



The Noncommutative Fractional Fourier Law in Bounded and Unbounded Domains

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Abstract

Using the spectral theory on the S -spectrum it is possible to define the fractional powers of a large class of vector operators. This possibility leads to new fractional diffusion and evolution problems that are of particular interest for nonhomogeneous materials where the Fourier law is not simply the negative gradient operator but it is a nonconstant coefficients differential operator of the form

$$T = \sum_{\ell=1}^3 e_{\ell} a_{\ell}(x) \partial_{x_{\ell}}, \quad x = (x_1, x_2, x_3) \in \overline{\Omega},$$

where, Ω can be either a bounded or an unbounded domain in \mathbb{R}^3 whose boundary $\partial\Omega$ is considered suitably regular, $\overline{\Omega}$ is the closure of Ω and e_{ℓ} , for $\ell = 1, 2, 3$ are the imaginary units of the quaternions \mathbb{H} . The operators $T_{\ell} := a_{\ell}(x) \partial_{x_{\ell}}$, for $\ell = 1, 2, 3$, are called the components of T and $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ are the coefficients of T . In this paper we study the generation of the fractional powers of T , denoted by $P_{\alpha}(T)$ for $\alpha \in (0, 1)$, when the operators T_{ℓ} , for $\ell = 1, 2, 3$ do not commute among themselves. To define the fractional powers $P_{\alpha}(T)$ of T we have to consider the weak formulation of a suitable boundary value problem associated with the pseudo S -resolvent operator of T . In this paper we consider two different boundary conditions. If Ω is unbounded we consider Dirichlet boundary conditions. If Ω is bounded we consider the natural Robin-type boundary conditions associated with the generation of

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the fractional powers of T represented by the operator $\sum_{\ell=1}^3 a_{\ell}^2(x) n_{\ell}(x) \partial_{x_{\ell}} + a(x)I$, for $x \in \partial\Omega$, where I is the identity operator, $a : \partial\Omega \rightarrow \mathbb{R}$ is a given function and $n = (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$. The Robin-type boundary conditions associated with the generation of the fractional powers of T are, in general, different from the Robin boundary conditions associated to the heat diffusion problem which leads to operators of the type $\sum_{\ell=1}^3 a_{\ell}(x) n_{\ell}(x) \partial_{x_{\ell}} + b(x)I$, $x \in \partial\Omega$. For this reason we also discuss the conditions on the coefficients $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ of T and on the coefficient $b : \partial\Omega \rightarrow \mathbb{R}$ so that the fractional powers of T are compatible with the physical Robin boundary conditions for the heat equations.

Keywords Fractional powers of vector operators · S -spectrum · S -spectrum approach · Fractional diffusion processes · Robin boundary conditions

Mathematics Subject Classification 47A10 · 47A60

1 Introduction

Fractional diffusion and fractional evolution equations take into account nonlocal phenomena giving a better description of the physical reality with respect to differential laws. The most successful variation of the heat equation that takes into account nonlocal effects is the fractional heat equation where the Laplace operator is replaced by the fractional Laplacian. There are several ways to define fractional powers of operators which are, in general, not equivalent. Using the spectral theory on the S -spectrum, see [3,16,17,22], a new class of fractional diffusion and evolution problems can be considered. In particular the S -spectrum approach to fractional diffusion problems has been considered in [16] where the fractional powers of quaternionic operators are systematically treated.

Using these new techniques based on the S -spectrum we can generate the fractional Fourier laws starting from the differential Fourier law and the associated boundary conditions. *This method has the advantage to modify only the Fourier law without changing the conservation of energy laws in the fractional heat equation for nonhomogeneous materials.* To recall this method and its advantages we need some notation. An element in the quaternions \mathbb{H} is of the form $s = s_0 + s_1 e_1 + s_2 e_2 + s_3 e_3$, where s_0, s_{ℓ} are real numbers ($\ell = 1, 2, 3$), $\text{Re}(s) := s_0$ denotes the real part of s and e_{ℓ} , for $\ell = 1, 2, 3$, are the imaginary units which satisfy the relations: $e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1$. The modulus of s is defined as $|s| = (s_0^2 + s_1^2 + s_2^2 + s_3^2)^{1/2}$ and the conjugate is given by $\bar{s} = s_0 - s_1 e_1 - s_2 e_2 - s_3 e_3$. In the sequel we will denote by \mathbb{S} the unit sphere of purely imaginary quaternions, an element j in \mathbb{S} is such that $j^2 = -1$.

