Toward Foundations of Near Sets: (Pre-)Sheaf Theoretic Approach

Marcin Wolski

Received: 30 August 2012 / Revised: 30 November 2012 / Accepted: 13 December 2012 / Published online: 19 February 2013
© The Author(s) 2013. This article is published with open access at Springerlink.com

Abstract The formal content of near set theory can be summarised in terms of three concepts: a perceptual system, a nearness relation and a near set. Perceptual systems and different forms of nearness relations have been already successfully related to important mathematical structures (e.g., approach spaces) and described in the frameworks of general topology and category theory. However, since near sets actually do not form any regular structure, there is lack of similar results about the concept of a near set. The main goal of this paper is to fill this gap and provide a mathematical basis for near sets on a similar abstract level as it has been already done for nearness relations. However, we take in the paper a down-to-earth approach; that is, instead of seeking the richest mathematical structures which express our intuitions (call it up-to-sky approach), we present in the paper the simplest category theoretic structures which, however, are rich enough to convey our ideas. Thus, although many times we actually deal with sheaves, we speak about them as pre-sheaves. The main reason is that a sheaf (in contrast to a pre-sheaf) embodies the idea of gluing (in the very similar way like manifolds, which are obtainable by gluing open subsets of Euclidean space), which is irrelevant to the study of near sets and rough sets. Furthermore, the concept a pre-sheaf is much simpler than a sheaf and should be easily “digested” by readers who are not very familiar with category theory.

Keywords Near set · Rough set · Pre-sheaf · Topology

Mathematics Subject Classification (2000) Primary 54B34; Secondary 54H05

1 Introduction

In these days of “soft mathematics”—or, according to its more sophisticated name, “soft computing paradigm”—intuitive and simple concepts of set and set membership have been extended to more complex mathematical objects such as fuzzy sets (e.g., [1]) or rough sets (e.g., [2]). However, after the scientific success of fuzzy set and rough set methodologies, there has been introduced a plethora of different types of sets: fuzzy rough sets, rough fuzzy sets, intuitionistic fuzzy sets, grey sets or interval-valued sets (just to list a few of them). Sometimes, “soft mathematics” resembles modern philosophy where the glut of two or three concepts forms not only a new notion, but also a new

M. Wolski (✉)
Department of Logic and Philosophy of Science, Maria Curie Skłodowska University,
Pl. Marii Curie-Skłodowskiej 4, Lublin, Poland
e-mail: marcin.wolski@umcs.lublin.pl

subject of study; e.g., interval-valued rough-grey sets [3]. Sometimes, as one can expect, these gluts bring nothing important; as was observed in [4]:

[...] at every level of existence—powerset level, topological fibre level, categorical level—interval-valued sets, interval-valued “intuitionistic” sets, and “intuitionistic” fuzzy sets and fuzzy topologies are redundant and represent unnecessarily complicated, strictly special subcases of standard fixed-basis set theory and topology. It therefore follows that theoretical workers should stop working in these restrictive and complicated programs and instead turn their efforts to substantial problems in the simpler and more general fixed-basis and variable-basis set theory and topology, while applied workers should carefully document the need or appropriateness of interval-valued or “intuitionistic” notions in applications.

In other words, sometimes it turns out that scientists are similar to Mr. Jourdain, the hero of *The Middle-Class Aristocrat* by Moliere, who during a philosophy lesson is surprised and delighted to learn that he has been speaking “prose” all his life without knowing it. So, many times a standard mathematical object is called (or defined) by some clumsy “prose”-like term and it is believed to be a new and important discovery.

The present paper is concerned with near sets which were introduced by Peters [5]. The term “near set” suggests that we deal with “soft mathematics”. However, as we demonstrate, similarly like rough sets, near set theory is worked out in basic set theory and topology (thus, it is not a “restrictive and complicated program”); furthermore, it offers an elegant mathematical interpretation of standard data tables (an alternative view to rough set theory, e.g., [2], formal concept analysis, e.g., [6], or Chu-spaces theory, e.g., [7]), which are basic objects of analysis in data mining. Finally, near set theory provides a “soft-mathematical” methodology, which has already been nicely embedded in a number of non-trivial categories, e.g., [8,9]. The theory of near sets may be summarised in terms of three simple concepts: a perceptual system, a nearness relation and a near set. We analyse all these concepts within the framework of category theory. We take in the paper a down-to-earth approach and use mainly simple category theoretic constructions so as to make presentation clear also for readers for whom, as for Rota [10], category theory is a controversial example of mathematical beauty. Our main idea, simply put, is to regard near set theory as an extension of rough set theoretic concepts of approximations of (undefinable) sets from the level of sets to the level of families of sets. In category theoretic language, our idea is expressible by a concept of a pre-sheaf, that is a contravariant functor from a topological space (regarded as a poset of open sets) to the category of sets. (It is worth a mention that here or there we actually deal with richer structures than pre-sheaves, that is, sheaves; however, a sheaf—additionally to a pre-sheaf—embodies the idea of gluing (that is, the idea putting together small pieces into a single object), which is irrelevant to the study of near sets and rough sets, and would make definitions quite complex.) Firstly, a perceptual system is a simple example of a pre-sheaf, which is a sub-pre-sheaf of the most elementary sheaf considered in the literature. Secondly, near sets regarded as a functor also form a pre-sheaf. Thus, pre-sheaves may be viewed as a well-defined methodological perspective from which near set theory deals with data tables and concepts derivable from them. Under this view, a lower approximation operator may be naturally converted into a pre-sheaf, whereas a near set would be an analogue of an upper approximation operator; therefore near set theory may be considered as a kind of pre-sheafification of rough set theory.

2 Preliminaries

In this section we recall important notions from topology, category theory, rough set theory and near set theory. As usual, the reader familiar these topics may regard this section as an introduction of basic notation used in the paper.

2.1 Bits of Topology and Category Theory

In order to make the presentation self-contained, we recall here the very basic definitions from topology and category theory. However, we confine ourselves to necessary ones to make the paper intuitively clear for the reader not familiar with these topics. For a quick introduction to the areas at issue see, for instance, [11–13].

Definition 2.1 (*Topology*) Let U be a set. A *topology* in X is family τ of subsets such that:

- each union of members of τ is a member of τ ;
- each finite intersection of members of τ is also a member of τ ;
- U and $\emptyset$ are members of τ .

If τ is closed under arbitrary intersections, then τ is called an *Alexandrov topology*. A couple (U, τ) is called a *topological space*; members of τ are called *open sets*. A set $X \subseteq U$ is *closed* if $U \setminus X \in \tau$.

