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A Characterisation of Some Z-Like Logics

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Abstract. In Béziau (Log Log Philos 15:99–111, 2006) a logic **Z** was defined with the help of the modal logic **S5**. In it, the negation operator is understood as meaning 'it is not necessary that'. The strong soundness–completeness result for **Z** with respect to a version of Kripke semantics was also given there. Following the formulation of **Z** we can talk about **Z**-*like logics* or *Beziau-style logics* if we consider other modal logics instead of **S5**—such a possibility has been mentioned in [1]. The correspondence result between modal logics and respective Beziau-style logics has been generalised for the case of normal logics naturally leading to soundness–completeness results [see Marcos (Log Anal 48(189–192):279–300, 2005) and Mruczek-Nasieniewska and Nasieniewski (Bull Sect Log 34(4):229–248, 2005)]. In Mruczek-Nasieniewska and Nasieniewski (Bull Sect Log 37(3–4):185–196, 2008), (Bull Sect Log 38(3–4):189–203, 2009) some partial results for non-normal cases are given. In the present paper we try to give similar but more general correspondence results for the non-normal-worlds case. To achieve this aim we have to enrich original Beziau's language with an additional negation operator understood as 'it is necessary that not'.

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1. Introduction

The idea comes from [1], where a logic **Z** was defined with the use of the normal logic **S5**. The definition of the logic **Z** was inspired by the definition of Jaśkowski's logic $\mathbf{D_2}$ (see [4,11]). Beziau's original formulation has been generalized for the case of other normal logics [7,8], and also for some regular

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and quasi-regular ones, where the case of non-normal-world semantics was considered [9,10]. In particular, it has been shown that there is a general way to move from completeness results for normal modal logics to completeness results for respective Beziau logics. A question arises: to what extent can those results be repeated for regular logics. Our answer (see Theorem 7.14) requires an extension of the language with a negation operator 'it is necessary that not'.

2. Béziau's Logic Z

2.1. Syntax

Let Var be a set of propositional variables and For—the set of formulas built up from elements of Var in the language $\{\sim, \wedge, \vee, \rightarrow\}$.

Definition 2.1 [1]. Axioms of the *system* **HZ** are

- all theses of positive classical logic[1]
- and for any $A, B \in$ For the following formulas:

$$A \vee \sim A \tag{AZ1}$$

$$(A \wedge \sim B) \wedge \sim(A \wedge \sim B) \rightarrow (A \wedge \sim A) \tag{AZ2}$$

$$\sim(A \wedge B) \rightarrow (\sim A \vee \sim B) \tag{AZ3}$$

$$\sim \sim A \rightarrow A \tag{AZ4}$$

Rules of the system **HZ** are:

$$\frac{A \rightarrow B \qquad A}{B} \tag{MP}$$

$$\frac{A \rightarrow B}{\sim(A \wedge \sim B)} \tag{RZ}$$

A formula A is a thesis of **HZ** (in symbols $\vdash_{\mathbf{Z}} A$) iff there is a finite sequence of formulas $B_1, \ldots, B_n$ such that $B_n = A$ and B_i $(1 \leq i \leq n)$ is an axiom of **HZ** or it is a result of application of a rule of **HZ**, whose all premisses are among formulas $B_1, \ldots, B_m$ $(m < i)$.

2.2. Semantics

Definition 2.2 [1].

1. A **Z**-*model*[2] is a non-empty set **C** of valuations such that: $\alpha \in \mathbf{C}$ iff classical conditions for ($\wedge$), ($\vee$), ($\rightarrow$) hold, while for $\sim$ we have:
$$\alpha(\sim A) = 0 \ \text{ iff } \ \forall_{\beta \in \mathbf{C}}\, \beta(A) = 1. \tag{$\sim$f}$$
2. A formula A is *valid in* **Z**-*model* **C** iff $\forall_{\alpha \in \mathbf{C}}\, \alpha(A) = 1$.
3. A formula A is **Z**-*valid* ($\vDash_{\mathbf{Z}} A$ for short) iff A is valid in all **Z**-models.

Theorem 2.3 (Completeness, [1]). $\vDash_{\mathbf{Z}} A$ *iff* $\vdash_{\mathbf{Z}} A$.[3]

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[1] In [1] an axiomatisation of positive classical logic is used, but it is not essential there.
[2] In [1], it was called '**Z**-*cosmos*'.
[3] In [1] a strong adequacy theorem is proved.

3. A Class $\mathcal{K}$

3.1. Syntax

Definition 3.1. Let $\mathcal{K}$ denote a family of logics such that each of them

– contains positive classical logic in the language $\{\wedge, \vee, \rightarrow\}$,
– contains the de Morgan law:

$$\sim(p \wedge q) \rightarrow (\sim p \vee \sim q) \tag{dM1$_{\rightarrow}$}$$

– the following version of *Ex falso quodlibet*:

$$\sim(p \rightarrow p) \rightarrow p \tag{EFQ}$$

– and is closed under *modus ponens* (MP), substitution and the rule of contraposition:

$$\frac{\vdash A \rightarrow B}{\vdash \sim B \rightarrow \sim A} \tag{CONTR}$$

Let $\mathbf{P_K}$ be the smallest logic in $\mathcal{K}$. Notice that the definition of $\mathbf{P_K}$ corresponds to the formulation of Došen's system $HK\Box'$ (see [2]) which is an extension of Heyting propositional calculus (where $\neg$ is the intuitionistic negation) with the axiom (dM1$_{\rightarrow}$), where $\Box'$ is used as a symbol of the second negation (in our notation '$\sim$'), an axiom $\neg\Box'(A \rightarrow A)$ [that corresponds to our (EFQ)] and the rule of contraposition with $\Box'$ used again as a negation symbol. Connections between these systems deserve a closer look but they will not be discussed in this paper.