With our approach Ω can be either a bounded or an unbounded domain in \mathbb{R}^3 whose boundary $\partial\Omega$ is sufficiently regular, $\overline{\Omega}$ denotes the closure of Ω . We consider vector operators of the form

$$T = \sum_{\ell=1}^3 e_{\ell} T_{\ell}, \tag{1.1}$$

where the components T_{ℓ} of T , $\ell = 1, 2, 3$, are defined by $T_{\ell} := a_{\ell}(x)\partial_{x_{\ell}}$, $x \in \overline{\Omega}$, and we suppose that the coefficient $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ of T are not necessarily nonconstant. From the physical point of view the operator T , defined in (1.1), can represent the Fourier law for nonhomogeneous materials, but it can represent also different physical laws. In general the operator T models the way the flux varies in a linear isotropic or anisotropic diffusion problem. Our goal is to generate the fractional powers of T when the operators T_{ℓ} , for $\ell = 1, 2, 3$ do not commute among themselves. The vector part of the fractional powers $P_{\alpha}(T)$, for $\alpha \in (0, 1)$, of T is called the fractional Fourier law associated with T .

It is important to observe that using the spectral theory on the S -spectrum to define the fractional powers of a vector operator T , one has to specify the boundary conditions associated with the operator T . When T is the Fourier law for the heat diffusion problems with the homogeneous Dirichlet boundary condition, there are no further boundary conditions that are necessary to generate the fractional powers of T . In the case Ω is bounded we studied this problem in the papers [14,18,19]. In this paper we consider the case in which Ω is unbounded.

In the paper [20] we have studied the fractional powers of T with Robin-type boundary conditions, where Ω is bounded, and the components T_{ℓ} , of T , for $\ell = 1, 2, 3$ commute among themselves. It turns out that the Robin-type boundary conditions necessary to generate the fractional powers of T and the classical Robin boundary conditions of the heat equation are different. In this paper we study the generation of the fractional powers when the components T_{ℓ} , of T , for $\ell = 1, 2, 3$ do not commute among themselves and the relation between the two type Robin boundary conditions.

In order to set the problem we need some results of the spectral theory on the S -spectrum. We will work in an Hilbert space but our techniques allow to define the fractional powers of operators in quaternionic Banach spaces.

2 Problems and main results on the fractional powers of vector operators

We consider a two-sided quaternionic Banach space V and we denote the set of closed quaternionic right linear operators on V by $\mathcal{K}(V)$. The Banach space of all bounded right linear operators on V is indicated by the symbol $\mathcal{B}(V)$ and is endowed with the natural operator norm. For $T \in \mathcal{K}(V)$, we define the operator associated with the S -spectrum as:

$$Q_s(T) := T^2 - 2Re(s)T + |s|^2\mathcal{I}, \quad \text{for } s \in \mathbb{H} \tag{2.1}$$

where $Q_s(T) : \mathcal{D}(T^2) \rightarrow V$, where $\mathcal{D}(T^2)$ is the domain of T^2 . We define the S -resolvent set of T as

$$\rho_S(T) := \{s \in \mathbb{H} : Q_s(T) \text{ is invertible and } Q_s(T)^{-1} \in \mathcal{B}(V)\}$$

and the S -spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

The operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo S -resolvent operator. For $s \in \rho_S(T)$, the left S -resolvent operator is defined as

$$S_L^{-1}(s, T) := \mathcal{Q}_s(T)^{-1}\bar{s} - T\mathcal{Q}_s(T)^{-1} \tag{2.2}$$

and the right S -resolvent operator is given by

$$S_R^{-1}(s, T) := -(T - T\bar{s})\mathcal{Q}_s(T)^{-1}. \tag{2.3}$$

The fractional powers of T , denoted by $P_\alpha(T)$, are defined as follows: for any $j \in \mathbb{S}$, for $\alpha \in (0, 1)$ and $v \in \mathcal{D}(T)$ we set

