Familiarizing Students with Definition of Lebesgue Integral: Examples of Calculation Directly from Its Definition Using Mathematica

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Abstract We present in this paper several examples of Lebesgue integral calculated directly from its definitions using Mathematica. Calculation of Riemann integrals directly from its definitions for some elementary functions is standard in higher mathematics education. But it is difficult to find analogical examples for Lebesgue integral in the available literature. The paper contains Mathematica codes which we prepared to calculate symbolically Lebesgue sums and limits of sums. We also visualize the graphs of simple functions used for approximation of the integrals. We also show how to calculate the needed sums and limits by hand (without CAS). We compare our calculations in Mathematica with calculations in some other CAS programs such as wxMaxima, MuPAD and Sage for the same integrals.

Keywords Higher education · Lebesgue integral · Application of CAS · Mathematica · Mathematical didactics

Mathematics Subject Classification 97R20 · 97I50 · 97B40

1 Introduction

Young man, in mathematics you don't understand things. You just get used to them

John von Neumann

I hear and I forget. I see and I remember. I do and I understand

Chinese Quote

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In popular books of calculus, for example [3, 7, 12], we can find many examples of Riemann integral calculated directly from its definition. The aim of these examples is to familiarize students with the definition of Riemann integral. But we cannot find analogical examples for Lebesgue integral.

In this article, with similar aim but for Lebesgue integral definition, we present the following examples of calculation directly from its definition: $\int_0^{\pi/2} \sin x \, dm(x)$, $\int_0^1 \exp(x) \, dm(x)$, $\int_0^\pi \ln(1 - 2r \cos x + r^2) \, dm(x)$ ($r > 1$), $\int_0^1 x^k \, dm(x)$ ($k \in \mathbb{N}$), $\int_{-1}^1 x^2 \, dm(x)$ where $dm(x)$ denotes the Lebesgue measure on the real line. The way we presented we may also calculate e.g. $\int_0^1 x \, dm(x)$, $\int_0^1 x^2 \, dm(x)$, $\int_0^1 x^3 \, dm(x)$. We also consider the calculation of integrals $\int_0^1 x^2 \, dm(x)$, $\int_0^1 x^{1/m} \, dm(x)$ ($m \in \mathbb{N}$), $\int_a^b \chi_{[a,b] \setminus \mathbb{Q}} \, dm(x)$, $\int_1^e \ln x \, dm(x)$ using “partition the range of f ” Lebesgue philosophy. We calculate sums, limits and plot graphs of needed simple functions using Mathematica.

2 Definitions of Lebesgue Integral

The following two definitions of Lebesgue integral are used in the first part of this article:

Let $(\mathbb{R}, \mathfrak{M}, m)$ be the measure space, where $\mathfrak{M}$ is σ -algebra of Lebesgue measurable subsets in $\mathbb{R}$, and m is the Lebesgue measure on $\mathbb{R}$.

Let $\mathbb{R}$ be extended real numbers $\mathbb{R}$ with two more elements adjoined, denoted $+\infty$ and $-\infty$. Let A be Lebesgue measurable subset of $\mathbb{R}$. A real-valued function f on A is called simple if it assumes only finitely many distinct values. Let $f: A \rightarrow \mathbb{R}$ be measurable nonnegative function (we’ve omitted the definition of Lebesgue integral for simple real measurable functions).

Definition 1 (See [4, 6, 8, 10, 13])

$$\int_A f \, dm(x) = \sup \left\{ \int_A s \, dm(x) : 0 \leq s \leq f, s : A \mapsto \mathbb{R} \text{ is simple measurable function} \right\}. \quad (2.1)$$

Definition 2 (See [1, 11, 15]) Let $s_n : A \mapsto \mathbb{R}$ be nondecreasing sequence of nonnegative simple measurable functions such that $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ for every $x \in A$. Then:

$$\int_A f \, dm(x) = \lim_{n \rightarrow \infty} \int_A s_n \, dm(x). \quad (2.2)$$

We stress the fact that the value of the Lebesgue integral of function f in the Definition 2 is independent of the choice of the sequence s_n . The proof of this fact can be found in [1, p. 130, Theorem 17.5.] and in [11, p. 300], [15, p. 289, Theorem 4.6].

The equivalence of the two definitions follows from the Lebesgue’s Monotone convergence theorem (See [13, Theorem 11.28, p. 318], [6]) or it could be proved more elementary (sketch of the proof):

From the basic properties of Lebesgue integral for simple measurable function we have that $\int_A s_n \, dm(x)$ is nondecreasing sequence of real numbers. Hence the limit $\lim_{n \rightarrow \infty} \int_A s_n \, dm(x)$ exists (finite or ∞). Suppose that $\lim_{n \rightarrow \infty} \int_A s_n \, dm(x) = a \geq 0$ for some nondecreasing sequence s_n of nonnegative simple measurable functions such that $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ for every $x \in A$.

Directly from properties of the least upper bound we have that

$$\sup \left\{ \int_A s \, dm(x) : 0 \leq s \leq f, s : A \mapsto \mathbb{R} \text{ is simple measurable function} \right\} \geq a.$$

When $a = \infty$ then the equivalence is obvious. Suppose that $a < \infty$ and that $\sup \left\{ \int_A s \, dm(x) : 0 \leq s \leq f, s : A \mapsto \mathbb{R} \text{ is simple measurable function} \right\} > a$. That means that there exists simple measurable function s such that $0 \leq s \leq f$ and $\int_A s(x) \, dm(x) > a$. Let $t_n(x) = \max\{s(x), s_n(x)\}$ for all $x \in A$ and for $n = 1, 2, \dots$

It can be shown that t_n are simple measurable functions for $n = 1, 2, \dots$, $s(x) \leq t_n(x) \leq t_{n+1}(x)$ for all $x \in A$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} t_n(x) = f(x)$ for all $x \in A$.

Because the limit in the Definition 2 is independent of the choice of the sequence s_n and $a < \int_A s(x) dm(x) \leq \int_A t_n(x) dm(x)$ we have:

$$a < \int_A s(x) dm(x) \leq \lim_{n \rightarrow \infty} \int_A t_n(x) dm(x) = \lim_{n \rightarrow \infty} \int_A s_n(x) dm(x)$$

which contradicts the assumption $\lim_{n \rightarrow \infty} \int_A s_n dm(x) = a$. So it must be:

$$\sup \left\{ \int_A s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} = \lim_{n \rightarrow \infty} \int_A s_n(x) dm(x).$$

We will consider six examples of calculating Lebesgue integral directly from its definition.