3.2. Semantics

Definition 3.2. 1. A *frame* is an ordered pair $\langle W, R\rangle$, where W is a non-empty set and R is a binary relation on W (accessibility relation).

2. A *valuation* is any function $v : \mathrm{Var} \longrightarrow 2^W$.
3. A *model* is any triple $\langle W, R, v\rangle$, where $\langle W, R\rangle$ is a frame and v—a valuation. We say that $\langle W, R, v\rangle$ is built over the frame $\langle W, R\rangle$.

Definition 3.3 [8]. 1. A formula $A \in \mathrm{For}$ is *true* in a world $w \in W$ under a valuation v (notation: $w \Vdash_v A$) iff

- if A is a variable, then
  $w \Vdash_v A \Longleftrightarrow w \in v(A)$.
- if A is of the form $\sim B$ for some B, then
  $w \Vdash_v A \Longleftrightarrow$ there is a world w' such that wRw' and it is not the case that $w' \Vdash_v B$ (i.e. $w' \nVdash_v B$).
- if A is of the form $B \wedge C$, for some B and C, then
  $w \Vdash_v A \Longleftrightarrow w \Vdash_v B$ and $w \Vdash_v C$.
- if A is of the form $B \vee C$, for some B and C, then
  $w \Vdash_v A \Longleftrightarrow w \Vdash_v B$ or $w \Vdash_v C$.
- if A is of the form $B \rightarrow C$, for some B and C, then
  $w \Vdash_v A \Longleftrightarrow w \nVdash_v B$ or $w \Vdash_v C$.

2. A formula A is *true* in a model $M = \langle W, R, v\rangle$ (notation $M \Vdash A$) iff $w \Vdash_v A$ for every $w \in W$.

3. A formula A is *valid* in a frame $\langle W, R\rangle$ iff it is true in every model built over $\langle W, R\rangle$.

3.3. The Completeness of $\mathbf{P_K}$

Theorem 3.4 (Completeness of $\mathbf{P_K}$, [8]). *For any* $A \in \mathrm{For}$. $A \in \mathbf{P_K}$ *iff* A *is valid in every frame.*

4. A General Result for the Normal Case

Let $\mathrm{For}^{\mathrm{M}}$ be the set of all propositional modal formulas in the language $\{\neg, \wedge, \vee, \rightarrow, \Diamond, \Box\}$.

Definition 4.1 [8]. Let $-^u : \mathrm{For}^{\mathrm{M}} \longrightarrow \mathrm{For}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}$ the following conditions:

1. $(a)^u = a$,
2. $(\neg A)^u = ((A)^u \rightarrow \sim(p \rightarrow p))$,
3. $(A \S B)^u = (A^u \S B^u)$, for $\S \in \{\wedge, \vee, \rightarrow\}$,
4. $(\Diamond A)^u = \sim((A)^u \rightarrow \sim(p \rightarrow p))$,
5. $(\Box A)^u = (\sim((A)^u) \rightarrow \sim(p \rightarrow p))$.

Definition 4.2. For $X \subseteq \mathrm{For}^{\mathrm{M}}$, let $\mathbf{K}[X]$ be the smallest normal modal logic containing the logic $\mathbf{K}$ and the set X. For a given logic $\boldsymbol{S} = \mathbf{K}[X]$, let $\mathbf{P}_{\mathbf{K}[X]}$ be the smallest element in $\mathcal{K}$ which contains $\mathbf{P_K}$ and the set of 'new' axioms $X^u = \{A^u : A \in X\}$.

Let us recall

Theorem 4.3 [8]. *Let* $\boldsymbol{S} = \mathbf{K}[X]$. *If the logic* $\boldsymbol{S}$ *is complete with respect to some class of frames with accessibility relation fulfilling a given condition* C, *then for the logic* $\mathbf{P}_{\boldsymbol{S}}$ *the following holds:*

For any $A \in \mathrm{For}$: A *is true in every frame with accessibility relation fulfilling the condition* C *iff* A *is a theorem of the logic* $\mathbf{P}_{\boldsymbol{S}}$.

5. Examples

5.1. The Logic $\mathbf{P_T}$

Let us recall that the logic $\mathbf{P_T}$ is obtained by adding to $\mathbf{P_K}$ a single extra axiom:

$$(\Box p \rightarrow p)^u\,,$$

i.e. $(\sim p \rightarrow \sim(p \rightarrow p)) \rightarrow p$.

Theorem 5.1 (Adequacy for $\mathbf{P_T}$, [8]).

1. *A formula* A *is true in every frame with reflexive accessibility relation iff* A *is a theorem of the logic* $\mathbf{P_T}$.
2. *The logic* $\mathbf{P_T}$ *is the smallest logic in* $\mathcal{K}$ *containing the formula* $p \vee \sim p$.

5.2. The Logic $\mathbf{P_{K5}}$

The logic $\mathbf{P_{K5}}$ is obtained by adding an extra axiom to $\mathbf{P_K}$:

$$(\Diamond p \to \Box\Diamond p)^u .$$

Corollary 5.2 (Adequacy for $\mathbf{P_{K5}}$, [8]). *A formula A is true in every frame with Euclidean accessibility relation iff A is theorem of the logic $\mathbf{P_{K5}}$.*

Theorem 5.3 [8]. *The logic $\mathbf{P_{K5}}$ is the smallest logic in $\mathcal{K}$ containing the formula $\sim p \wedge \sim\sim p \to q$.*

Corollary 5.4. *The logic* **Z** *is equal the smallest logic in $\mathcal{K}$ that contains*

$$p \vee \sim p$$
$$\sim p \wedge \sim\sim p \to q.$$

6. Class $\mathcal{R}$

6.1. Syntax

Definition 6.1. Let $\mathcal{R}$ be the class of all logics that contain positive classical logic in the language with $\{\wedge, \vee, \to\}$, that include ($\mathrm{dM1}_{\to}$), and are closed under modus ponens, substitution, and contraposition (CONTR).