As usual, the smallest closed set containing X is denoted by $Cl(X)$, and the largest open set contained in X is denoted by $Int(X)$. Operator $Cl : 2^U \rightarrow 2^U$ sending X to $Cl(X)$ is called a *closure*, whereas an operator $Int : 2^U \rightarrow 2^U$ sending X to $Int(X)$ is called an *interior*. Thus, $Cl(X)$ is called a closure of X and $Int(X)$ is called an interior of X .

Definition 2.2 (*Category*) A *category* $\mathbf{C}$ consists of:

- a class of objects denoted by $|\mathbf{C}|$;
- a class of arrows (or morphisms) from $\mathbf{a}$ to $\mathbf{b}$, denoted by $\mathbf{C}(\mathbf{a}, \mathbf{b})$, for all $\mathbf{a}, \mathbf{b} \in |\mathbf{C}|$;
- a composition operation $\circ : \mathbf{C}(\mathbf{b}, \mathbf{c}) \times \mathbf{C}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{C}(\mathbf{a}, \mathbf{c})$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathbf{C}|$;
- the identity arrows $id_{\mathbf{a}} \in \mathbf{C}(\mathbf{a}, \mathbf{a})$, for all $\mathbf{a} \in |\mathbf{C}|$;

such that, for all $f \in \mathbf{C}(\mathbf{a}, \mathbf{b})$, $g \in \mathbf{C}(\mathbf{b}, \mathbf{c})$, $h \in \mathbf{C}(\mathbf{c}, \mathbf{d})$, the following equations are satisfied:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$f \circ id_{\mathbf{a}} = f = id_{\mathbf{b}} \circ f$$

If $\mathbf{C}$ is a category, then its *dual*, denoted by $\mathbf{C}^{op}$, has the same objects as $\mathbf{C}$ but its morphisms are reversed. A category $\mathbf{C}$ is *small*, if $|\mathbf{C}|$ is a set in the sense of Gödel-Bernays set theory. $\mathbf{C}$ is called *locally small*, if $\mathbf{C}(\mathbf{a}, \mathbf{b})$ is a set for all its objects $\mathbf{a}, \mathbf{b}$. A morphism $f \in \mathbf{C}(\mathbf{a}, \mathbf{b})$ (denoted also by $f : \mathbf{a} \rightarrow \mathbf{b}$) is called a *monomorphism*, if for all objects $\mathbf{c}$ and all arrows $g, h : \mathbf{c} \rightarrow \mathbf{a}$, $f \circ g = f \circ h$ implies $g = h$. An arrow $f : \mathbf{a} \rightarrow \mathbf{b}$ is an *epimorphism*, if for all objects $\mathbf{c}$ and all arrows $g, h : \mathbf{b} \rightarrow \mathbf{c}$, $g \circ f = h \circ f$ implies $g = h$. An arrow $f : \mathbf{a} \rightarrow \mathbf{b}$ is an *isomorphism* or *iso*, if there exists $g : \mathbf{b} \rightarrow \mathbf{a}$ such that $g \circ f = id_{\mathbf{a}}$ and $f \circ g = id_{\mathbf{b}}$.

A standard example of a category is **Set** which has sets as objects and total functions as arrows. Let us recall that for any topological space (U, τ) its open sets are partially ordered by set inclusion $\subseteq$. Now, we can regard τ as a small category where there is an arrow from $X \in \tau$ to $Y \in \tau$ iff $X \subseteq Y$. Such a category will be denoted by $\mathbf{C}(\tau)$.

In any category, an object $\top$ (resp. $\perp$) is called a *terminal* (resp. *initial*) object, if for every object A , there is exactly one arrow $f : A \rightarrow \top$ (resp. $f : \perp \rightarrow A$). Any two terminal (resp. initial) objects are isomorphic. In category **Set**, the empty set $\emptyset$ is an initial object, whereas any singleton set $\{*\}$ is a terminal object. In category $\mathbf{C}(\tau)$, $\emptyset$ is an initial object and U is a terminal object.

Definition 2.3 (*Functor*) A functor F from a category $\mathbf{A}$ to a category $\mathbf{B}$ consists of:

- a mapping $|\mathbf{A}| \rightarrow |\mathbf{B}|$ of objects; the image of $\mathbf{a} \in |\mathbf{A}|$ is denoted by $F\mathbf{a}$;
- a mapping $\mathbf{A}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{B}(F\mathbf{a}, F\mathbf{b})$ of arrows, for all $\mathbf{a}, \mathbf{b} \in |\mathbf{A}|$; the image of $f \in \mathbf{A}(\mathbf{a}, \mathbf{b})$ is denoted Ff ;

such that, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathbf{A}|$, $f \in \mathbf{A}(\mathbf{a}, \mathbf{b})$, $g \in \mathbf{A}(\mathbf{b}, \mathbf{c})$, the following conditions are satisfied:

$$F(g \circ f) = Fg \circ Ff \text{ and } F id_{\mathbf{a}} = id_{F\mathbf{a}}.$$

To give an example, let $\mathbf{C}(\tau_1)$ and $\mathbf{C}(\tau_2)$ be two categories defined above. Then a functor $F : \mathbf{C}(\tau_1) \rightarrow \mathbf{C}(\tau_2)$ is an order preserving function: if $X \subseteq Y$, then $F(X) \subseteq F(Y)$, for all $X, Y \in \tau_1$. A functor F from a category $\mathbf{A}^{op}$ to a category $\mathbf{B}$ is called a *contravariant functor* from $\mathbf{A}$ to $\mathbf{B}$.

Definition 2.4 (*Presheaf*) Let (U, τ) be a topological space and $\mathbf{A}$ be a category; a presheaf F on U with values in $\mathbf{A}$ is a functor from $\mathbf{C}(\tau)^{op}$ to $\mathbf{A}$.

In other words, a presheaf $\mathbb{F}$ assigns to each open set X of U an object $\mathbb{F}X$ of $\mathbf{A}$, and if $X \subseteq Y$, where X and Y are open sets of U , then there exists a morphisms in $\mathbf{A}$, often denoted by ρ_X^Y and called a restriction map, which takes $\mathbb{F}Y$ to $\mathbb{F}X$, in such a way that for all open sets X, Y, Z of U , if

$$X \subseteq Y \subseteq Z \quad \text{then} \quad \rho_X^Z = \rho_X^Y \circ \rho_Y^Z.$$

Thus $\rho_X^Y \in \mathbf{A}(\mathbb{F}Y, \mathbb{F}X)$. For any $X \in \tau$ elements of $\mathbb{F}X$ are called sections over X and $\rho_X^Y(s)$, for $s \in \mathbb{F}X$, is usually denoted by $s|_X$.

