BVP (1.1) has unique solution in $\overline{K}_c$.

Corollary 3.1 Suppose conditions (H1)–(H4) hold. If

$$\mu + 1 < p,$$

then there exists a constant $c > 1$ such that the BVP (1.1) has the maximal and minimal solutions x^* and y^* , which are positive and concave such that

$$0 < \|x^*\| \leq c, \quad 0 < \|y^*\| \leq c.$$

Moreover for initial values $x_0 = c$, $y_0 = 0$, the iterative sequences $\{x_n\}$, $\{y_n\}$ defined by (3.2) converge uniformly to $x^*(t)$ and $y^*(t)$ on $t \in [0, 1]$, respectively.

Proof It follows from $\mu + 1 < p$ that

$$\lim_{x \rightarrow \infty} \frac{x^{\mu+1}}{x^p} = 0,$$

which implies that there exists $c > 1$ such that

$$\frac{c^\mu}{\phi_p(c)} = \frac{c^{\mu+1}}{c^p} < \frac{1}{\phi_p(C)\rho},$$

i.e.,

$$\phi_p(C) \leq \frac{\phi_p(c)}{\rho c^\mu}.$$

By Theorem 3.1, the conclusion of Corollary 3.1 holds. $\square$

Remark 3.2 The result of Corollary 3.1 is very interesting, as we can obtain twin positive and concave solutions for the BVP (1.1) only by comparing p with $\mu + 1$. Moreover p and μ are irrelative, which is easy to be satisfied.

4 Main examples

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} (|x'(t)|x'(t))' + \frac{1}{\sqrt{t}}(e^t x^{\frac{4}{3}} + 4) = 0, & 0 < t < 1, \\ x(0) = \frac{1}{3}x(\frac{1}{3}) + \frac{1}{3}x(\frac{2}{3}), & x(1) = \frac{1}{3}x(\frac{1}{3}) + \frac{3}{8}x(\frac{2}{3}). \end{cases} \quad (4.1)$$

Obviously, $p = 3$. Let

$$f(t, x) = e^t x^{\frac{4}{3}} + 4, \quad q(t) = \frac{1}{\sqrt{t}}, \quad \mu = \frac{1}{2},$$

then for any $0 \leq \lambda \leq 1$ and $x \in [0, +\infty)$, $t \in [0, 1]$, we have

$$f(t, \lambda x) = \lambda^{\frac{4}{3}} e^t x^{\frac{4}{3}} + 4 \geq \lambda^{\frac{4}{3}} e^t x^{\frac{4}{3}} + 4\lambda^{\frac{4}{3}} = \lambda^{\frac{4}{3}} f(t, x) \geq \lambda^{\frac{1}{2}} f(t, x),$$

which implies that **(H3)** holds.

On the other hand, it is clear that **(H1)**, **(H2)** and **(H4)** are satisfied, and

$$\rho = \max_{t \in [0, 1]} f(t, 1) = e + 4.$$

Next we compute C . Since $q(t) = \frac{1}{\sqrt{t}}$, $\alpha_1 = \alpha_2 = \frac{1}{3}$, we have

$$C = \frac{\phi_p^{-1}(\int_0^1 q(s)ds)}{1 - \sum_{i=1}^{m-2} \alpha_i} = \frac{\phi_p^{-1}(\int_0^1 \frac{1}{\sqrt{s}}ds)}{1 - \frac{2}{3}} = \frac{\phi_p^{-1}(2)}{1 - \frac{2}{3}} = 3\phi_p^{-1}(2).$$

So

$$\phi_p(C) = 2\phi_p(3) = 18.$$

Take

$$c = [20(e+4)]^{\frac{2}{3}},$$

then

$$\frac{\phi_p(c)}{\rho c^\mu} = \frac{([20(e+4)]^{\frac{2}{3}})^2}{(e+4)([20(e+4)]^{\frac{2}{3}})^{\frac{1}{2}}} = 20 > 18 = \phi_p(C).$$

It follows from Theorem 3.1 that the BVP (4.1) has at least two positive and concave solutions x^* and y^* such that

$$0 < \|x^*\| \leq [20(e+4)]^{\frac{2}{3}}, \quad 0 < \|y^*\| \leq [20(e+4)]^{\frac{2}{3}}.$$

Moreover for initial values $x_0 = \|x^*\| \leq [20(e+4)]^{\frac{2}{3}}$, $y_0 = 0$, the corresponding iterative sequences hold.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} (|u'(t)|^4 u'(t))' + \left\{ \frac{1}{\sqrt{t}} + \frac{1}{\sqrt[3]{t-\frac{1}{2}}} \right\} (t^2 u^3 + 2tu^{\frac{1}{2}} + 1) = 0, & t \in (0, 1), \\ u(0) = \frac{1}{4}u(\frac{1}{4}) + \frac{1}{6}u(\frac{1}{2}) + \frac{1}{7}u(\frac{3}{4}), \\ u(1) = \frac{1}{5}u(\frac{1}{4}) + \frac{2}{5}u(\frac{1}{2}) + \frac{1}{5}u(\frac{3}{4}). \end{cases} \quad (4.2)$$

Let

$$q(t) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt[3]{t-\frac{1}{2}}}, \quad f(t, x) = t^2 x^3 + 2tx^{\frac{1}{2}} + 1.$$

Then **(H1)**, **(H2)** and **(H4)** hold.

On the other hand, choose $\mu = 3$, then for any $0 \leq \lambda \leq 1$,

$$f(t, \lambda x) = \lambda^3 t^2 x^3 + \lambda^{\frac{1}{2}} 2tx^{\frac{1}{2}} + 1 \geq \lambda^3 (t^2 x^3 + 2tx^{\frac{1}{2}} + 1) = \lambda^3 f(t, x).$$

Thus **(H3)** holds. Since

$$\mu + 1 = 4 < 5 = p,$$

by Corollary 3.1, the BVP (4.2) has at least two positive and concave solutions.

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