We easily obtain:

Fact 6.2. For any $\boldsymbol{L} \in \mathcal{R}$, $\boldsymbol{L}$ contains:

$$(\sim p \vee \sim q) \to \sim(p \wedge q) \qquad (\mathrm{dM1}_{\leftarrow})$$
$$\sim(p \vee q) \to (\sim p \wedge \sim q) \qquad (\mathrm{dM2}_{\to})$$

6.2. Semantics

Definition 6.3.
1. A *frame* is any triple $\langle W, R, N\rangle$, where W is a set, N is a non-empty subset of W and R is a binary relation on W. Elements of W, N, and $W \backslash N$ are called respectively: *possible worlds*, *normal worlds*, and *non-normal worlds*.
2. A *model* is a quadruple $\langle W, R, N, v\rangle$, where $\langle W, R, N\rangle$ is a frame, and v—a valuation. We say that $\langle W, R, N, v\rangle$ is built over the frame $\langle W, R, N\rangle$.

Definition 6.4 [9]. A formula A is *true* in $w \in W$ under a valuation v ($w \Vdash_v A$ for short) iff

- if A is of the form $\sim B$ for some B, then
  - for $w \in W \backslash N$
    $w \Vdash_v A$
  - for $w \in N$:
    $w \Vdash_v A \iff$ there is w' such that wRw' and it is not the case that $w' \Vdash_v B$ ($w' \nVdash_v B$ for short).

  Other cases stay unchanged with respect to Definition 3.3.

Definition 6.5.
1. A is *valid* in a model $M = \langle W, R, N, v\rangle$ ($M \Vdash_{\mathcal{R}} A$) iff $w \Vdash_v A$ for each $w \in W$.

2. A formula A is *valid* in a frame $\langle W, R, N\rangle$ iff it is valid in each model built over $\langle W, R, N\rangle$.

Since classical logic belongs to $\mathcal{R}$, there is a smallest logic in $\mathcal{R}$, denoted as $\mathbf{R_{C2}}$. Thus, we obtain:

Corollary 6.6 (Adequacy for $\mathbf{R_{C2}}$, [9]). *For any $A \in \mathrm{For}$, $A \in \mathbf{R_{C2}}$ iff A is true in every frame $\langle W, R, N\rangle$.*

6.3. Problems with Generalisation for the Non-normal Case

Any formula of the form $\Diamond A$ is valid in every non-normal world, even a formula $\Diamond\neg(p \to p)$. Bearing in mind the applied translations, we cannot use $\sim(p \to p)$ as a bottom constant as we did in the case of normal world semantics which is adequate for logics from the class $\mathcal{K}$.

To obtain a general result—corresponding to the Theorem 4.3—for the case in which non-normal worlds are allowed, we will use a formula that gives after translation a formula of the form $\Box A$. Such formulas are false in every non-normal world (see [6]). A natural candidate to assist in this task is a 'it is necessary that not' operator which is denoted in what follows by $\dot\sim$. Moreover, a formula $\dot\sim(p \to p)$ will be treated as false also in every normal world with a non-empty set of alternative worlds. We have to remember that even this formula is valid in worlds with the empty set of alternatives. This will limit the generality of our solution.

7. A Solution: The Class $\mathcal{R}^{\dot\sim}$

First, let us consider a set $\mathrm{For}^{\dot\sim}$ of formulas in the language with the two negations $\sim$ and $\dot\sim$, and positive connectives $\wedge, \vee, \to$.

Definition 7.1 (Counterpart of the regular logic D2[4]). Let $\mathcal{R}^{\dot\sim}$ be the class of all logics that are non-trivial subsets of $\mathrm{For}^{\dot\sim}$, containing full positive classical logic in the language $\{\wedge, \vee, \to\}$, including the following formulas:

$$\dot\sim p \wedge \dot\sim q \to \dot\sim(p \vee q), \qquad (\mathrm{dM2}^{\dot\sim}_{\leftarrow})$$

$$\sim p \to (\dot\sim(p \to \dot\sim(q \to q)) \to \dot\sim(q \to q)) \qquad (\mathrm{df}^{\sim}_{\to})$$

$$(\dot\sim(p \to \dot\sim(q \to q)) \to \dot\sim(q \to q)) \to \sim p \qquad (\mathrm{df}^{\sim}_{\leftarrow})$$

$$\dot\sim p \to \sim p \qquad (\mathrm{D}^{\dot\sim}_{\sim})$$

$$((p \to \dot\sim(q \to q)) \to \dot\sim(q \to q)) \to p \qquad (\mathrm{dneg})$$

and closed under modus ponens, $\mathrm{CONTR}^{\dot\sim}$:

$$\frac{\vdash A \to B}{\vdash \dot\sim B \to \dot\sim A} \qquad (\mathrm{CONTR}^{\dot\sim})$$

and any substitution.

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[4] Notice that 'D2' has nothing to do with Jaskowski's logic D_2. It was introduced by Lemmon, see Definition 7.7 and [6].

Observe that the above system is an extension of the system **N** of Došen [3], defined by the axioms of positive intuitionistic logic, rules (MP) and ($\mathrm{CONTR}^{\dot\sim}$). On the other hand in [2] a system $HK\Diamond'$ analogous to previously mentioned system $HK\Box'$ had been defined this time with ($\mathrm{dM2}^{\dot\sim}_{\leftarrow}$), the additional axiom $\dot\sim\neg(A \to A)$ (here $\neg$ is the intuitionistic negation) and was closed under the rule ($\mathrm{CONTR}^{\dot\sim}$).