It is worth emphasising that since $\emptyset$ is an initial object of $\mathbf{C}(\tau)$ and $\mathbb{F}$ is a contravariant functor, $\mathbb{F}(\emptyset) = \{*\}$.

The standard example is as follows: given a topological space (U, τ) and any set D , let $\mathbb{F}_{\rightarrow D}(X)$ – for any $X \in \tau$ – be the set of all functions from X to D ; then $\mathbb{F}_{\rightarrow D}$ is a pre-sheaf, where restriction maps are the restrictions of functions in the usual sense. If D is a topological space, then one can take $\mathbb{C}_{\rightarrow D}(X)$ to be a set of all continuous functions. Of course, $\mathbb{C}_{\rightarrow D}$ is a pre-sheaf. The most classical example, as one can guess, is $\mathbb{C}_{\rightarrow \mathbb{R}}$, where $\mathbb{R}$ is a set of real numbers with the standard topology.

2.2 Rough Sets: Basic Concepts

In the present section we briefly recall basic concepts from rough set theory which are relevant to our study [2, 14, 15].

Definition 2.5 (*Information System*) A quadruple $\mathcal{I} = \langle U, Att, Val, f \rangle$ is called an *information system*, where:

- U is a non-empty finite set of objects;
- Att is a non-empty finite set of attributes;
- $Val = \bigcup_{A \in Att} Val_A$, where Val_A is the (non-empty) value-domain of the attribute A ;
- $f : U \times Att \mapsto Val$ is a partial information function, such that for all $A \in Att$ and $a \in U$, when $f(a, A)$ is defined, then $f(a, A) \in Val_A$.

If f is a total function, i.e. $f(a, A)$ is defined for all $a \in U$ and $A \in Att$, then the information system $\mathcal{I}$ is called *complete*; otherwise, it is called *incomplete*. The reader may also consult [16], where the concept of information system was elaborated.

When f is generalised to a function from $U \times Att$ to $\mathcal{P}(Val)$, where $\mathcal{P}(Val)$ is the powerset of Val , then the information system is *nondeterministic*. In what follows we focus our attention on complete and deterministic systems.

If we distinguish in an information system two disjoint classes of attributes C and D , called condition and decision attributes, respectively, then the system will be called a *decision table*.

An information system $\mathcal{I}$ gives rise to an equivalence relation E , called an *indiscernibility relation*, defined as:

$$E = \{(a, b) : \forall A \in Att. \forall X \in Val (f(a, A) = X \Leftrightarrow f(b, A) = X)\}.$$

Customarily, E is often written as $IND(Att)$, the partition induced by the relation $IND(Att)$ is denoted by $U/IND(Att)$, and $[a]_{IND(Att)}$ denotes the equivalence class of $IND(Att)$ defined by $a \in U$. A simple generalisation of $(U, IND(Att))$ is given by the concept of an approximation space:

Definition 2.6 (*Approximation Space*) A pair (U, E) , where U is a non-empty set and E is an equivalence relation on U , is called an *approximation space*. A subset $X \subseteq U$ is called *definable*, if $X = \bigcup \mathcal{Y}$ for some $\mathcal{Y} \subseteq U/E$, where U/E is the family of equivalence classes of E (the quotient set of E).

Definition 2.7 (*Approximation Operators*) Let (U, E) be an approximation space. For every concept $X \subseteq U$, its E -lower and E -upper approximations are defined as follows, respectively:

$$\underline{X} = \{a \in U : [a]_E \subseteq X\};$$

$$\overline{X} = \{a \in U : [a]_E \cap X \neq \emptyset\}.$$

By the usual abuse of language and notation, the operator $\underline{} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ sending X to $\underline{X}$ will be called the *lower approximation operator*, whereas the operator $\overline{} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ sending X to $\overline{X}$ will be called the *upper approximation operator*.

Definition 2.8 (*Increasing Representation of Rough Sets*) For an approximation space (U, E) and $X \subseteq U$, a pair $(\underline{X}, \overline{X})$ is called an *increasing representation* of X .

Definition 2.9 (*Disjoint Representation of Rough Sets*) For an approximation space (U, E) and $X \subseteq U$, a pair $(\underline{X}, U \setminus \overline{X})$ is called a *disjoint representation* of X .

The set $U \setminus \overline{X}$ is often called an *exterior* of X and denoted by $Ext(X)$, whereas $Bnd(X) = \overline{X} \setminus \underline{X}$ is the boundary region of X . Of course, the choice of representation depends on a context of application.

Definition 2.10 (*Approximation Topological Space*) A topological space (U, τ_E) where U/E , the family of all equivalence classes of E , is a subbasis of τ_E and Int is given by

$$Int(X) = \bigcup \{[x]_E \in U/E : x \in U \text{ and } [x]_E \subseteq X\}$$

is called an *approximation topological space*.

Proposition 2.11 Let (U, E) be an approximation space and (U, τ_E) be the corresponding approximation topological space. Then (U, τ_E) is an Alexandrov topological space such that:

$$\underline{X} = Int(X) \text{ and } \overline{X} = Cl(X)$$

for all $X \subseteq U$.

2.3 Near Set Theory: Basic Concepts

Near sets were introduced by J. Peters [5]. The algebraic properties of near sets are described in [17].

Definition 2.12 (*Perceptual System*) A *perceptual system* is a pair $\langle U, \mathbb{F} \rangle$, where U is a non-empty finite set of perceptual objects and $\mathbb{F}$ is a countable set of probe functions $\phi_i : U \rightarrow \mathbb{R}$.

The probe functions describes physical features of objects and usually are regarded as sensors. They also give rise to a number of relations between objects. Let $|\alpha - \beta|$ denote the absolute value of the difference of $\alpha, \beta \in \mathbb{R}$. Then one can formulate:

Definition 2.13 (*Perceptual Indiscernibility Relation*) Let $\langle U, \mathbb{F} \rangle$ be a perceptual system. For every $\mathcal{B} \subseteq \mathbb{F}$, the *perceptual indiscernibility relation* $\sim_{\mathcal{B}}$ is defined as follows:

$$\sim_{\mathcal{B}} = \{(x, y) \in U \times U : \text{for all } \phi_i \in \mathcal{B}, \phi_i(x) - \phi_i(y) = 0\}.$$

Of course, this relation is the counterpart of the original indiscernibility relation given by Pawlak in [14]. This induces perceptual elementary sets of the following form:

$$[x]_{\sim_{\mathcal{B}}} = \{x' \in U \mid x' \sim_{\mathcal{B}} x\}.$$

Now, one can define perceptual approximation operators as in Definition 2.7. However, the theory of near sets is actually focused on weaker relations.