$$P_\alpha(T)v := \frac{1}{2\pi} \int_{-j\mathbb{R}} S_L^{-1}(s, T) ds_j s^{\alpha-1}Tv, \tag{2.4}$$

or

$$P_\alpha(T)v := \frac{1}{2\pi} \int_{-j\mathbb{R}} s^{\alpha-1} ds_j S_R^{-1}(s, T)Tv, \tag{2.5}$$

where $ds_j = ds/j$. These formulas are a consequence of the quaternionic version of the H^∞ -functional calculus based on the S -spectrum, see the book [16] for more details. For the generation of the fractional powers $P_\alpha(T)$ a crucial assumption on the S -resolvent operators is that, for $s \in \mathbb{H} \setminus \{0\}$ with $\text{Re}(s) = 0$, the estimates

$$\|S_L^{-1}(s, T)\|_{\mathcal{B}(V)} \leq \frac{\Theta}{|s|} \quad \text{and} \quad \|S_R^{-1}(s, T)\|_{\mathcal{B}(V)} \leq \frac{\Theta}{|s|}, \tag{2.6}$$

hold with a constant $\Theta > 0$ that does not depend on the quaternion s . It is important to observe that the conditions (2.6) assure that the integrals (2.4) and (2.5) are convergent and so the fractional powers are well defined.

For the definition of the fractional powers of the operator T we can use equivalently the integral representation in (2.4) or the one in (2.5). Moreover, they correspond to a modified version of Balakrishnan’s formula that takes only spectral points with positive real part into account.

A crucial problem is to determine the conditions on the coefficients $a_1, a_2, a_3 : \bar{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$, of the operator T defined in (1.1), such that the purely imaginary quaternions are in the S -resolvent set $\rho_S(T)$. This is a necessary condition, see formulas (2.4) and (2.5), since in the quaternionic case the map $s \mapsto s^\alpha$, for $\alpha \in (0, 1)$ is not defined for $s \in (-\infty, 0)$ and, unlike in the complex setting, it is not possible to choose different branches of s^α in order to avoid this problem. For this reason it is of great importance to assume the condition $\text{Re}(s) \geq 0$ that avoids the half real line $(-\infty, 0]$.

Regarding the boundary conditions of Robin-type, we will study the following problem associated with the fractional powers of the operator T .

Problem 2.1 (*Existence of the fractional powers with Robin-like boundary conditions*). Let Ω be a bounded domain. Let T be the vector operators defined in (1.1) where the coefficients $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ are suitable regular functions. Let $F : \Omega \rightarrow \mathbb{H}$ be a given function and denote by $u : \Omega \rightarrow \mathbb{H}$ the unknown function satisfying the boundary value problem:

$$\begin{cases} (T^2 - 2s_0T + |s|^2\mathcal{I})u(x) = F(x), & x \in \Omega, \\ \sum_{\ell=1}^3 a_\ell^2(x)n_\ell(x)\partial_{x_\ell}u(x) + a(x)u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

where $a : \partial\Omega \rightarrow \mathbb{R}$ is a given function and $n = (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$.

- (I) Determine the conditions on the coefficients $a : \partial\Omega \rightarrow \mathbb{R}, a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the boundary value problem (2.7) has a unique solution in a suitable function space when $Re(s) = 0$.
- (II) Under the conditions in (I) determine the conditions on the coefficients such that the S -resolvent operators satisfy the estimates (2.6).
- (III) Given the stationary heat equation for nonhomogeneous materials with Robin boundary conditions:

$$\begin{cases} \operatorname{div} T(x)v(x) = 0, & x \in \Omega, \\ b(x)v(x) + \sum_{\ell=1}^3 a_\ell(x)n_\ell(x)\partial_{x_\ell}v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.8}$$

where $v : \Omega \rightarrow \mathbb{R}, n = (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$ and $b : \partial\Omega \rightarrow \mathbb{R}$ is a given continuous function, determine the conditions on the coefficients $a, b : \partial\Omega \rightarrow \mathbb{R}, a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the boundary condition in (2.8) implies the boundary condition in (2.7) (see Remark 2.2).