Similarly, as in [8] we have:

Fact 7.2. For any $\boldsymbol{L} \in \mathcal{R}^{\dot\sim}$, $\boldsymbol{L}$ is closed under the rule of extensionality:

$$\frac{\vdash (A \to B) \wedge (B \to A)}{\vdash (\dot\sim A \to \dot\sim B) \wedge (\dot\sim B \to \dot\sim A)} \qquad (\mathrm{EXT}^{\dot\sim})$$

and contains the following versions of de Morgan laws:

$$(\dot\sim p \vee \dot\sim q) \to \dot\sim(p \wedge q) \qquad (\mathrm{dM1}^{\dot\sim}_{\leftarrow})$$

$$\dot\sim(p \vee q) \to (\dot\sim p \wedge \dot\sim q) \qquad (\mathrm{dM2}^{\dot\sim}_{\rightarrow})$$

Proof. The fact that $\boldsymbol{L}$ is closed on the rule of extensionality is obvious. Theses ($\mathrm{dM1}^{\dot\sim}_{\leftarrow}$) and ($\mathrm{dM2}^{\dot\sim}_{\rightarrow}$) are obtained by modus ponens, substitution, ($\mathrm{CONTR}^{\dot\sim}$), and positive logic. $\Box$

We have

Fact 7.3. For any $\boldsymbol{L} \in \mathcal{R}^{\dot\sim}$, $\boldsymbol{L} \cap \mathrm{For} \in \mathcal{R}$.

Proof. First we prove that $\boldsymbol{L}$ is closed under (CONTR). Assume that $\ulcorner A \to B\urcorner \in \boldsymbol{L}$. By positive logic we have $\ulcorner(B \to \dot\sim(q \to q)) \to (A \to \dot\sim(q \to q))\urcorner \in \boldsymbol{L}$ and using ($\mathrm{CONTR}^{\dot\sim}$) we obtain $\ulcorner\dot\sim(A \to \dot\sim(q \to q)) \to \dot\sim(B \to \dot\sim(q \to q))\urcorner \in \boldsymbol{L}$. Again by positive logic we see that $\ulcorner(\dot\sim(B \to \dot\sim(q \to q)) \to \dot\sim(q \to q)) \to (\dot\sim(A \to \dot\sim(q \to q)) \to \dot\sim(q \to q))\urcorner \in \boldsymbol{L}$ and by ($\mathrm{df}^{\sim}_{\rightarrow}$), ($\mathrm{df}^{\sim}_{\leftarrow}$), substitution and positive logic we have that $\ulcorner\sim B \to \sim A\urcorner \in \boldsymbol{L}$.

Now, it is enough to show that ($\mathrm{dM1}_{\rightarrow}$)$\in \boldsymbol{L}$. By ($\mathrm{df}^{\sim}_{\rightarrow}$) and substitution we have: $\ulcorner\sim(p\wedge q) \to (\dot\sim((p\wedge q) \to \dot\sim(q \to q)) \to \dot\sim(q \to q))\urcorner \in \boldsymbol{L}$. By positive logic and ($\mathrm{EXT}^{\dot\sim}$) we see that $\ulcorner\sim(p \wedge q) \to \big(\dot\sim((p \to \dot\sim(q \to q)) \vee (q \to \dot\sim(q \to q))) \to \dot\sim(q \to q)\big)\urcorner \in \boldsymbol{L}$. Due to ($\mathrm{dM2}^{\dot\sim}_{\leftarrow}$) and substitution we obtain $\dot\sim(p \to \dot\sim(q \to q))\wedge\dot\sim(q \to \dot\sim(q \to q)) \to \dot\sim((p \to \dot\sim(q \to q))\vee(q \to \dot\sim(q \to q)))$ as a thesis of $\boldsymbol{L}$ and by positive logic we conclude that $\ulcorner\sim(p \wedge q) \to \big(\dot\sim(p \to \dot\sim(q \to q)) \wedge \dot\sim(q \to \dot\sim(q \to q)) \to \dot\sim(q \to q)\big)\urcorner \in \boldsymbol{L}$ and next $\ulcorner\sim(p \wedge q) \to (\dot\sim(p \to \dot\sim(q \to q)) \to \dot\sim(q \to q)) \vee (\dot\sim(q \to \dot\sim(q \to q)) \to \dot\sim(q \to q))\urcorner \in \boldsymbol{L}$. The thesis follows by ($\mathrm{df}^{\sim}_{\leftarrow}$) and again positive logic. $\Box$

We have to extend Definition 6.4 for the case of $\dot\sim$.

Definition 7.4. A formula $A \in \mathrm{For}^{\dot\sim}$ is *true* in a world $w \in W$ under a valuation v (notation: $w \Vdash_v A$) iff

- if A has the form $\dot\sim B$, for some formula B, then
  - for $w \in N$:
    $w \Vdash_v A \iff$ for every world w' such that wRw', it is not the case that $w' \Vdash_v B$;

– for $w \in W \backslash N$:
$w \not\Vdash_v A$,

Other cases stay unchanged with respect to Definition 6.4.

Validity in a model and in a frame behaves as in Definition 6.5
We extend the function $-^m$ used in [8].

Definition 7.5. Let $-^m : \mathrm{For}^{\dot\sim} \longrightarrow \mathrm{For}^{\mathrm{M}}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}^{\mathrm{M}}$ the following conditions:

1. $(a)^m = a$,
2. $(\sim A)^m = \Diamond\neg((A)^m)$,
3. $(\dot\sim A)^m = \Box\neg((A)^m)$,
4. $(A \mathbin{\S} B)^m = (A^m \mathbin{\S} B^m)$, for $\S \in \{\wedge, \vee, \rightarrow\}$.

Now we redefine 4.1 using a new bottom constant. Although it is not essential, to be able to easier formulate some statements, below we consider the modal language with material equivalence.