Definition 2.14 (*Perceptual Weak Indiscernibility Relation*) Let $\langle U, \mathbb{F} \rangle$ be a perceptual system. For every $\mathcal{B} \subseteq \mathbb{F}$, the *perceptual weak indiscernibility relation* $\simeq_{\mathcal{B}}$ is defined as follows:

$$\simeq_{\mathcal{B}} = \{(x, y) \in U \times U : \text{for some } \phi_i \in \mathcal{B}, \phi_i(x) - \phi_i(y) = 0\}.$$

Fig. 1 A standard data table, i.e. an information system

Patient	BMI	Obese
1	12	No
2	18	No
3	23	No
4	25	Yes
5	27	Yes
6	39	Yes

Definition 2.15 (*Perceptual Tolerance Relation*) Let $\langle U, \mathbb{F} \rangle$ be a perceptual system and let $\varepsilon \in \mathbb{R}$. For every $\mathcal{B} \subseteq \mathbb{F}$, the *perceptual tolerance relation* $\cong_{\mathcal{B}}$ is defined as follows:

$$\cong_{\mathcal{B}, \varepsilon} = \{(x, y) \in U \times U : \text{for all } \phi_i \in \mathcal{B}, |\phi_i(x) - \phi_i(y)| \leq \varepsilon\}.$$

For notational convenience, this relation is often denoted by $\cong_{\mathcal{B}}$ instead of $\cong_{\mathcal{B}, \varepsilon}$ with the assumption that ε is inherent to the definition of the tolerance relation. Note that the sets of the form $x/\cong_{\mathcal{B}}$ cover U instead of partitioning it.

Definition 2.16 (*Weak Nearness Relation*) Let $\langle U, \mathbb{F} \rangle$ be a perceptual system and let $X, Y \subseteq U$. A set X is said to be *weakly near to* a set Y within the perceptual system $\langle U, \mathbb{F} \rangle$ iff there are $x \in X$ and $y \in Y$ and there is $\mathcal{B} \subseteq \mathbb{F}$ such that $x \sim_{\mathcal{B}} y$. If a perceptual system is understood, then one can say shortly that a set X is *weakly near to* a set Y , written $X \bowtie_{\mathbb{F}} Y$.

Object recognition problems and the problem of the nearness of objects have motivated the introduction of near sets.

Definition 2.17 (*Perceptual Near Set*) Let $\langle U, \mathbb{F} \rangle$ be a perceptual system and let $X \subseteq U$. A set X is a *perceptual near set* iff there is $Y \subseteq U$ such that $X \bowtie_{\mathbb{F}} Y$. The family of near sets of a perceptual system $\langle U, \mathbb{F} \rangle$ is denoted by $\text{Near}_{\mathbb{F}}(U)$.

3 Near Sets: A Unifying Approach

As said in Introduction, near set theory is based on three fundamental concepts: a perceptual system, a nearness relation, a near set. In what follows, we discuss all these concepts against the background of rough set theory. The main reason to make this comparison is that at the first reading fundamental concepts of near set theory may be viewed as a case for Ockham's razor: *entities must not be multiplied beyond necessity*.¹ Especially the two first concepts seem to multiply of what is present in rough set theory, and many readers may see no necessity to do so. However, as we show, a specific representation of data has a natural mathematical context, which, in actual fact, validates the introduction of new definitions.

3.1 Perceptual System

In data analysis the usual representation of gathered pieces of information is a data table, e.g., Fig. 1.

Of course, this table is a simple example of an information system with just two attributes *BMI* and *Obese*, and six objects. One can easily convert it into a perceptual system by replacing *No* by, e.g., 0, and *Yes* by, e.g., 1, as depicted by Fig. 2. Hence, mutatis mutandis, both representations are the same, and one can ask whether we need another representation of the same structure.

¹ In actual fact these words are absent in works of Ockham and probably come from papers of John Punch; however, in this particular form the Ockham razor is often presented on philosophy classes.

Fig. 2 A property system

Patient	BMI	Obese
1	12	0
2	18	0
3	23	0
4	25	1
5	27	1
6	39	1

The answer is that these representations draw our attention into different aspects of a data table and allow us to regard it as different mathematical structures.

A concept of an information system $\mathcal{I} = \langle U, Att, Val, f \rangle$ involves a single information function $f : U \times Att \rightarrow Val$, which for a complete information system is onto. Furthermore, given a system $\mathcal{I}$ one actually works with the partition $U/IND(Att)$, whose blocks are used to compute approximations of sets. From the standpoint mathematics, any function f from U onto V (in category theoretic terms, such f is called an *epimorphism*), induces a partition π whose blocks are $f^{-1}(v)$, for $v \in V$. Thus, one would say that any partition of a set U is actually an epimorphism $f : U \rightarrow V$ in **Set** and vice versa. In particular case of information systems, it suffices to define $f_A(x) \stackrel{df}{=} f(x, A)$, for $x \in U$ and $A \in Att$. Then a partition π_A is determined by $f^{-1}_A(v)$, for $v \in Val_A$. As one can easily see, U/E , or better still, $U/IND(Att)$, is equal to $\bigcap_{A \in Att} \pi_A$. That is what we have at the start while working with rough set theory.

The dual notion to epimorphism is a monomorphism; “dual” means that an epimorphism in a category **C** is a monomorphism in the dual category **C^{op}** and vice versa. Interestingly, similarly like epimorphisms are linked to partitions, monomorphisms in **Set** are linked to relations of set membership $\in$ and set inclusion $\subseteq$. Thus, a partition is actually a dual concept (in the above sense) to a concept of a subset (or an element of a set). The reader familiar with rough set theory can easily see that this duality is close to the main idea of rough sets that the relations of set membership and set inclusion should be viewed (defined) in terms of partitions.