Remark 2.2 The operator

$$\sum_{\ell=1}^3 a_\ell^2(x)n_\ell(x)\partial_{x_\ell}$$

is associated with the boundary condition of problem (2.7) that naturally arises in the definition of the bilinear form associated with the existence of the pseudo S -resolvent operator as a bounded linear operator, while the operator

$$n \cdot T(x) = \sum_{\ell=1}^3 a_\ell(x)n_\ell(x)\partial_{x_\ell}$$

is associated with the boundary condition of the problem (2.8) that naturally arises as a physical flux condition.

Remark 2.3 In the paper [20] we have investigated some possible solutions of the boundary value problem (2.7), in different Hilbert spaces, depending on the spectral

parameter $s \in \mathbb{H}$ where the operator $T = \sum_{\ell=1}^3 e_\ell T_\ell$, defined in (1.1), has commuting components T_ℓ , for $\ell = 1, 2, 3$. Such analysis can be done also when the components T_ℓ , for $\ell = 1, 2, 3$ do not commute. In this paper we focus our attention on the spectral problem where $s \in \mathbb{H}$ and $\text{Re}(s) = 0$ because this is the case of interest for the definitions (2.4) with (2.5) so that we can generate the fractional powers of T .

Regarding the boundary Dirichlet conditions for the unbounded domains, we will study the following problem associated with the fractional powers of the operator T .

Problem 2.4 (*Existence of the fractional powers with Dirichlet boundary conditions for unbounded domains*) Let Ω be an unbounded domain. Let T be the vector operator defined in (1.1) where the coefficients $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ are suitable regular functions. Let $F : \Omega \rightarrow \mathbb{H}$ be a given function and denote by $u : \Omega \rightarrow \mathbb{H}$ the unknown function satisfying the boundary value problem:

$$\begin{cases} (T^2 - 2s_0T + |s|^2\mathcal{I})u(x) = F(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.9}$$

- (I) Determine the conditions on the coefficients $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the boundary value problem (2.9) has a unique solution in a suitable function space when $\text{Re}(s) = 0$.
- (II) Under the conditions in (I) determine the conditions on the coefficients such that the S -resolvent operators satisfy the estimates (2.6).

2.1 Summary of the main results of the paper

In Sect. 3 we give the weak formulation of Problems 2.1 and 2.4. In Sect. 4 we prove, under the condition $a \in C^0(\partial\Omega, \mathbb{R})$ and on the coefficients $a_1, a_2, a_3 \in C^1(\overline{\Omega}, \mathbb{R})$ of the operator T defined in (1.1), the existence and the uniqueness of the weak solutions of the problems and suitable estimates on the pseudo S -resolvent operators. Precisely we summarize the results in the following points.

(A) The existence and uniqueness of the weak solution of Problem 2.1 is stated in Theorem 4.4 where we define the constants

$$C_T := \min_{\ell=1,2,3} \inf_{x \in \Omega} (a_\ell^2(x)), \quad C'_T := \sum_{i,\ell=1}^3 \|a_\ell \partial_{x_i} a_i\|_\infty, \quad K_{a,\Omega} := C_{\partial\Omega}^2 \|a\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the sup norm, and we assume

$$C_T - C'_T C_P - K_{a,\Omega} (1 + C_P^2) > 0 \quad \text{and} \quad C_T > 0,$$

where C_P is the Poincaré-Wirtinger constant and $C_{\partial\Omega}$ are a given constant that depends on $\partial\Omega$. Under the above conditions the boundary value Problem (2.7) has a unique weak solution $u \in \mathcal{H}(\Omega, \mathbb{H}) := \{u \in H^1(\Omega, \mathbb{H}) : \int_\Omega u(x) dx = 0\}$, for $s \in \mathbb{H} \setminus \{0\}$ with $\text{Re}(s) = 0$.

(B) In the case we work on unbounded domains the weak solution to Problems 2.4 is stated in Theorem 4.9, i.e., the boundary value Problem (2.9) has a unique weak solution $u \in H_0^1(\Omega, \mathbb{H})$, for $s \in \mathbb{H} \setminus \{0\}$ with $Re(s) = 0$ when we assume that $a_1, a_2, a_3 \in C^1(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega)$

$$M := \sum_{i,j=1}^3 \|a_i \partial_{x_i}(a_j)\|_{L^3(\Omega)} < +\infty, \quad C_T - 4M > 0, \quad C_T > 0.$$

Observe that the condition $M < +\infty$, in the case of the unbounded domains, is necessary to get the estimate (4.25) through the Sobolev-Gagliardo-Nirenberg inequality.