Definition 7.6. Let $-^{u\dot\sim} : \mathrm{For}^{\mathrm{M}} \longrightarrow \mathrm{For}^{\dot\sim}$ be a function satisfying for any $a \in \mathrm{Var}, A, B \in \mathrm{For}$ the following conditions:

1. $(a)^{u\dot\sim} = a$
2. $(\neg A)^{u\dot\sim} = ((A)^{u\dot\sim} \rightarrow \dot\sim(p \rightarrow p))$
3. $(A \S B)^{u\dot\sim} = (A^{u\dot\sim})\S(B^{u\dot\sim})$, for $\S \in \{\wedge, \vee, \rightarrow\}$
4. $(A \leftrightarrow B)^{u\dot\sim} = (A^{u\dot\sim} \rightarrow B^{u\dot\sim}) \wedge (B^{u\dot\sim} \rightarrow A^{u\dot\sim})$,
5. $(\Diamond A)^{u\dot\sim} = (\dot\sim((A)^{u\dot\sim}) \rightarrow \dot\sim(p \rightarrow p))$,
6. $(\Box A)^{u\dot\sim} = \dot\sim((A)^{u\dot\sim} \rightarrow \dot\sim(p \rightarrow p))$.

Let us recall that a regular logic is any set of modal formulas $\boldsymbol{L}$ that contains classical logic, includes (df $\Diamond$): $\Diamond p \leftrightarrow \neg\Box\neg p$, (K): $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, and is closed under modus ponens, substitution and the monotonicity rule (mon): if $\ulcorner A \rightarrow B \urcorner \in \boldsymbol{L}$ then $\ulcorner \Box A \rightarrow \Box B \urcorner \in \boldsymbol{L}$.

Definition 7.7 [6]. **D2** is the smallest regular logic containing the axiom (D) : $\Box p \rightarrow \Diamond p$ (equivalently $\Diamond(p \rightarrow p)$).

Lemma 7.8. *For any* $A \in \mathrm{For}^{\mathrm{M}}$*:*

$$\vdash_{\mathbf{D2}} A \leftrightarrow ((A)^{u\dot\sim})^m \qquad (\mathrm{ext}_{\mathbf{D2}})$$

Proof. The proof goes by induction on the complexity of a modal formula A. The initial step is obvious since

$$((a)^{u\dot\sim})^m = a$$

for any variable a. For the inductive step notice that every regular logic is closed under the rule of replacement:

$$\frac{A \leftrightarrow B}{C[{}^{A}/{}_{B}] \leftrightarrow C} \qquad (\mathrm{ext})$$

Besides, for the case of $\neg$, $\Diamond$ and $\Box$ it is enough to observe that for any formulas $A, B \in \mathrm{For}^{\mathrm{M}}$, the following formulas belong to **D2**:

$$\neg A \leftrightarrow (A \to \Box\neg(p \to p))$$
$$\Diamond A \leftrightarrow (\Box\neg A \to \Box\neg(p \to p))$$
$$\Box A \leftrightarrow \Box\neg(A \to \Box\neg(p \to p))$$

To prove the last statement, it is enough to observe that the following formulas

$$A \leftrightarrow (A \wedge \neg\Box\neg(p \to p))$$
$$\Diamond A \leftrightarrow (\neg\Box\neg(p \to p) \to \neg\Box\neg A)$$
$$A \leftrightarrow \neg(A \to \Box\neg(p \to p))$$

are also theses of **D2**.

The cases of positive classical connectives are obvious. $\Box$

Let $\mathbf{R_{D2}}$ be the smallest logic in $\mathcal{R}^{\dot\sim}$.

Lemma 7.9. *For any formula* $A \in \mathrm{For}^{\dot\sim}$

$$\vdash_{\mathbf{R_{D2}}} (A \to ((A)^m)^{u\dot\sim}) \wedge (((A)^m)^{u\dot\sim} \to A) \qquad (\mathrm{ext}_{\mathbf{R_{D2}}})$$

Proof. The proof goes by induction on the complexity of a formula A. The initial step is obvious since

$$((a)^m)^{u\dot\sim} = a$$

for any variable a. For the inductive step using Facts 7.2, 7.3, and positive logic we can replace in any formula, any its subformula B, by a formula C whenever $\ulcorner B \to C \urcorner \in \mathbf{R_{D2}}$ and $\ulcorner C \to B \urcorner \in \mathbf{R_{D2}}$. Thus, for the cases of $\sim$ and $\dot\sim$ it is enough to observe that every logic from $\mathcal{R}^{\dot\sim}$ has as its theses the following formulas:

$$\sim A \to (\dot\sim(A \to \dot\sim(p \to p)) \to \dot\sim(p \to p)),$$
$$(\dot\sim(A \to \dot\sim(p \to p)) \to \dot\sim(p \to p)) \to \sim A,$$
$$\dot\sim A \to \dot\sim((A \to \dot\sim(p \to p)) \to \dot\sim(p \to p)),$$
$$\dot\sim((A \to \dot\sim(p \to p)) \to \dot\sim(p \to p)) \to \dot\sim A,$$

We see that first two formulas are substitutions of $(\mathrm{df}_{\overset{\sim}{\to}})$ and $(\mathrm{df}_{\overset{\sim}{\leftarrow}})$, respectively. The third one is obtained by $(\mathrm{CONTR}^{\dot\sim})$ from a substitution of (dneg), while the fourth one follows again by $(\mathrm{CONTR}^{\dot\sim})$ from a substitution to *modus ponendo ponens*: $p \to ((p \to q) \to q)$. The cases of positive connectives are obvious. $\Box$

Definition 7.10. For $X \subseteq \mathrm{For}^{\mathrm{M}}$, let $\mathbf{D2}[X]$ be the smallest regular modal logic containing the logic **D2** and the set X. For a given logic $\boldsymbol{S} = \mathbf{D2}[X]$, let $\mathbf{R}_{\boldsymbol{S}}$ denote the smallest element in $\mathcal{R}^{\dot\sim}$ which contains $\mathbf{R_{D2}}$ and the set of 'new' axioms $X^{u\dot\sim} = \{(A)^{u\dot\sim} : A \in X\}$.