3 Example: $\int_0^{\pi/2} \sin x dm(x)$

Let us consider the function: $f(x) = \sin x$, $x \in [0, \pi/2)$. For the rest of this example we will restrict our consideration to $x \in [0, \pi/2)$.

We will calculate $\int_0^{\pi/2} \sin x dm(x)$ applying directly Definition 1.

Consider

$$s_n(x) = \sum_{k=0}^{2^n-1} \sin\left(\frac{k}{2^{n+1}}\pi\right) \chi_{\left[\frac{k}{2^{n+1}}\pi, \frac{k+1}{2^{n+1}}\pi\right)}(x), \quad \text{for } x \in [0, \pi/2), n = 1, 2, \dots$$

$$\text{and } \bar{s}_n(x) = \sum_{k=1}^{2^n} \sin\left(\frac{k}{2^{n+1}}\pi\right) \chi_{\left[\frac{k-1}{2^{n+1}}\pi, \frac{k}{2^{n+1}}\pi\right)}(x), \quad \text{for } x \in [0, \pi/2), n = 1, 2, \dots$$

Using Wolfram Mathematica we get the following Figs. 1 and 2:

Fig. 1 Graphs of functions f , s_1 , s_2 . We can see that $s_1(x) \leq s_2(x)$ for $x \in [0, \pi/2)$

Fig. 2 Graphs of functions f , $\underline{s}_2$, $\underline{s}_3$. We can see that $\underline{s}_2(x) \leq \underline{s}_3(x)$ for $x \in [0, \pi/2)$

It is clear that $\underline{s}_n, \bar{s}_n$ are sequences of nonnegative simple measurable functions and that $\underline{s}_n \leq f$ and $\bar{s}_n \geq f$ on $[0, \pi/2)$ for all $n = 1, 2, \dots$

Using Wolfram Mathematica we get:

Listing 1 Mathematica code:

```
In[1]:= Simplify[ $\frac{\pi}{2^{n+1}} \sum_{k=0}^{2^n-1} \text{Sin}[\frac{\pi k}{2^{n+1}}]$ ]
Out[1]=  $2^{-2-n} \pi (-1 + \text{Cot}[2^{-2-n} \pi])$ 

In[2]:= Limit[%,  $n \rightarrow \infty$ ]
Out[2]= 1
```

So

$$a_n = \int_0^{\pi/2} \underline{s}_n \, dm(x) = \sum_{k=0}^{2^n-1} \sin \frac{k\pi}{2^{n+1}} \cdot \frac{1}{2^{n+1}} \pi = 2^{-2-n} \pi (-1 + \cot(2^{-2-n} \pi)) \rightarrow 1 \quad (3.1)$$

Similarly

Listing 2 Mathematica code:

```
In[3]:= Simplify[ $\frac{\pi}{2^{n+1}} \sum_{k=1}^{2^n} \text{Sin}[\frac{\pi k}{2^{n+1}}]$ ]
Out[3]=  $2^{-2-n} \pi (1 + \text{Cot}[2^{-2-n} \pi])$ 

In[4]:= Limit[%,  $n \rightarrow \infty$ ]
Out[4]= 1
```

So

$$\bar{a}_n = \int_0^{\pi/2} \bar{s}_n dm(x) = \sum_{k=1}^{2^n} \sin \frac{k\pi}{2^{n+1}} \cdot \frac{1}{2^{n+1}} \pi = 2^{-2-n} \pi \left(1 + \cot(2^{-2-n} \pi)\right) \rightarrow 1 \quad (3.2)$$

Of course we could use the following formulae: $\sum_{k=1}^n \sin(kx) = \frac{\sin \frac{n+1}{2} x \sin \frac{n}{2} x}{\sin \frac{x}{2}}$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ instead of the code in Listings 1 and 2 to get the results in formulae (3.1) and (3.2).

Using formulae (3.1) and (3.2), basic properties of least upper, greatest lower bounds and properties of Lebesgue integral of simple measurable functions we will prove that:

$$\sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} \geq 1 \quad (3.3)$$

and

$$\sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} \leq 1. \quad (3.4)$$

Inequalities (3.3) and (3.4) give

$$\sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} = 1,$$

which means that $\int_0^{\pi/2} f dm(x) = \int_0^{\pi/2} \sin x dm(x) = 1$.

Because $\lim_{n \rightarrow \infty} \bar{a}_n = 1$, so $\sup \left\{ \int_0^{\pi/2} \bar{s}_n dm(x) : n = 1, 2, \dots \right\} = \sup \bar{a}_n \geq 1$.

From $\left\{ \int_0^{\pi/2} \bar{s}_n dm(x) : n = 1, 2, \dots \right\} \subset \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\}$ follows

$$\sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} \geq 1. \quad (3.5)$$

For every simple measurable function s such that $s \leq f$ we have also that $s \leq \bar{s}_n$, $n = 1, 2, \dots$. Hence for every simple measurable function s such that $s \leq f$ we have $\int_0^{\pi/2} s dm(x) \leq \int_0^{\pi/2} \bar{s}_n dm(x)$, $n = 1, 2, \dots$, hence

$$\begin{aligned} \sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} &\leq \int_0^{\pi/2} \bar{s}_n dm(x), n = 1, 2, \dots, \text{ hence} \\ \sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} \\ &\leq \inf \left\{ \int_0^{\pi/2} \bar{s}_n dm(x), n = 1, 2, \dots \right\} \leq 1. \end{aligned} \quad (3.6)$$

The last inequality follows because $\lim_{n \rightarrow \infty} \bar{a}_n = 1$, which means $\inf \left\{ \int_0^{\pi/2} \bar{s}_n dm(x), n = 1, 2, \dots \right\} = \inf \bar{a}_n \leq 1$.

Inequalities (3.5) i (3.6) give

$$\sup \left\{ \int_0^{\pi/2} s dm(x) : 0 \leq s \leq f, s \text{ is simple measurable function} \right\} = 1,$$

and that means $\int_0^{\pi/2} f dm(x) = \int_0^{\pi/2} \sin x dm(x) = 1$.