To be more precise, a *subobject* of an object X in category **C** is a monomorphism $m : M \rightarrow X$. So, for a category theorist there is no difference between a subset of a set X and a monomorphism to the set X . In **Set**, every subobject of X is represented by a unique subset of X and hence the duality between epimorphisms and monomorphisms actually bears the duality between partitions and subsets. As usual, $x \in X$ if and only if $\{x\} \subseteq X$, and hence x is a member of X only if $\{x\}$ is a subobject of X . Of course, $\{x\}$ is a terminal object in **Set**, thus we have the unique morphism $\top \rightarrow X$. In general, when X is an object of a category **C** and $\top$ is a terminal object of **C**, then the arrow $\top \rightarrow X$ is called a *point of X*. Every $\top \rightarrow X$ is obviously a monomorphism. So, one can say that an element of an object is a trivial subobject. In category **Set**, every point of X “picks up” exactly one element of X and each element of X is the value of exactly one point. So, the concepts of epimorphism, partition, monomorphism, subset and element of a set are very closely related, and in category theory each of these concepts “fires up” the others.

Given that $x \in X$ only if $\{x\} \subseteq X$, one can generalise this statement and say that x is an element of X if some set which includes x is a subobject of X . In the above case, this set is, of course, $\{x\}$. In rough set theory, $\{x\}$ is replaced by an equivalence class (a block) $[x] = f^{-1}(f(x))$ for some epimorphism f . That is, a monomorphism $\{x\} \rightarrow X$, is replaced by a monomorphism $[x] \rightarrow X$. Formally we can define: $x \in_{rough} X$ if $[x]$ a subobject of X . It is worth to recall in this place a discussion concerning individuals in set theory. Objects from U in data analysis are considered as individuals rather than sets, and it rises a general problem of distinguishing the empty set $\emptyset$ from an individual (object) $x \in U$. Formally, an object x shares with $\emptyset$ the property of having no members (and it is not an easy task to find any other property of $\emptyset$ or x). Frege and Quine suggested, to solve this problem in set theory, to identify an individual x with a set $\{x\}$ (for further discussion of this issue see, e.g., [18]). Not going deeply into this topic, we can however see that the solutions offered by category theory and – to an extent – by rough set theory have also some philosophical motivations. The set of elements of X which belong to X in the sense of $\in_{rough}$ is, in actual fact, the lower approximation of X , i.e., $\underline{X}$. The set of elements which belong to the complement of X

is exactly the exterior of X , i.e., $Ext(X)$. Finally, the elements which belong neither to X nor to $U \setminus X$ form the boundary of X , i.e., $Bnd(X)$. Let us emphasise once again our main point: a specific representation of a data table is embedded into a specific context of this or that theory; in our case it is category theory. Rough sets, which are based on partitions are naturally embedded into the duality between epimorphisms and monomorphisms; or, in other words, the duality between partitions and subsets. When we intuitively paste these two concepts we obtain basic notions of rough set theory: a lower approximation, an exterior, and a boundary. In what follows, we shall see that a different representation offered by near set theory has a different context which is embedded into, and hence allows one to develop a different theory than rough sets.

As already said, a perceptual system $\langle U, \mathbb{F} \rangle$ is a set of objects equipped with a set $\mathbb{F}$ of real valued functions $f : U \rightarrow \mathbb{R}$. The reader has already seen this type of structures in Sect. 2.1. Let us recall that the classic example of a pre-sheaf is $\mathbb{F}_{\rightarrow \mathbb{R}}$, which assigns to every open set X the set of continuous functions $f : X \rightarrow \mathbb{R}$. Thus, one can take the powerset 2^U of U as a discrete topological space (U, τ) and ask how close $\mathbb{F}$ is to $\mathbb{F}_{\rightarrow \mathbb{R}}$. Or, better still, what do we need to make $\mathbb{F}$ a pre-sheaf? Firstly, each function from $\mathbb{F}$ is defined for all points of U ; that is, for every $f \in \mathbb{F}$, we have $f : U \rightarrow \mathbb{R}$. Thus, to every non-empty set $X \subseteq U$, we may assign the standard restrictions of functions (from $\mathbb{F}$) to X : $\mathbb{F}_{\mathbb{F} \rightarrow \mathbb{R}}(X) = \{f|_X : f \in \mathbb{F}\}$. Secondly, it should be noticed that $\emptyset \in 2^U$, and since $\mathbb{F}_{\mathbb{F} \rightarrow \mathbb{R}}$ must be defined for all open sets, we have to add to $\mathbb{F}$ the empty function $\emptyset \rightarrow \mathbb{R}$, and assign $\mathbb{F}_{\mathbb{F} \rightarrow \mathbb{R}}(\emptyset) = \{\emptyset \rightarrow \mathbb{R}\}$. Since $\emptyset$ is the initial object in **Set**, there is exactly one such function. Needless to say, $f|_{\emptyset} = \emptyset \rightarrow \mathbb{R}$, for all $f \in \mathbb{F}$. That's all!

Proposition 3.1 *Let $\langle U, \mathbb{F} \rangle$ be a perceptual system; then $\mathbb{F}_{\mathbb{F} \rightarrow \mathbb{R}} : \mathbf{C}(\tau)^{op} \rightarrow \mathbf{Set}$ defined as above is a pre-sheaf.*

To the initial object $\emptyset$ of $\mathbf{C}(\tau)$, we have assigned a terminal object $\{*\}$ of **Set**, and the restrictions of functions work in the obvious way. Thus, every perceptual system $\langle U, \mathbb{F} \rangle$ can be regarded as a simple pre-sheaf $\mathbb{F}_{\mathbb{F} \rightarrow \mathbb{R}}$ which is a sub pre-sheaf of $\mathbb{F}_{\rightarrow \mathbb{R}}$. It is worth noting that $\langle U, \mathbb{F} \rangle$ “fires up” a pre-sheaf $\mathbb{F}_{\mathbb{F} \rightarrow \mathbb{R}}$ in a very natural way: one does not need to make some advanced study to see it. As partitions are naturally related to concepts of subset and set membership, perceptual systems are linked to pre-sheaves. Actually, it is a sheaf, but, as already said, our aim is to take a down-to-earth approach and keep the things simple; therefore we do not go further into the sheaf direction.

Coming to the end of this section, we want to emphasise once again that the form of perceptual systems bears easy to observe connections to pre-sheaves. In what follows, we shall argue that the structure of a pre-sheaf is indeed universal for near set theory.