We obtain an analogue with Lemma 16 in [8].

Lemma 7.11. *Let* $\boldsymbol{S} = \mathbf{D2}[X]$. *For* $A \in \mathrm{For}^{\mathrm{M}}$ *we have* $A \in \boldsymbol{S}$ *iff* $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (A)^{u\dot\sim}$.

Proof. (⇒). Let $A \in \mathrm{For}^{\mathrm{M}}$ and $A \in \boldsymbol{S}$. Consider a proof of A: $C_1, \ldots, C_k$. We prove by induction on i that for any $1 \leq i \leq k$: $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_i)^{u\dot{\sim}}$. Let us take any i such that $1 \leq i \leq k$ and assume that for any $1 \leq j < i$ it holds that $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_j)^{u\dot{\sim}}$. For the initial step we repeat argumentation for the cases of axioms of the logic $\boldsymbol{S}$ (points 1–6 given below) together with the fact that classical logic can be axiomatised by positive logic and the strong law of contraposition. We consider the cases:

1. C_i is a thesis of classical positive logic; then $(C_i)^{u\dot{\sim}}$ is also a thesis of classical positive logic, so $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_i)^{u\dot{\sim}}$.
2. C_i is of the form $(\neg C \to \neg D) \to (D \to C)$ for some $C, D \in \mathrm{For}^{\mathrm{M}}$. We prove that $\vdash_{\mathbf{R}_{\boldsymbol{S}}} \big(((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to ((D)^{u\dot{\sim}} \to \dot{\sim}(p \to p))\big) \to ((D)^{u\dot{\sim}} \to (C)^{u\dot{\sim}})$ i.e. $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_i)^{u\dot{\sim}}$. By (dneg) and substitution we have $(((C)^{u\dot{\sim}} \to \dot{\sim}(q \to q)) \to \dot{\sim}(q \to q)) \to (C)^{u\dot{\sim}}$ while by positive logic we get $\big((D)^{u\dot{\sim}} \to (((C)^{u\dot{\sim}} \to \dot{\sim}(q \to q)) \to \dot{\sim}(q \to q))\big) \to ((D)^{u\dot{\sim}} \to (C)^{u\dot{\sim}})$, so the required thesis follows by commutation of antecedents.
3. C_i is of the form $\Diamond C \leftrightarrow \neg\Box\neg C$ for some $C \in \mathrm{For}^{\mathrm{M}}$. We need to prove that $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (\dot{\sim}(C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to \big(\dot{\sim}(((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to \dot{\sim}(p \to p)) \to \dot{\sim}(p \to p)\big)$ and $\vdash_{\mathbf{R}_{\boldsymbol{S}}} \big(\dot{\sim}(((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to \dot{\sim}(p \to p)) \to \dot{\sim}(p \to p)\big) \to (\dot{\sim}(C)^{u\dot{\sim}} \to \dot{\sim}(p \to p))$. But the first formula follows by a substitution to *modus ponendo ponens*: $(C)^{u\dot{\sim}} \to (((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to \dot{\sim}(p \to p))$, (CONTR$^{\dot{\sim}}$) and positive logic, while the second one is obtained by a substitution to (dneg): $(((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to \dot{\sim}(p \to p)) \to (C)^{u\dot{\sim}}$, and again (CONTR$^{\dot{\sim}}$) and positive logic.
4. C_i is of the form $\Box(C \to D) \to (\Box C \to \Box D)$ for some $C, D \in \mathrm{For}^{\mathrm{M}}$. We have to infer the formula $\dot{\sim}((C \to D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to (\dot{\sim}((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to \dot{\sim}((D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)))$. Indeed, by positive logic, Definition 7.6 and substitution we obtain $((D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to ((C \to D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \vee ((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p))$, and the required formula can be proved by (CONTR$^{\dot{\sim}}$), a substitution to (dM2$^{\dot{\sim}}_{\leftarrow}$), transitivity of implication and exportation.
5. C_i is an instance of the axiom (D). We need to prove $\dot{\sim}((D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to (\dot{\sim}((D)^{u\dot{\sim}}) \to \dot{\sim}(p \to p))$. From (df$^{\sim}_{\to}$) by commutation of antecedents and substitution we have: $\dot{\sim}((D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to (\sim((D)^{u\dot{\sim}}) \to \dot{\sim}(p \to p))$. The rest follows by (D$^{\dot{\sim}}_{\sim}$) and the transitivity of implication: $(\dot{\sim}((D)^{u\dot{\sim}}) \to \sim((D)^{u\dot{\sim}})) \to ((\sim((D)^{u\dot{\sim}}) \to \dot{\sim}(p \to p)) \to (\dot{\sim}((D)^{u\dot{\sim}}) \to \dot{\sim}(p \to p)))$.
6. C_i is a specific axiom of the logic $\boldsymbol{S}$ i.e. $C_i \in X$. By the definition of $\mathbf{R}_{\boldsymbol{S}}$: $(C_i)^{u\dot{\sim}} \in \mathbf{R}_{\boldsymbol{S}}$.
7. C_i is obtained by (MP) from $C_k = C_j \to C_i$, where $k, j < i$. By the inductive hypothesis $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_k)^{u\dot{\sim}}$ and $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_j \to C_i)^{u\dot{\sim}}$. By Definition 7.6 we have $(C_j \to C_i)^{u\dot{\sim}} = (C_j)^{u\dot{\sim}} \to (C_i)^{u\dot{\sim}}$ and $(C_i)^{u\dot{\sim}}$ also arises from $(C_k)^{u\dot{\sim}}$ and $(C_j)^{u\dot{\sim}}$ by (MP).
8. C_i is obtained by the monotonicity rule from some formula $C_j = C \to D$ where $j < i$, so C_i equals $\Box C \to \Box D$. By the inductive hypothesis $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C \to D)^{u\dot{\sim}}$. By the definition of the function $(-)^{u\dot{\sim}}$ and transitivity

of $\to$ we obtain $((D)^{u\dot{\sim}} \to \dot{\sim}(p \to p)) \to ((C)^{u\dot{\sim}} \to \dot{\sim}(p \to p))$. By (CONTR$^{\dot{\sim}}$) we get: $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (C_i)^{u\dot{\sim}}$.