Let calculate $\int_0^{\pi/2} \sin x dm(x)$ applying directly Definition 2.

We can see that $\underline{s}_n(x) \leq \underline{s}_{n+1}(x)$ for $x \in [0, \pi/2)$ and for all $n = 1, 2, \dots$. In Figs. 1 and 2 we can see that $\underline{s}_1(x) \leq \underline{s}_2(x)$ and $\underline{s}_2(x) \leq \underline{s}_3(x)$ for $x \in [0, \pi/2)$. We can also see that $\lim_{n \rightarrow \infty} \underline{s}_n(x) = \sin(x)$ for all $x \in [0, \pi/2)$. So $\underline{s}_n$ is nondecreasing sequence of nonnegative simple measurable functions and $\underline{s}_n$ converges pointwise to f .

So by formula (3.1) and directly by Definition 2 we get

$$\int_0^{\pi/2} \sin x \, dm(x) = \int_0^{\pi/2} f \, dm(x) = \lim_{n \rightarrow \infty} \int_0^{\pi/2} \underline{s}_n \, dm(x) = \lim_{n \rightarrow \infty} a_n = 1.$$

In the following Sects. 4–8 we omitted graphs of function f , $\underline{s}_n(x)$, $\bar{s}_n(x)$ because they are similar to the previous ones in Sect. 3.

4 Example: $\int_0^1 \exp x \, dm(x)$

Let consider the function: $f(x) = \exp x$, $x \in [0, 1)$. For the rest of this example we will restrict our consideration to $x \in [0, 1)$.

We will calculate $\int_0^1 \exp x \, dm(x)$ applying directly Definition 1.

Consider

$$\underline{s}_n(x) = \sum_{k=0}^{2^n-1} \exp\left(\frac{k}{2^n}\right) \chi_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}(x), \quad \text{for } x \in [0, 1), n = 1, 2, \dots$$

$$\text{and } \bar{s}_n(x) = \sum_{k=1}^{2^n} \exp\left(\frac{k}{2^n}\right) \chi_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x), \quad \text{for } x \in [0, 1), n = 1, 2, \dots$$

It is clear that $\underline{s}_n, \bar{s}_n$ are sequences of nonnegative simple measurable functions and that $\underline{s}_n \leq f$ and $\bar{s}_n \geq f$ on $[0, 1)$ for all $n = 1, 2, \dots$

Using Wolfram Mathematica we get:

Listing 3 Mathematica code:

```
In[5]:= Simplify[1/2^n Sum[Exp[k/2^n], {k, 0, 2^n-1}]]
```

```
Out[5]= (2^-n (-1 + e))/(-1 + e^(2^-n))
```

```
In[6]:= Limit[%, n -> Infinity]
```

```
Out[6]= -1 + e
```

So

$$a_n = \int_0^1 \underline{s}_n \, dm(x) = \sum_{k=0}^{2^n-1} \exp \frac{k}{2^n} \cdot \frac{1}{2^n} = 2^{-n} (1 - e) / (-1 + e^{2^{-n}}) \rightarrow e - 1 \quad (4.1)$$

Similarly

Listing 4 Mathematica code:

```
In[7]:= Simplify[1/2^n Sum[Exp[k/2^n], {k, 1, 2^n}]]
Out[7]= (2^-n (-1 + e) e^(2^-n)) / (-1 + e^(2^-n))

In[8]:= Limit[%, n -> Infinity]
Out[8]= -1 + e
```

So

$$\bar{a}_n = \int_0^1 \bar{s}_n dm(x) = \sum_{k=1}^{2^n} \exp \frac{k}{2^n} \cdot \frac{1}{2^n} = 2^{-n} e^{2^{-n}} (1 - e) / (-1 + e^{2^{-n}}) \rightarrow e - 1 \quad (4.2)$$

Of course we could use the following formulae: $\sum_{k=0}^n q^k = \frac{1-q^{n+1}}{1-q}$ ($q \neq 1$) and $\lim_{x \rightarrow 0} \frac{\exp x - 1}{x} = 1$ instead of the code in Listings 3 and 4 to get the results in formulae (4.1) and (4.2).

Using formulae (4.1) and (4.2) and similar reasoning like in the previous example (Sect. 3) we get:

$$\int_0^1 f dm(x) = \int_0^1 \exp x dm(x) = e - 1 \text{ directly from Definition 1}$$

and

$$\int_0^1 \exp x dm(x) = \int_0^1 f dm(x) = \lim_{n \rightarrow \infty} \int_0^1 \underline{s}_n dm(x) = \lim_{n \rightarrow \infty} \underline{a}_n = e - 1 \text{ directly from Definition 2.}$$

5 Example: $\int_0^\pi \ln(1 - 2r \cos x + r^2) dm(x)$ ($r > 1$)

Let consider the function: $f(x) = \ln(1 - 2r \cos x + r^2)$, $x \in [0, \pi)$, $r > 1$. For the rest of this example we will restrict our consideration to $x \in [0, \pi)$.

We will calculate $\int_0^\pi \ln(1 - 2r \cos x + r^2) dm(x)$ applying directly Definition 1.

Consider

$$s_n(x) = \sum_{k=0}^{2^n-1} f\left(\frac{k}{2^n}\pi\right) \chi_{\left[\frac{k}{2^n}\pi, \frac{k+1}{2^n}\pi\right)}(x), \quad \text{for } x \in [0, \pi), n = 1, 2, \dots$$

$$\text{and } \bar{s}_n(x) = \sum_{k=1}^{2^n} f\left(\frac{k}{2^n}\pi\right) \chi_{\left[\frac{k-1}{2^n}\pi, \frac{k}{2^n}\pi\right)}(x), \quad \text{for } x \in [0, \pi), n = 1, 2, \dots$$

It is clear that $\underline{s}_n, \bar{s}_n$ are sequences of nonnegative simple measurable functions and that $\underline{s}_n \leq f$ and $\bar{s}_n \geq f$ on $[0, \pi)$ for all $n = 1, 2, \dots$

In the Mathematica code below we use the fact that if s_k is a sequence of positive values with convergent sum, then we have $\sum_k s_k = \ln(\exp(\sum_k s_k)) = \ln(\prod_k \exp(s_k))$.