3.2 Near Sets and Nearness Relations

When one checks the concept of nearness in mathematical textbooks (e.g., [19]), one can find that, e.g., a nearness space (originally introduced in [20]) is a pair (U, ν) , where U is a set of points (objects) and ν a family of coverings of U (which, of course, must satisfy some additionally properties). When we compare it with a concept of topological space (U, τ) , then τ can be regarded as a single covering. Thus, when dealing with the concept of nearness, we make a shift from a family of sets to a family of families of sets (or, better still, we go to the powerset of the original set). Since a *set* is a collection of some elements, it would be natural (within the above context) to define a *near set* as a collection of sets of elements. It is even more natural, when one checks definitions of representations of rough sets (Sect. 2.2), which are pairs of sets (i.e. sets of sets). However, as one already saw in Sect. 2.3, a near set is defined as a set: a set is near provided that it is near to some other sets. In what follows we regard a near set along the line sketched above, that is, as a collection of sets which are near to it, or better still, as an operator:

$$\mathcal{N}(X) = \{Y \subseteq U : Y \text{ is near to } X\}. \quad (3.1)$$

Let us recall that a set X is said to be (weakly) near to a set Y within the perceptual system $\langle U, \mathbb{F} \rangle$ only if there are objects $x \in X$ and $y \in Y$ and there is $\mathcal{B} \subseteq \mathbb{F}$ such that $x \sim_{\mathcal{B}} y$. The relation $\sim_{\mathcal{B}}$ was defined as follows:

$$\sim_{\mathcal{B}} = \{(x, y) \in U \times U : \text{for all } \phi_i \in \mathcal{B}, \phi_i(x) - \phi_i(y) = 0\}.$$

It is obvious that two sets X and Y are actually near if there are two objects $x \in X$ and $y \in Y$ such that $x \simeq_{\mathbb{F}} y$, where

$$\simeq_{\mathbb{F}} = \{(x, y) \in U \times U : \text{for some } \phi_i \in \mathbb{F}, \phi_i(x) - \phi_i(y) = 0\}.$$

The relation $\simeq_{\mathbb{F}}$ is, of course, reflexive and symmetric; that is, it is a tolerance relation. The standard generalisation of rough set approximations from an equivalence relation E to a tolerance relation T consists in replacing an equivalence class $[x]_E$ by a relational image of x , i.e.,

$$\langle x \rangle_T = \{y \in U : (x, y) \in T\}$$

in their original definitions:

$$low_T(X) = \{x \in U : \langle x \rangle_T \subseteq X\} \text{ and } upp_T(X) = \{x \in U : \langle x \rangle_T \cap X \neq \emptyset\}. \quad (3.2)$$

Now, the operator $\mathcal{N} : 2^U \rightarrow 2^{2^U}$ (see Eq. 3.1) may be formally written as:

$$\mathcal{N}(X) = \{Y \subseteq U : upp_{\simeq_{\mathbb{F}}}(Y) \cap X \neq \emptyset\}.$$

For a moment let us come back to an approximation space (U, E) . Under above notation we get:

$$\underline{X} = low_E(X) = \bigcup \{Y \subseteq U : upp_E(Y) \subseteq X\}.$$

If we regard $low_E(X)$ as a family of sets and drop out the sign $\bigcup$, then we will define an operator $\mathcal{L} : 2^U \rightarrow 2^{2^U}$:

$$\mathcal{L}(X) = \{Y \subseteq U : upp_E(Y) \subseteq X\},$$

$$\mathcal{N}(X) = \{Y \subseteq U : upp_E(Y) \cap X \neq \emptyset\}.$$

Below $\mathcal{L}$ we have written $\mathcal{N}$ generated by the approximation space (U, E) ; as the reader may see these operators resemble definitions of lower and upper approximations, regarded as families of sets (see Definition 2.7 and Eq. 3.2). Hence, we could regard a near set (or a near set operator) as an upper approximation (or an upper approximation operator) shifted to the higher level of granulation.

As in the (previous) Sect. 3.1, one can regard 2^U as a discrete topological space (U, τ) and ask whether both operators $\mathcal{L}$ and $\mathcal{N}$ may be converted into functors:

$$\mathcal{L} : \mathbf{C}(\tau)^{op} \rightarrow \mathbf{Set} \text{ and } \mathcal{N} : \mathbf{C}(\tau)^{op} \rightarrow \mathbf{Set}.$$

Firstly, let us consider $\mathcal{L}$. To the empty set $\emptyset$ (the initial object of $\mathbf{C}(\tau)$) the operator $\mathcal{L}$ assigns $\{\emptyset\}$, that is $\{*\}$, which is a terminal object of $\mathbf{Set}$ as required. So, what is left to define are restriction functions $\rho_X^Y : \mathcal{L}(Y) \rightarrow \mathcal{L}(X)$, for $X \subseteq Y$. Since $\mathcal{L}$ is monotonic, that is, $\mathcal{L}(X) \subseteq \mathcal{L}(Y)$ for $X \subseteq Y$, all elements of $\mathcal{L}(X)$ belong to $\mathcal{L}(Y)$, so to obtain $\mathcal{L}(X)$ it suffices to “pick them up” from $\mathcal{L}(Y)$:

$$\rho_X^Y(W) = \begin{cases} W & \text{if } upp_E(W) \subseteq X, \\ \emptyset & \text{otherwise.} \end{cases}$$

Under this definition it is obvious that the following condition holds:

$$X \subseteq Y \subseteq Z \text{ then } \rho_X^Z = \rho_X^Y \circ \rho_Y^Z.$$

Proposition 3.2 *Let (U, E) be an approximation space; then $\mathcal{L} : \mathbf{C}(\tau)^{op} \rightarrow \mathbf{Set}$ defined as above is a pre-sheaf.*

The operator $\mathcal{N}$ fails to be a pre-sheaf at the very beginning: $\mathcal{N}(\emptyset) = \emptyset$; $\mathcal{N}$ assigns to the initial object of $\mathbf{C}(\tau)$ the initial object of $\mathbf{Set}$. In other words, using the nomenclature from Sect. 2.3, the empty set $\emptyset$ is not a near set (it is far from every set, including itself). Let us recall that in general topology there is defined a nearness relation $\delta \subseteq U \times 2^U$ between a point x and a set X , a subject to the following conditions:

- $(x, \emptyset) \notin \delta$,
- if $x \in Y$, then $(x, Y) \in \delta$,
- $(x, Y \cup Z) \in \delta$ iff $(x, Y) \in \delta$ or $(x, Z) \in \delta$,
- for all $z \in Z$, if $(x, Z) \in \delta$ and $(z, Y) \in \delta$, then $(x, Y) \in \delta$.