9. C_i is obtained from a formula C_j where $j < i$ by a substitution of some formulas $D^i_1, \ldots, D^i_k \in \mathrm{For}^{\mathrm{M}}$. Since $\mathbf{R}_{\boldsymbol{S}}$ is closed under the substitution it is enough to observe using induction on the complexity of the formula that $(C_i)^{u\dot{\sim}}$ is equivalent on the basis of $\mathbf{R}_{\boldsymbol{S}}$ to a formula (in the sense that implications in both directions are theses of $\mathbf{R}_{\boldsymbol{S}}$) which arises from $(C_j)^{u\dot{\sim}}$ by the appropriate substitution of formulas $(D^i_1)^{u\dot{\sim}}, \ldots, (D^i_k)^{u\dot{\sim}}$.

($\Leftarrow$). Let $A \in \mathrm{For}^{\mathrm{M}}$ and $\vdash_{\mathbf{R}_{\boldsymbol{S}}} A^{u\dot{\sim}}$. Consider a proof of $A^{u\dot{\sim}}$: $C_1, \ldots, C_k$. We prove by induction on i that for any $1 \leq i \leq k$: $(C_i)^m \in \boldsymbol{S}$. Let us take any i: $1 \leq i \leq k$. We consider the following cases:

1. C_i is a thesis of classical positive logic; then $(C_i)^m$ is just C_i and of course $(C_i)^m \in \mathbf{D2}$.
2. C_i is $\dot{\sim} p \wedge \dot{\sim} q \to \dot{\sim}(p \vee q)$. By definition $(\dot{\sim} p \wedge \dot{\sim} q \to \dot{\sim}(p \vee q))^m$ equals $\Box\neg p \wedge \Box\neg q \to \Box\neg(p \vee q)$. The proof that the last formula belongs to **D2** ($\subseteq \boldsymbol{S}$) is a standard task.
3. C_i is $\sim p \to (\dot{\sim}(p \to \dot{\sim}(q \to q)) \to \dot{\sim}(q \to q))$. Its translation by $(-)^m$ is just the formula $\Diamond\neg p \to (\Box\neg(p \to \Box\neg(q \to q)) \to \Box\neg(q \to q))$ which belongs to $\mathbf{D2} \subseteq \boldsymbol{S}$. This can be easily observed by the following thesis of **D2**: $\Diamond\neg p \to (\Diamond(q \to q) \to \Diamond(\Diamond(q \to q) \to \neg p))$.
4. C_i is a formula $(\dot{\sim}(p \to \dot{\sim}(q \to q)) \to \dot{\sim}(q \to q)) \to \sim p$. By translation we have $(\Box\neg(p \to \Box\neg(q \to q)) \to \Box\neg(q \to q)) \to \Diamond\neg p$. We can see that this formula is a thesis of **D2** since also the following formula is a thesis of **D2** $(\Diamond(q \to q) \to \Diamond(\Diamond(q \to q) \to \neg p)) \to \Diamond\neg p$.
5. C_i is a formula $\dot{\sim} p \to \sim p$. Its translation is a substitution of (D).
6. C_i is a formula $((p \to \dot{\sim}(q \to q)) \to \dot{\sim}(q \to q)) \to p$. Its translation has the form $((p \to \Box\neg(q \to q)) \to \Box\neg(q \to q)) \to p$. It is enough to see that the last formula can be easily inferred from the following theorem of **D2**: $(\Diamond(q \to q) \to (p \wedge \Diamond(q \to q))) \to p$.
7. C_i is obtained by (MP) from $C_k = C_j \to C_i$, where $k, j < i$. By the inductive hypothesis $(C_k)^m \in \boldsymbol{S}$ and $(C_j \to C_i)^m \in \boldsymbol{S}$. By Definition 7.5 we have $(C_j \to C_i)^m = (C_j)^m \to (C_i)^m$ and $(C_i)^m$ also arises from $(C_k)^m$ and $(C_j)^m$ by (MP).
8. $C_i = \dot{\sim} B \to \dot{\sim} C$ is obtained by (CONTR$^{\dot{\sim}}$) from $C_k = C \to B$, where $k < i$ and $C, B \in \mathrm{For}$. By the inductive hypothesis $(C \to B)^m = C^m \to B^m \in \boldsymbol{S}$, while by contraposition and monotonicity we obtain $(C_i)^m = \ulcorner\Box\neg((B)^m) \to \Box\neg((C)^m)\urcorner \in \boldsymbol{S}$.
9. C_i is obtained by a substitution of some formulas $D^i_1, \ldots, D^i_k \in \mathrm{For}$ into a formula C_j where $j < i$. Since $\boldsymbol{S}$ is closed under any substitution, it is enough to observe that that $(C_i)^m$ arises from $(C_j)^m$ by the appropriate substitution of formulas $(D^i_1)^m, \ldots, (D^i_k)^m$.
10. $C_i \in X^{u\dot{\sim}}$, i.e. C_i is of the form $(C)^{u\dot{\sim}}$, for some $C \in X$. Thus, $(C_i)^m = ((C)^{u\dot{\sim}})^m$ and by Lemma 7.8 it is equivalent on the basis of **D2** to $C \in X \subseteq \boldsymbol{S} = \mathbf{D2}[X]$. So, also $(C_i)^m \in \boldsymbol{S}$.