Using Wolfram Mathematica we get:

Listing 5 Mathematica code:

```

In[1]:= g[x_] = 1 - 2 r Cos[x] + r^2;
In[2]:= pr=Simplify[ $\prod_{k=0}^{n-1} g\left[\frac{k\pi}{n}\right]$  /. {n -> 2^n}]
Out[2]=  $\frac{(-1+r)(-1+r^{2^{1+n}})}{1+r}$ 
In[3]:= d = r^{2^{n+1}};
In[4]:= Limit[ $\frac{\pi}{2^n} \text{Log}\left[\frac{pr}{d}\right]$ , n -> ∞, Assumptions -> r > 1] + Limit[ $\frac{\pi}{2^n} \text{Log}[d]$ , n -> ∞, Assumptions -> r > 1]
Out[4]= 2π Log[r]

```

So

$$a_n = \int_0^\pi \underline{s}_n \, dm(x) = \sum_{k=0}^{2^n-1} f\left(\frac{k\pi}{2^{n+1}}\right) \cdot \frac{1}{2^n} \pi \rightarrow 2\pi \ln(r) \quad (5.1)$$

Similarly

Listing 6 Mathematica code:

```

In[5]:= pr1=Simplify[ $\frac{pr * g[\pi]}{g[0]}$ ]
Out[5]=  $\frac{(1+r)(-1+r^{2^{1+n}})}{-1+r}$ 
In[6]:= Limit[ $\frac{\pi}{2^n} \text{Log}\left[\frac{pr1}{d}\right]$ , n -> ∞, Assumptions -> r > 1] + Limit[ $\frac{\pi}{2^n} \text{Log}[d]$ , n -> ∞, Assumptions -> r > 1]
Out[6]= 2π Log[r]

```

In listings 5, 6 we used the substitution rule (n->2^n) because when we used directly 2^n instead n, Mathematica could not simplify the expression. We cannot calculate these limits in one step using Mathematica. But using other CAS (wxMaxima, MuPAD) we cannot calculate these limits even in two steps in any way.

So

$$\bar{a}_n = \int_0^\pi \bar{s}_n \, dm(x) = \sum_{k=1}^{2^n} f\left(\frac{k\pi}{2^{n+1}}\right) \cdot \frac{1}{2^n} \pi \rightarrow 2\pi \ln(r) \quad (5.2)$$

Of course we could use the following formulae:

$$z^{2^n} - 1 = (z^2 - 1) \prod_{k=1}^{n-1} \left(1 - 2z \cos(k\pi/n) + z^2\right)$$

instead of the code in Listings 5 and 6 to get the results in formulae (5.1) and (5.2).

Using formulae (5.1) and (5.2) and similar reasoning like in the previous example (Sect. 3) we get:

$\int_0^\pi f \, dm(x) = \int_0^\pi \ln(1 - 2r \cos x + r^2) \, dm(x) = 2\pi \ln(r)$ directly from Definition 1 and
 $\int_0^\pi \ln(1 - 2r \cos x + r^2) \, dm(x) = \int_0^\pi f \, dm(x) = \lim_{n \rightarrow \infty} \int_0^\pi \underline{s}_n \, dm(x) = \lim_{n \rightarrow \infty} \underline{a}_n = 2\pi \ln(r)$ directly from Definition 2.

6 Example: $\int_0^1 x^m dm(x)$ ($m \in \mathbb{N}$)

Let consider the function: $f(x) = x^m$, $x \in [0, 1)$, $m \in \mathbb{N}$. For the rest of this example we will restrict our consideration to $x \in [0, 1)$.

We will calculate $\int_0^1 x^m dm(x)$ applying directly Definition 1.

Consider

$$\underline{s}_n(x) = \sum_{k=0}^{2^n-1} \left(\frac{k}{2^n}\right)^m \chi_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}(x), \quad \text{for } x \in [0, 1), n = 1, 2, \dots$$

$$\text{and } \bar{s}_n(x) = \sum_{k=1}^{2^n} \left(\frac{k}{2^n}\right)^m \chi_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x), \quad \text{for } x \in [0, 1), n = 1, 2, \dots$$

As in the previous examples it is clear that $\underline{s}_n, \bar{s}_n$ are sequences of nonnegative simple measurable functions and that $\underline{s}_n \leq f$ and $\bar{s}_n \geq f$ on $[0, 1)$ for all $n = 1, 2, \dots$

Using Wolfram Mathematica we get:

Listing 7 Mathematica code:

```
In[1]:= Limit[1/2^n Sum[(j/2^n)^m, {j, 0, 2^n-1}], n -> Infinity, Assumptions -> m ∈ Integers && m > 0]
Out[1]= 1/(m+1)
```

So

$$\underline{a}_n = \int_0^1 \underline{s}_n dm(x) = \sum_{k=0}^{2^n-1} \left(\frac{k}{2^n}\right)^m \cdot \frac{1}{2^n} \rightarrow \frac{1}{m+1} \quad (6.1)$$

Similarly

Listing 8 Mathematica code:

```
In[2]:= Limit[1/2^n Sum[(j/2^n)^m, {j, 1, 2^n}], n -> Infinity, Assumptions -> m ∈ Integers && m > 0]
Out[2]= 1/(m+1)
```

So

$$\bar{a}_n = \int_0^1 \bar{s}_n dm(x) = \sum_{k=1}^{2^n} \left(\frac{k}{2^n}\right)^m \cdot \frac{1}{2^n} \rightarrow \frac{1}{m+1} \quad (6.2)$$

Of course we could use the Stolz and binomial theorems instead of the code in Listings 7 and 8 to get the results in formulae (6.1) and (6.2).

Using formulae (6.1) and (6.2) and similar reasoning like in the previous example (Sect. 3) we get:

$$\int_0^1 f dm(x) = \int_0^1 x^m dm(x) = \frac{1}{m+1} \text{ directly from Definition 1}$$

and

$$\int_0^1 x^m dm(x) = \int_0^1 f dm(x) = \lim_{n \rightarrow \infty} \int_0^1 \underline{s}_n dm(x) = \lim_{n \rightarrow \infty} \underline{a}_n = \frac{1}{m+1} \text{ directly from Definition 2.}$$

7 Example: $\int_{-1}^1 x^2 dm(x)$

Let consider the function: $f(x) = x^2$, $x \in [-1, 1)$. For the rest of this example we will restrict our consideration to $x \in [-1, 1)$.