The standard reading of nearness x to Y is that x is at *distance zero* from Y . A topological nearness relation θ defines a closure operator in a very natural way:

$$\theta(X) = \{x \in U : (x, X) \in \theta\}.$$

Thus, a closed set is a set for which the nearness relation reduces to the containment relation. Of course, $Cl_\theta(\emptyset) = \emptyset$ and hence the empty set $\emptyset$ is trivially closed under nearness relation θ . Furthermore, all points near to $\emptyset$, i.e. $Cl_\theta(\emptyset)$, are included in any set X (are near to X) – thus, it seems natural to regard $\emptyset$ as a set near to any set X . However, every non empty set X should be far from the empty set $\emptyset$ since every point $x \in X$ is far from $\emptyset$ (according to the first condition put upon δ). Following these intuitions let us redefine the operator $\mathcal{N}$:

$$\mathcal{N}_\emptyset(X) = \{Y \subseteq U : upp_E(Y) \cap X \neq \emptyset\} \cup \{\emptyset\}.$$

This time we obtain $\mathcal{N}_\emptyset(\emptyset) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{*\}$, as required. In other words, the only set near to the empty set $\emptyset$ is the empty set itself. Trivially, for all $X \subseteq U$, $\emptyset \in \mathcal{N}_\emptyset(X)$, which means that the empty set $\emptyset$ is near to every set (as suggested by the topological nearness δ). The operator $\mathcal{N}_\emptyset$ is monotonic as the operator $\mathcal{L}$, therefore we can apply the same method of defining the restriction functions:

$$\rho_X^Y(W) = \begin{cases} W & \text{if } upp_E(W) \cap X \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proposition 3.3 *Let (U, E) be an approximation space; then $\mathcal{N}_\emptyset : \mathbf{C}(\tau)^{op} \rightarrow \mathbf{Set}$ defined as above is a pre-sheaf.*

As one can notice, $\mathcal{L}(X) \subseteq \mathcal{N}_\emptyset(X)$, for all $X \subseteq U$; however, these two pre-sheaves have different restriction functions and therefore $\mathcal{L}$ is not a sub-pre-sheaf of $\mathcal{N}_\emptyset$. Nevertheless, if $\mathcal{N}_\emptyset$ assigns to X sets which are near to X , then $\mathcal{L}$ assigns to X sets which are intuitively even nearer (one can say that $\mathcal{L}$ is more conservative than $\mathcal{N}_\emptyset$, or that it is a safer pre-sheaf).

Both pre-sheaves $\mathcal{L}$ and $\mathcal{N}_\emptyset$ are parameterised by upp_E , which, as one can guess, can be generalised to upp_T for any reflexive relation T . Let us regard such a relation T as a nearness relation between points (objects) of U . Thus, $(x, y) \in T$ is read as “ x is near to y ”, and $(x, y) \notin T$ is read as “ x is far from y ”. Then a T -closure $Cl(X)$ of X may be defined in two intuitive ways:

1. Closure can be defined with respect to nearness:

$$Cl_{near}(X) = \{y : y \text{ is near to some } x \in X\}.$$

2. We can define closure with respect to farness: let us call Y closed iff for all $x \in U \setminus Y$, it holds that x is far from every member of Y . We denote the family of closed set of this form by $\mathcal{C}_{far}$ and define:

$$Cl_{far}(X) = \bigcap \{Y : X \subseteq Y \text{ and } Y \in \mathcal{C}_{far}\}.$$

Both operators are interestingly related to each other.

Proposition 3.4 *For a reflexive relation $T \subseteq U \times U$, Cl_{far} is a Kuratowski closure; that is, it satisfies the following conditions:*

- *extensivity:* $X \subseteq Cl_{far}(X)$,
- *$\cup$ -distributivity:* $Cl_{far}(X \cup Y) = Cl_{far}(X) \cup Cl_{far}(Y)$,
- *$\emptyset$ -preservation:* $Cl_{far}(\emptyset) = \emptyset$,
- *idempotency:* $Cl_{far}(Cl_{far}(X)) = Cl_{far}(X)$,

for all $X, Y \subseteq U$. Cl_{near} satisfies the first three conditions, but it lacks idempotency.

The case of Cl_{near} is quite intuitive. So, let us see how Cl_{far} works. Given a set X , we may check whether X is a closed set (with respect to farness). If $U \setminus X$ contains a point y which is near to some $x \in X$, then X is not closed. So, we must add to X all points of $U \setminus X$ near to some elements of X . Doing so we in actual fact

convert X into $Cl_{near}(X)$. Now, we can check whether $Cl_{near}(X)$ is closed (with respect to Cl_{far}). So, once again we check whether $U \setminus Cl_{near}(X)$ includes elements near to $Cl_{near}(X)$. If so, we repeat the procedure and obtain $Cl_{near}(Cl_{near}(X))$. Doing these steps, we eventually obtain a set Y whose complement has only points far from Y ; that is, we obtain a set Y such that $Cl_{far}(Y) = Y$. But the procedure consists in applying Cl_{near} operator and therefore for any Y such that $Cl_{far}(Y) = Y$ it must hold that $Cl_{near}(Y) = Y$. It also follows that:

$$Cl_{near}(X) \subseteq Cl_{near}(Cl_{near}(X)) \text{ and } Cl_{near}(X) \subseteq Cl_{far}(X). \quad (3.3)$$

In topology a set X is called closed if $Cl(X) = X$. Let us denote the family of Cl_{near} closed sets by $\mathcal{C}_{near}$. From the above considerations it follows that:

Proposition 3.5 *For a reflexive relation $T \subseteq U \times U$, it holds that $\mathcal{C}_{far} = \mathcal{C}_{near}$. For a preorder relation T , it holds that $Cl_{near}(X) = Cl_{far}(X)$, for all $X \subseteq U$.*

Proposition 3.6 *Let $T \subseteq U \times U$ be a tolerance relation and $E \subseteq U \times U$ the minimal equivalence relation such that $T \subseteq E$. Let also Cl_{near} and Cl_{far} be defined with respect to T . Then:*

$$upp_T(X) = Cl_{near}(X) \text{ and } upp_E = Cl_{far}(X)$$

for all $X \subseteq U$.