Finally, let $A \in \mathrm{For}^{\mathrm{M}}$. If $\vdash_{\mathbf{R}_{\boldsymbol{S}}} (A)^{u\dot{\sim}}$ than by the above reasoning we have $((A)^{u\dot{\sim}})^m \in \boldsymbol{S}$ but by Lemma 7.8 also $A \in \boldsymbol{S} = \mathbf{D2}[X]$. □

Let us recall that **D2** is sound and complete with respect to the class of serial frames,[5] where a modal formula is valid in a model iff this formula is true in every world including non-normal worlds, i.e. worlds where every formula of the form $\Diamond A$ is true and every formula of the form $\Box A$ is false.

Below the fact that a modal formula A is true in a world w under a valuation v is denoted by $w \models_v A$. We can easily verify that

Lemma 7.12. *For any $A \in \mathrm{For}$, any model $\mathcal{M} = \langle W, R, N, v\rangle$, and any $w \in W$ the following holds: $w \Vdash_v A$ iff $w \models_v (A)^m$.*

Theorem 7.13. $\mathbf{R}_{\mathbf{D2}}$ *is sound and complete with respect to the class of all serial frames.*

Proof. Let $A \in \mathrm{For}^{\dot{\sim}}$. We have: $A \in \mathbf{R}_{\mathbf{D2}}$ iff (by Lemma 7.9) $((A)^m)^{u\dot{\sim}} \in \mathbf{R}_{\mathbf{D2}}$ iff (by Lemma 7.11) $(A)^m \in \mathbf{D2}$ iff (by completeness for **D2**) $(A)^m$ is valid in every serial frame iff (by Lemma 7.12) A is valid in every serial frame. □

Theorem 7.14. *Let $\boldsymbol{S} = \mathbf{D2}[X]$. If a modal logic $\boldsymbol{S}$ is sound and complete with respect to some class of frames in the sense of Definition 6.3 (i.e. where non-normal worlds are allowed) with accessibility relations fulfilling a given condition C, then for the logic $\mathbf{R}_{\boldsymbol{S}}$ the following holds:*

For any $A \in \mathrm{For}^{\dot{\sim}}$: A is valid in every frame with an accessibility relation fulfilling the condition C iff A is a theorem of the logic $\mathbf{R}_{\boldsymbol{S}}$.

Proof. Assume that A is true in every frame with an accessibility relation fulfilling the condition C. By Lemma 7.12 it holds iff $(A)^m$ is valid in every frame with an accessibility relation fulfilling the condition C and, given assumption about $\boldsymbol{S}$, it is true iff $(A)^m \in \boldsymbol{S}$. By Lemma 7.11 it is equivalent to the fact that $\vdash_{\mathbf{R}_{\boldsymbol{S}}} ((A)^m)^{u\dot{\sim}}$ while by Lemma 7.9 and the definition of $\mathbf{R}_{\boldsymbol{S}}$ it holds iff $\vdash_{\mathbf{R}_{\boldsymbol{S}}} A$. □

References

[1] Béziau, J.-Y.: The paraconsistent logic Z. Log. Log. Philos. **15**, 99–111 (2006)

[2] Došen, K.: Negative modal operators in intuitionistic logic. Publ. L'Inst. Math. Nouvelle Sér. **35**(49), 3–14 (1984)

[5] A frame (also a relation) is serial iff for any normal world, the set of alternative worlds is non-empty.

[3] Došen, K.: Negation as modal operator. Rep. Math. Log. **20**, 15–27 (1986)
[4] Jaśkowski, S.: Rachunek zdań dla systemów dedukcyjnych sprzecznych. Studia Societatis Scientiarum Torunensis Sect. A, I(5), 57–77 (**in Polish**). Propositional Calculus for Contradictory Deductive Systems. Studia Logica 24, 143–157 (1969) and A Propositional Calculus for Inconsistent Deductive systems. Logic and Logical Philosophy **7**, 35–56 (1999). https://doi.org/10.12775/LLP.1999.003
[5] Kripke, S.A.: Semantical analysis of modal logic II. Non-normal modal propositional calculi. In: Symposium on the Theory of Models, pp. 206–220. North-Holland Publ. Co., Amsterdam (1965)
[6] Lemmon, E.J.: New fundations for lewis modal systems. J. Symb. Log. **22**(2), 176–186 (1957). https://doi.org/10.2307/2964179
[7] Marcos, J.: Nearly every normal modal logic is paranormal. Log. Anal. **48**(189–192), 279–300 (2005)
[8] Mruczek-Nasieniewska, K., Nasieniewski, M.: Syntactical and semantical characterization of a class of paraconsistent logics. Bull. Sect. Log. **34**(4), 229–248 (2005)
[9] Mruczek-Nasieniewska, K., Nasieniewski, M.: Paraconsistent logics obtained by J.-Y. Béziau's method by means of some non-normal modal logics. Bull. Sect. Log. **37**(3–4), 185–196 (2008)
[10] Mruczek-Nasieniewska, K., Nasieniewski, M.: Béziau's logics obtained by means of quasi-regular logics. Bull. Sect. Log. **38**(3–4), 189–203 (2009)
[11] Stanisław, J.: O koniunkcji dyskusyjnej w rachunku zdań dla systemów dedukcyjnych sprzecznych. Studia Societatis Scientiarum Torunensis, Sect. A, I(8), 171–172 (1942) (**in Polish**). On The Discussive Conjunction in the Propositional Calculus for Inconsistent Deductive Systems. Logic and Logical Philosophy 7, 57–59 (1999). https://doi.org/10.12775/LLP.1999.004

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