We will calculate $\int_{-1}^1 x^2 dm(x)$ applying directly Definition 1.

Consider

$$s_n(x) = \sum_{k=0}^{2^n-1} \left(\frac{k}{2^n}\right)^2 \chi_{\left[-\frac{k+1}{2^n}, -\frac{k}{2^n}\right) \cup \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}(x), \text{ for } x \in [-1, 1), n = 1, 2, \dots$$

$$\text{and } \bar{s}_n(x) = \sum_{k=1}^{2^n} \left(\frac{k}{2^n}\right)^2 \chi_{\left[-\frac{k}{2^n}, -\frac{k-1}{2^n}\right) \cup \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x), \text{ for } x \in [-1, 1), n = 1, 2, \dots$$

As in the previous examples it is clear that $s_n, \bar{s}_n$ are sequences of nonnegative simple measurable functions and that $s_n \leq f$ and $\bar{s}_n \geq f$ on $[-1, 1)$ for all $n = 1, 2, \dots$

Using Wolfram Mathematica we get:

Listing 9 Mathematica code:

```
In[1]:= 2/2^3n Sum[k^2, {k, 1, 2^n-1}]
Out[1]= 1/3 * 2^-2n (-1 + 2^n) (-1 + 2^1+n)
In[2]:= Limit[%, n -> Infinity]
Out[2]= 2/3
```

So

$$a_n = \int_{-1}^1 s_n dm(x) = \sum_{k=0}^{2^n-1} \left(\frac{k}{2^n}\right)^2 \cdot \frac{2}{2^n} \rightarrow \frac{2}{3} \quad (7.1)$$

Similarly

Listing 10 Mathematica code:

```
In[3]:= 2/2^3n Sum[k^2, {k, 1, 2^n}]
Out[3]= 1/3 * 2^-2n (1 + 2^n) (1 + 2^1+n)
In[4]:= Limit[%, n -> Infinity]
Out[4]= 2/3
```

So

$$\bar{a}_n = \int_{-1}^1 \bar{s}_n dm(x) = \sum_{k=1}^{2^n} \left(\frac{k}{2^n}\right)^2 \cdot \frac{2}{2^n} \rightarrow \frac{2}{3} \quad (7.2)$$

Of course we could use the formula $\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$ instead of the code in Listings 9 and 10 to get the results in formulae (7.1) and (7.2).

Using formulae (7.1) and (7.2) and similar reasoning like in the previous example (Sect. 3) we get:

$$\int_{-1}^1 f \, dm(x) = \int_{-1}^1 x^2 \, dm(x) = \frac{2}{3} \text{ directly from Definition 1}$$

and

$$\int_{-1}^1 x^2 \, dm(x) = \int_{-1}^1 f \, dm(x) = \lim_{n \rightarrow \infty} \int_{-1}^1 \underline{s}_n \, dm(x) = \lim_{n \rightarrow \infty} \underline{a}_n = \frac{2}{3} \text{ directly from Definition 2.}$$

8 Example: $\int_0^1 f(x) \, dm(x)$, where $f(x)$ is Thomae's function

Let us consider Thomae's function

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in [0, 1] \cap \mathbb{Q}, \frac{p}{q} \text{ is in lowest terms,} \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases} \quad (8.1)$$

The Thomae's function $f(x)$ is Riemann integrable over $[0, 1]$ and $R \int_0^1 f(x) \, dx = 0$. Let us calculate $\int_0^1 f(x) \, dm(x)$ directly from Definition 1.

Let $0 < \frac{1}{2^n} < \frac{2}{2^n} < \dots < \frac{2^n-1}{2^n} < 1$ be partition of $[0, 1]$. Let $A_k = \{x \in [0, 1] : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\} = f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n}))$ for $k = 1, 2, \dots, 2^n - 1$ and $A_{2^n} = \{x \in [0, 1] : \frac{2^n-1}{2^n} \leq f(x) \leq 1\} = f^{-1}([\frac{2^n-1}{2^n}, 1])$.

One can see that $A_1 = [0, 1] \setminus B$, where $B \subset \mathbb{Q}$ and $A_k \subset [0, 1] \cap \mathbb{Q}$ for $k = 2, 3, \dots, 2^n$. Hence $m(A_1) = 1$, $m(A_k) = 0$ for $k = 2, 3, \dots, 2^n$, where m is the Lebesgue measure on $\mathbb{R}$.

Consider

$$\underline{s}_n(x) = \sum_{k=1}^{2^n} \frac{k-1}{2^n} \chi_{A_k}(x), \quad \text{for } x \in [0, 1], n = 1, 2, \dots$$

$$\text{and } \bar{s}_n(x) = \sum_{k=1}^{2^n} \frac{k}{2^n} \chi_{A_k}(x), \quad \text{for } x \in [0, 1], n = 1, 2, \dots$$

As in the previous examples it is clear that $\underline{s}_n, \bar{s}_n$ are sequences of nonnegative simple measurable functions and that $\underline{s}_n \leq f$ and $\bar{s}_n \geq f$ on $[0, 1]$ for all $n = 1, 2, \dots$

One can see that:

$$\underline{a}_n = \int_0^1 \underline{s}_n \, dm(x) = \sum_{k=1}^{2^n} \frac{k-1}{2^n} \cdot m(A_k) = 0. \quad (8.2)$$

and

$$\bar{a}_n = \int_0^1 \bar{s}_n \, dm(x) = \sum_{k=1}^{2^n} \frac{k}{2^n} \cdot m(A_k) = \frac{1}{2^n} \rightarrow 0. \quad (8.3)$$

Using formulae (8.2) and (8.3) and similar reasoning like in the previous example (Sect. 3) we get:

$$\int_0^1 f \, dm(x) = 0 \text{ directly from Definition 1}$$

and

$$\int_0^1 f \, dm(x) = \lim_{n \rightarrow \infty} \int_0^1 \underline{s}_n \, dm(x) = 0 \text{ directly from Definition 2.}$$

9 The Lebesgue Philosophy

The Lebesgue philosophy was well described in literature e.g. we will follow the description in [5, Chapter 3].