As already said, both pre-shaves $\mathcal{L}$ and $\mathcal{N}_\emptyset$ are parameterised by upp_E . Let us denote by $\mathcal{L}_{Cl_{near}}$ a pre-sheaf in whose definition we use Cl_{near} instead of upp_E . In similar way, we define also $\mathcal{L}_{Cl_{near}(Cl_{near})}$, $\mathcal{L}_{Cl_{far}}$, and $\mathcal{N}_\emptyset$ -versions of these operators. Thus, given a reflexive relation T one can actually define a number of different pre-sheaves. It is worth noticing that due to Eq. 3.3, it holds that:

$$\mathcal{L}_{Cl_{far}}(X) \subseteq \mathcal{L}_{Cl_{near}}(X) \subseteq \mathcal{N}_{\emptyset_{Cl_{near}}}(X) \subseteq \mathcal{N}_{\emptyset_{Cl_{far}}}(X), \text{ for all } X \subseteq U.$$

Let us recall that $\mathcal{N}_\emptyset$ was intuitively interpreted as a pre-sheaf which to a given set assigns sets near to it. We also observed that $\mathcal{L}$ would be interpreted as a more conservative pre-sheaf. Along this line we may also say that $\mathcal{L}_{Cl_{far}}(X)$ is more exact (conservative or safer) than $\mathcal{L}_{Cl_{near}}(X)$. Thus, given a reflexive relation T we can form a number of pre-sheaves of different exactness. As written earlier, our suggestion is to call – by abuse of language – pre-sheaves obtainable from a relation T , T -near sets. Then the T -lower approximation operator regarded as a family of sets would be an example of a T -near set. In this way, near set theory would be viewed as a pre-sheafification of rough set theory, and, due to diversity of T -near sets, a much richer theory. It is worth emphasising that actually the only important difference between the original theory of near sets and its pre-sheafified version is how the empty set $\emptyset$ is counted. Originally, $\emptyset$ was far from every set; here it is close to every set, but every non-empty set is far from it. What we gain is a theory which has a strictly defined theoretical basis (contravariant functors obtainable from reflexive relations), methodological assumptions (relations are interpreted as nearness and collect sets into families called T -near sets) and clear connections to rough set theory (T -near sets generalise rough set approximations onto a higher level of granulation).

Coming to the end of this section let us observe an important difference between $\mathcal{L}$ and $\mathcal{N}_\emptyset$:

$$\bigcup \mathcal{N}_\emptyset(X) = U, \text{ for all } X \subseteq U.$$

That is, $\mathcal{N}_\emptyset(X)$ – in contrast to $\mathcal{L}(X)$ – may be regarded as a cover of U , and hence by providing every $X \subseteq U$ with $\mathcal{N}_\emptyset(X)$, we can define a family, say, ν of coverings of U . Hence, as in the case of nearness spaces or merotopic spaces [19,20], we deal with a family of coverings. Of course, the additional requirements put upon nearness or merotopic spaces are not satisfied by ν . However, if we regard ν merely as a family of some sub-covers of $\mathcal{N}_\emptyset(U)$, then, of course, a nearness space or a merotopic space would be an example of (U, ν) . Perceptual nearness space is not a merotopic space. However, it is a family of covers μ equipped with restriction functions in such a way that if there exists $f : \mathcal{N}_\emptyset(X) \rightarrow \mathcal{N}_\emptyset(Y)$ then $\mathcal{N}_\emptyset(Y)$ is a sub-cover of $\mathcal{N}_\emptyset(X)$. So, more precisely we can define perceptual nearness space as (U, ν, f) , where f is a set of restriction functions which is closed under composition. Since every sub-cover of a given cover is its refinement, we obtain that merotopic spaces and

4 Conclusions

In the paper we have described near set theory in terms of pre-sheaves. As noted above, a perceptual system is a simple example of a pre-sheaf. Interestingly, a family of near sets to a given set X also forms a pre-sheaf and bears structural resemblance to an upper approximation operator. Since a lower approximation operator may be converted into a pre-sheaf as well, a near set (understood as a pre-sheaf) becomes a pre-sheafification of an upper approximation operator. More generally, near set theory may be regarded as a pre-sheafification of rough set theory. In this way, near set theory becomes an elegant extension of rough set theory onto the higher level of granulation; in other words, near set theory “moves” rough set theory from the level of a set of granules to the level of a family of sets of granules. What is important, everything can be done is basic topology and category theory, that is, a theory is not unnecessarily complicated as it is often the case in “soft mathematics” [4].

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Zimmermann, H.: Fuzzy set theory and its applications. Springer, Verlag (2001)
2. Pawlak, Z.: Rough sets: theoretical aspects of reasoning about data. Kluwer Dordrecht, The Netherlands (1991)
3. Wu, S., Lin, L., Gao, Q.: A new grey-rough set model based on interval-valued grey sets. In: Proceeding SMC'09 Proceedings of the 2009 IEEE international conference on systems, Man and Cybernetics (2009)
4. Garcia, J., Rodabaugh, S.: Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued “intuitionistic” sets, “intuitionistic” fuzzy sets and topologies. *Fuzzy Set. Syst.* **156**(3), 445–484 (2005)
5. Peters, J.: Near sets. Special theory about nearness of objects. *Fundamenta Informaticae* **75**(1–4), 407–433 (2007)
6. Ganter, B., Wille, R.: Formal concept analysis. Springer, Verlag (1999)
7. Pratt, V.: Chu spaces from the representational viewpoint. *Ann. Pure Appl. Logic* **96**, 319–333 (1999)
8. Peters, J., Tiwari, S.: Approach merotopies and near filters. *Theory Appl. Gen. Math. Notes* **3**(1), 1–15 (2011)
9. Peters, J., Naimpally, S.: Approach spaces for near families. *Gen. Math. Notes* **2**(1), 159–164 (2011)
10. Rota, G.C.: The phenomenology of mathematical beauty. *Synthese* **111**(2), 171–182 (1997)
11. Blute, R., Scott, P.: Category theory for linear logicians. In: Ehrhard, T., Jirard, J., Ruet, P., Scott, P. (eds) *Linear Logic in Computer Science*, pp. 3–64. Cambridge University Press, Cambridge (2005)
12. Dugundji, J.: Topology. Allyn and Bacon (1996)
13. Lane, S.M., Moerdijk, I.: Sheaves in Geometry and Logic: a first introduction to Topos theory. Springer, Verlag (1992)
14. Pawlak, Z.: Information systems. Theoretical foundations. *Inform. Syst.* **6**(3), 205–218 (1981)
15. Pawlak, Z.: Rough sets. *Int. J. Comp. Inform. Sci.* **11**, 341–356 (1981)
16. Marek, W., Pawlak, Z.: Information storage and retrieval systems—mathematical foundations. *Theor. Comput. Sci.* **1**(4), 331–354 (1976)
17. Peters, J.F., Wasilewski, P.: Foundations of near sets. *Inform. Sci.* **179**(18), 3091–3109 (2009) <http://dx.doi.org/10.1016/j.ins.2009.04.018>
18. Potter, M.: Set theory and its philosophy. Oxford University Press, USA (2004)
19. Koslowski, J., Melton, A.: Categorical perspectives. Birkhauser Boston (2001)
20. Herrlich, H.: A concept of nearness. *Fuzzy Set. Syst.* **5**, 191–212 (1974)