(C) In both cases (A) and (B) we proved the following estimates

$$\|u\|_{L^2}^2 \leq \frac{1}{s^2} Re(b_s(u, u)), \quad \|T(u)\|_{L^2}^2 \leq c Re(b_s(u, u)),$$

where $c > 0$ is a given constant, $b_s(u, u)$ is the bilinear form associated with the weak formulation of the problems and the estimates hold for all $s \in \mathbb{H} \setminus \{0\}$ with $Re(s) = 0$.

(D) In Sect. 5, based on the estimates in point (C), we prove the estimates (2.6) for the \mathcal{S} -resolvent operators and we define the fractional powers of T using formula (2.4) or equivalently using (2.5).

(E) Finally, consider the point (III) of the Problem 2.1. Suppose that there exists a constant μ such that the functions a_1, a_2, a_3 satisfy the conditions

$$a_1(x) = a_2(x) = a_3(x) = \mu \text{ for all } x \in \partial\Omega \tag{2.10}$$

and the coefficients a and b are such that

$$a(x) = \mu b(x) \text{ for all } x \in \partial\Omega. \tag{2.11}$$

Then the relation $\sum_{\ell=1}^3 a_\ell(x) n_\ell(x) \partial_{x_\ell} + b(x)I = 0$, implies $\sum_{\ell=1}^3 a_\ell^2(x) n_\ell(x) \partial_{x_\ell} + a(x)I = 0$, for $x \in \partial\Omega$. Observe that, using (2.10) and (2.11), for $x \in \partial\Omega$, we have

$$\begin{aligned} \sum_{\ell=1}^3 a_\ell^2(x) n_\ell(x) \partial_{x_\ell} + a(x)I &= \mu^2 \sum_{\ell=1}^3 n_\ell(x) \partial_{x_\ell} \\ \partial_{x_\ell} + \mu b(x)I &= \mu \left(\sum_{\ell=1}^3 a_\ell(x) n_\ell(x) \partial_{x_\ell} + b(x)I \right). \end{aligned} \tag{2.12}$$

3 The weak formulation of the Problems 2.1 and 2.4

In the following Ω can be either a bounded or an unbounded domain of \mathbb{R}^3 according to the problem that we will consider. The boundary $\partial\Omega$ of Ω is assumed to be of class

C^1 even though for some lemmas in the sequel the conditions on the open set Ω can be weakened. We define

$$L^p := L^p(\Omega, \mathbb{H}) := \left\{ u : \Omega \rightarrow \mathbb{H} : \int_{\Omega} |u(x)|^p dx < +\infty \right\}.$$

The space L^2 with the scalar product:

$$\langle u, v \rangle_{L^2} := \langle u, v \rangle_{L^2(\Omega, \mathbb{H})} := \int_{\Omega} \overline{u(x)}v(x) dx,$$

where $u(x) = u_0(x) + u_1(x)e_1 + u_2(x)e_2 + u_3(x)e_3$ and $v(x) = v_0(x) + v_1(x)e_1 + v_2(x)e_2 + v_3(x)e_3$ for $x = (x_1, x_2, x_3) \in \Omega$ is a Hilbert space. We furthermore introduce the quaternionic Sobolev space

$$H^1 := H^1(\Omega, \mathbb{H}) := \left\{ u \in L^2(\Omega, \mathbb{H}) : \exists g_{\ell,j}(x) \in L^2(\Omega, \mathbb{R}), \ell = 1, 2, 3, j = 0, 1, 2, 3 \right.$$

$$\left. \text{such that } \int_{\Omega} u_j(x) \partial_{x_\ell} \varphi(x) dx = - \int_{\Omega} g_{\ell,j}(x) \varphi(x) dx, \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}) \right\},$$

where $C_c^\infty(\Omega, \mathbb{R})$ is the set of real-valued infinitely differentiable functions with compact support on Ω . If $u \in H^1$ then $\partial_{x_\ell}(u_j) = g_{\ell j}$ for $\ell = 1, 2, 3$, and $j = 0, 1, 2, 3$. With the quaternionic scalar product