Lebesgue's goal was to collect approximately equal values of f . He proceeded in the following way. Let $f : [a, b] \mapsto \mathbb{R}$ be a bounded Lebesgue measurable function, and consider the partition

$$Q : c = y_0 < y_1 < \cdots < y_n = d$$

where

$$c = \inf\{f(x) : x \in [a, b]\} \text{ and } d = \sup\{f(x) : x \in [a, b]\}$$

and the diameter of Q is

$$|Q| = \max\{|y_k - y_{k-1}| : k = 1, \dots, n-1\}.$$

If $A_k = \{x \in [a, b] : y_{k-1} \leq f(x) < y_k\}$, $k = 1, \dots, n-1$ and $A_n = \{x \in [a, b] : y_{n-1} \leq f(x) \leq d\}$ we define

$$s_Q = \sum_{k=1}^n y_{k-1} m(A_k) \text{ and } S_Q = \sum_{k=1}^n y_k m(A_k), \text{ where } m \text{ is the Lebesgue measure on } \mathbb{R}$$

and

$$\underline{L} \int_a^b f \, dm(x) = \sup_Q s_Q \quad \text{and} \quad \overline{L} \int_a^b f \, dm(x) = \inf_Q S_Q.$$

Definition 3 A bounded Lebesgue measurable function $f : [a; b] \mapsto \mathbb{R}$ is Lebesgue integrable if $\underline{L} \int_a^b f \, dm(x) = \overline{L} \int_a^b f \, dm(x)$. In this case, the Lebesgue integral of f over $[a; b]$ is

$$\underline{L} \int_a^b f \, dm(x) = \overline{L} \int_a^b f \, dm(x) = L \int_a^b f \, dm(x).$$

9.1 The Lebesgue Philosophy for $\int_0^1 x^2 \, dm(x)$

Let us calculate $\int_0^1 x^2 \, dm(x)$ directly from Definition 3.

Let $f(x) = x^2$ and $g(x) = f^{-1}(x) = \sqrt{x}$ for $x \in [0, 1]$.

Let $y_k = k/n$, $k = 0, 1, 2, \dots, n$ so $Q_n : c = 0 < \frac{1}{n} < \cdots < \frac{n-1}{n} < 1 = d$.

$$\begin{aligned} s_{Q_n} &= \sum_{k=1}^n y_{k-1} m(A_k) = \sum_{k=1}^n y_{k-1} (g(y_k) - g(y_{k-1})) = \sum_{k=2}^n \frac{k-1}{n} (g(y_k) - g(y_{k-1})) \\ &= \frac{1}{n} (ng(y_n) - g(y_1) - \sum_{k=2}^n g(y_k)) = \frac{1}{n} (ng(1) - g(1/n) - \sum_{k=2}^n g(k/n)) \\ &= \frac{1}{n} (n - \sqrt{1/n} - \sum_{k=2}^n \sqrt{k/n}) = 1 - \frac{1}{n\sqrt{n}} - \frac{1}{n\sqrt{n}} \sum_{k=2}^n \sqrt{k}. \end{aligned} \tag{9.1}$$

Using Stolz theorem we calculate the limit:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\sum_{k=2}^n \sqrt{k}}{n^{3/2}} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{(n+1)^{3/2} - n^{3/2}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}((n+1)^{3/2} + n^{3/2})}{(n+1)^3 - n^3} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}((n+1)^{3/2} + n^{3/2})}{3n^2 + 3n + 1} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+1/n}((1+1/n)^{3/2} + 1)}{3 + 3/n + 1/n^2} = 2/3.\end{aligned}\quad (9.2)$$

So $\lim_{n \rightarrow \infty} s_{Q_n} = 1 - 0 - 2/3 = 1/3$. Hence from the basic properties of s_Q , S_Q and least upper bound we have

$$\sup_Q s_Q \geq 1/3 \text{ and } S_Q \geq 1/3 \text{ for all partitions } Q. \quad (9.3)$$

Similarly

$$\begin{aligned}S_{Q_n} &= \sum_{k=1}^n y_k m(A_k) = \sum_{k=1}^n y_k (g(y_k) - g(y_{k-1})) = \sum_{k=1}^n \frac{k}{n} (g(y_k) - g(y_{k-1})) \\ &= \frac{1}{n} (ng(y_n) - g(y_0) - \sum_{k=0}^{n-1} g(y_k)) = \frac{1}{n} (ng(1) - g(0) - \sum_{k=0}^{n-1} g(k/n)) \\ &= \frac{1}{n} (n - \sqrt{0} - \sum_{k=0}^{n-1} \sqrt{k/n}) = 1 - \frac{1}{n^{3/2}} \sum_{k=1}^{n-1} \sqrt{k}.\end{aligned}\quad (9.4)$$

Using Stolz theorem we calculate the limit:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{n-1} \sqrt{k}}{n^{3/2}} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{(n+1)^{3/2} - n^{3/2}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}((n+1)^{3/2} + n^{3/2})}{(n+1)^3 - n^3} \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}((n+1)^{3/2} + n^{3/2})}{3n^2 + 3n + 1} = \lim_{n \rightarrow \infty} \frac{(1+1/n)^{3/2} + 1}{3 + 3/n + 1/n^2} = 2/3.\end{aligned}\quad (9.5)$$

So $\lim_{n \rightarrow \infty} S_{Q_n} = 1 - 2/3 = 1/3$. Hence similarly we have

$$\inf_Q S_Q \leq 1/3 \text{ and } s_Q \leq 1/3 \text{ for all partitions } Q. \quad (9.6)$$

From 9.3 and 9.6 follows that:

$$L \int_0^1 x^2 dm(x) = \sup_Q s_Q = 1/3 \quad \text{and} \quad L \int_0^1 x^2 dm(x) = \inf_Q S_Q = 1/3.$$

Hence directly from Definition 3 we have

$$L \int_0^1 x^2 dm(x) = \frac{1}{3}.$$

We cannot get Wolfram Mathematica, wxMaxima and MuPAD to calculate the above limits. In Figs. 3 and 4 below we present dynamic plots of Lebesgue lower and upper sums s_{Q_n} and S_{Q_n} for $L \int_0^1 x^2 dm(x)$ created using Wolfram Mathematica. We used Manipulate function.

The above calculation are much more difficult that the ones for Riemann integral $\int_0^1 x^2 dx$. Similarly we can calculate $\int_0^1 x^3 dm(x)$ and $\int_0^1 x^m dm(x)$ ($m \in \mathbb{N}$).

Fig. 3 Dynamic plots of Lebesgue lower sums s_{Q_n} for $L \int_0^1 x^2 dm(x)$ and $n \in \{20, 2, 4, 6, 10, 40, 1000\}$

Fig. 4 Dynamic plots of Lebesgue upper sums S_{Q_n} for $L \int_0^1 x^2 dm(x)$ and $n \in \{20, 2, 4, 6, 10, 40, 1000\}$

The dynamic versions of the Figs. 3 and 4 can be found in Electronic supplementary material.

Actually when we take the following range partition $y_k = \left(\frac{k}{n}\right)^2$, $k = 0, 1, \dots, n$ the calculation for the considered Lebesgue integral $\int_0^1 x^2 dm(x)$ are pretty the same like for the Riemann one (slightly different reasoning). Similarly when we take the following range partition $y_k = \sin\left(\frac{k}{2n}\pi\right)$, $k = 0, 1, \dots, n$ the calculation for the considered Lebesgue integral $\int_0^{\pi/2} \sin x dm(x)$ are pretty the same like for the Riemann case.

We can see graphical presentation of the Lebesgue philosophy done in [2,9]. But we could not find examples calculated directly from Definition 3 except for the Dirichlet like function.

9.2 The Lebesgue Philosophy for $\int_a^b \chi_{[a,b] \cap \mathbb{Q}} dm(x)$

In [5] is proved that Dirichlet function $\chi_{[a,b] \cap \mathbb{Q}}$ is not Riemann integrable but it is proved directly from Definition 3 that

$$L \int_a^b \chi_{[a,b] \cap \mathbb{Q}} dm(x) = 0.$$

9.3 The Lebesgue Philosophy for $\int_a^b \chi_{[a,b] \setminus \mathbb{Q}} dm(x)$

Let

$$f(x) = \chi_{[a,b] \setminus \mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in [a, b] \setminus \mathbb{Q}, \\ 0 & \text{if } x \in [a, b] \cap \mathbb{Q}. \end{cases}$$

Let us calculate $\int_a^b \chi_{[a,b] \setminus \mathbb{Q}} dm(x)$ directly from Definition 3.

Let $Q : 0 = y_0 < y_1 < \dots < y_n = 1$ be any partition of $[0, 1]$.

One can see that $A_1 = [a, b] \cap \mathbb{Q}$, $A_2 = A_3 = \dots = A_{n-1} = \emptyset$, $A_n = [a, b] \setminus \mathbb{Q}$.

Thus $s_Q = y_{n-1}(b-a)$, $S_Q = (b-a)$ for any partition Q .

So $\sup_Q s_Q = b-a$ and $\inf_Q S_Q = b-a$. Hence

$$L \int_a^b \chi_{[a,b] \setminus \mathbb{Q}} dm(x) = b-a \text{ directly from Definition 3.}$$

9.4 The Lebesgue Philosophy for $\int_0^1 f(x) dm(x)$, where $f(x)$ is Thomae's function

Let us consider Thomae's function $f(x)$ which is defined in formula 8.1 in Sect. 8.

Let us calculate $\int_0^1 f(x) dm(x)$ directly from Definition 3.

Let $Q : 0 = y_0 < y_1 < \dots < y_n = 1$ be any partition of $[0, 1]$.

One can see that $A_1 = [0, 1] \setminus B$, where $B \subset \mathbb{Q}$ and $A_2 \subset [0, 1] \cap \mathbb{Q}$, $A_3 \subset [0, 1] \cap \mathbb{Q}$, $\dots$, $A_n \subset [0, 1] \cap \mathbb{Q}$. Hence $m(A_1) = 1$, $m(A_2) = m(A_3) = \dots = m(A_n) = 0$, where m is the Lebesgue measure on $\mathbb{R}$.

Thus $s_Q = 0$, $S_Q = y_1$ for any partition Q .

So $\sup_Q s_Q = 0$ and $\inf_Q S_Q = 0$. Hence

$$L \int_0^1 f(x) dm(x) = 0 \text{ directly from Definition 3.}$$

9.5 The Lebesgue Philosophy for $\int_0^1 x^{1/m} dm(x)$; ($m \in \mathbb{N}$)

Let us calculate $\int_0^1 x^{1/m} dm(x)$; $m \in \mathbb{N}$ directly from Definition 3.

Let $f(x) = x^{1/m}$ and $g(x) = f^{-1}(x) = x^m$ for $x \in [0, 1]$.

Let $y_k = k/n$, $k = 0, 1, 2, \dots, n$ so $Q_n : c = 0 < \frac{1}{n} < \dots < \frac{n-1}{n} < 1 = d$.

$$\begin{aligned}
 s_{Q_n} &= \sum_{k=1}^n y_{k-1} m(A_k) = \sum_{k=1}^n y_{k-1} (g(y_k) - g(y_{k-1})) = \sum_{k=2}^n \frac{k-1}{n} (g(y_k) - g(y_{k-1})) \\
 &= \frac{1}{n} (ng(y_n) - g(y_1) - \sum_{k=2}^n g(y_k)) = \frac{1}{n} \left(ng(1) - g(1/n) - \sum_{k=2}^n g(k/n) \right) \\
 &= \frac{1}{n} \left(n - \frac{1}{n^m} - \sum_{k=2}^n (k/n)^m \right) = 1 - \frac{1}{n^{m+1}} - \frac{1}{n^{m+1}} \sum_{k=2}^n k^m.
 \end{aligned} \tag{9.7}$$

Similarly

$$\begin{aligned}
 S_{Q_n} &= \sum_{k=1}^n y_k m(A_k) = \sum_{k=1}^n y_k (g(y_k) - g(y_{k-1})) = \sum_{k=1}^n \frac{k}{n} (g(y_k) - g(y_{k-1})) \\
 &= \frac{1}{n} \left(ng(y_n) - g(y_0) - \sum_{k=0}^{n-1} g(y_k) \right) = \frac{1}{n} \left(ng(1) - g(0) - \sum_{k=0}^{n-1} g(k/n) \right) \\
 &= \frac{1}{n} \left(n - 0^m - \sum_{k=0}^{n-1} (k/n)^m \right) = 1 - \frac{1}{n^{m+1}} \sum_{k=1}^{n-1} k^m.
 \end{aligned} \tag{9.8}$$

Using Wolfram Mathematica we get:

Listing 11 Mathematica code:

```

In[1]:= Limit[ $\frac{1}{n} \sum_{k=2}^n \left(\frac{k}{n}\right)^m$ ,  $n \rightarrow \infty$ , Assumptions  $\rightarrow m \in \text{Integers} \&\& m > 0$ ]
Out[1]=  $\frac{1}{1+m}$ 

```

and

Listing 12 Mathematica code:

```

In[2]:= Limit[ $\frac{1}{n} \sum_{k=1}^{n-1} \left(\frac{k-1}{n}\right)^m$ ,  $n \rightarrow \infty$ , Assumptions  $\rightarrow m \in \text{Integers} \&\& m > 0$ ]
Out[2]=  $\frac{1}{1+m}$ 

```

From listings 11, 12 and we have

$$\lim_{n \rightarrow \infty} s_{Q_n} = 1 - \frac{1}{m+1} = \frac{m}{m+1} \quad \text{and} \quad \lim_{n \rightarrow \infty} S_{Q_n} = 1 - \frac{1}{m+1} = \frac{m}{m+1} \tag{9.9}$$

Of course we could use the Stolz and binomial theorems instead of the code in Listings 11 and 12 to get the results in formulae (9.9).

Similar reasoning like in the previous example gives:

$$L \int_0^1 x^{1/m} dm(x) = \sup_Q s_Q = \frac{m}{m+1} \quad \text{and} \quad L \int_0^1 x^{1/m} dm(x) = \inf_Q S_Q = \frac{m}{m+1}.$$

Hence directly from Definition 3 we have

$$L \int_0^1 x^{1/m} dm(x) = \frac{m}{m+1}.$$

9.6 The Lebesgue Philosophy for $\int_1^e \ln x dm(x)$

Let us calculate $\int_1^e \ln x dm(x)$ directly from Definition 3.

Let $f(x) = \ln x$ $x \in [1, e]$ and $g(x) = f^{-1}(x) = \exp x$ for $x \in [0, 1]$.

Let $y_k = k/n$, $k = 0, 1, 2, \dots, n$ so $Q_n : c = 0 < \frac{1}{n} < \dots < \frac{n-1}{n} < 1 = d$.

$$s_{Q_n} = \sum_{k=1}^n y_{k-1} m(A_k) = \sum_{k=1}^n y_{k-1} (g(y_k) - g(y_{k-1})) = \sum_{k=2}^n \frac{k-1}{n} (\exp(k/n) - \exp((k-1)/n)). \quad (9.10)$$

Similarly

$$S_{Q_n} = \sum_{k=1}^n y_k m(A_k) = \sum_{k=1}^n y_k (g(y_k) - g(y_{k-1})) = \sum_{k=1}^n \frac{k}{n} (\exp(k/n) - \exp((k-1)/n)). \quad (9.11)$$

Using Wolfram Mathematica we get:

Listing 13 Mathematica code:

```
In[1]:= Limit[1/n Sum[(k-1) (Exp[k/n] - Exp[(k-1)/n]), {k, 1, n}], n -> Infinity]
Out[1]= 1
```

and

Listing 14 Mathematica code:

```
In[2]:= Limit[1/n Sum[k (Exp[k/n] - Exp[(k-1)/n]), {k, 1, n}], n -> Infinity]
Out[2]= 1
```

Of course we could use the formula for sum of geometric series instead of the code in Listings 13 and 14.

From listings 13, 14 and similar reasoning like in the last section we have

$$L \int_1^e \ln x dm(x) = \sup_Q s_Q = 1 \quad \text{and} \quad L \int_1^e \ln x dm(x) = \inf_Q S_Q = 1.$$

Hence directly from Definition 3 we have

$$L \int_1^e \ln x dm(x) = 1.$$

10 Comparing the Calculations in Mathematica with Calculations in Some Other CAS Programs

The authors tried to repeat calculations for examples of Lebesgue integrals using some other CAS programs such as: wxMaxima, MuPAD and Sage. The following Lebesgue integrals: $\int_0^{\pi/2} \sin x \, dm(x)$, $\int_0^1 e^x \, dm(x)$, $\int_{-1}^1 x^2 \, dm(x)$ were calculated in analogical way in these CAS programs. We used standard procedures in these programs to calculate sums and limits. We could not calculate the integrals: $\int_0^1 x^m \, dm(x)$ ($m \in \mathbb{N}$) and $\int_0^\pi \ln(1 - 2r \cos x + r^2) \, dm(x)$ ($r > 1$). The use of procedures which we have used before (for integrals $\int_0^{\pi/2} \sin x \, dm(x)$, $\int_0^1 e^x \, dm(x)$, $\int_{-1}^1 x^2 \, dm(x)$) for these two integrals had no effect. We used standard procedures: sum, limit, simplification, assume.

11 Conclusions

In this paper the authors present several examples of Lebesgue integral calculated directly from its definition using Mathematica. We also consider the calculation of integrals using “partition the range of f ” Lebesgue-philosophy.

Familiarizing students with definition of an integral by calculation integrals directly from its definition is a standard approach in the case of Riemann integral.

Many examples of Riemann integral calculated using only its definition can be found in literature. We could not find any analogical examples in available literature for Lebesgue integral, so this paper is an attempt to fill this gap.

Using Mathematica or other CAS programs for calculation Lebesgue integrals directly from its definitions, seems to be didactically useful for students because of the possibility of symbolic calculation of sums, limits and plot graphs—checking our hand calculations. Moreover we get students used not only to definition of Lebesgue integral but also to CAS applications generally. The principles of programming in Mathematica language are given in [14, 16].

The six examples from Sects. 3–8 of our article could be used as a supplement to the textbooks [4, 6, 8, 10, 13] (when we use Definition 1) and also as a supplement to the textbooks [1, 11, 15] (when we use Definition 2). And this is the main reason why we in our article included the Definitions 1 and 2 of Lebesgue integral and calculated the examples of integral directly from these definitions.

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