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{H^1(\Omega, \mathbb{H})} := \langle u, v \rangle_{L^2} + \sum_{\ell=1}^3 \langle \partial_{x_\ell} u, \partial_{x_\ell} v \rangle_{L^2},$$

we have that $H^1(\Omega, \mathbb{H})$ becomes a quaternionic Hilbert space and the norm is defined by

$$\|u\|_{H^1}^2 := \|u\|_{H^1(\Omega, \mathbb{H})}^2 := \|u\|_{L^2}^2 + \|u\|_D^2,$$

where we have set

$$\|u\|_D^2 := \sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2}^2.$$

As usual the space $H_0^1(\Omega, \mathbb{H})$ is the closure of the space $C_0^\infty(\Omega, \mathbb{H})$ in $H^1(\Omega, \mathbb{H})$ with respect to the norm $\|\cdot\|_{H^1}$. Now we give to the problems (2.7) and (2.9) the weak formulations in order to apply the Lax-Milgram lemma in the space $H^1(\Omega, \mathbb{H})$ and

$H_0^1(\Omega, \mathbb{H})$, respectively. From the Definition 1.1 of the operator T , we have

$$\begin{aligned} \mathcal{Q}_s(T) &= T^2 - 2s_0T + |s|^2\mathcal{I} \\ &= -(a_1\partial_{x_1})^2 - (a_2\partial_{x_2})^2 - (a_3\partial_{x_3})^2 \\ &\quad + e_1(a_3\partial_{x_3}(a_2)\partial_{x_2} - a_2\partial_{x_2}(a_3)\partial_{x_3}) + e_2(a_3\partial_{x_3}(a_1)\partial_{x_1} - a_1\partial_{x_1}(a_3)\partial_{x_3}) \\ &\quad + e_3(a_1\partial_{x_1}(a_2)\partial_{x_2} - a_2\partial_{x_2}(a_1)\partial_{x_1}) - 2s_0T + |s|^2\mathcal{I}, \end{aligned}$$

where

$$\text{Scal}(\mathcal{Q}_s(T)) := -(a_1(x)\partial_{x_1})^2 - (a_2(x_2)\partial_x)^2 - (a_3(x_3)\partial_x)^2 + |s|^2\mathcal{I}$$

is the scalar part of $\mathcal{Q}_s(T)$ and

$$\begin{aligned} \text{Vect}(\mathcal{Q}_s(T)) &:= e_1(a_3\partial_{x_3}(a_2)\partial_{x_2} - a_2\partial_{x_2}(a_3)\partial_{x_3}) + e_2(a_3\partial_{x_3}(a_1)\partial_{x_1} - a_1\partial_{x_1}(a_3)\partial_{x_3}) \\ &\quad + e_3(a_1\partial_{x_1}(a_2)\partial_{x_2} - a_2\partial_{x_2}(a_1)\partial_{x_1}) - 2s_0T \end{aligned}$$

is the vector part. We consider the bilinear form

$$\langle \mathcal{Q}_s(T)u, v \rangle_{L^2} = \int_{\Omega} \overline{\mathcal{Q}_s(T)u(x)}v(x) dx$$

for functions u, v in class $\mathcal{C}^2(\overline{\Omega}, \mathbb{H})$. Using the definition of $\mathcal{Q}_s(T)$ we have

$$\langle \mathcal{Q}_s(T)u, v \rangle_{L^2} = \langle T^2u, v \rangle_{L^2} - 2s_0\langle Tu, v \rangle_{L^2} + |s|^2\langle u, v \rangle_{L^2}.$$

Integrating by parts we obtain

$$\begin{aligned} \langle \text{Scal}(\mathcal{Q}_s(T))u, v \rangle_{L^2} &= \sum_{\ell=1}^3 \int_{\Omega} \overline{a_{\ell}(x)\partial_{x_{\ell}}(u(x))} (\partial_{x_{\ell}}a_{\ell}(x)) v(x) dx \\ &\quad + \sum_{\ell=1}^3 \int_{\Omega} \overline{a_{\ell}(x)\partial_{x_{\ell}}(u(x))} a_{\ell}(x)\partial_{x_{\ell}} v(x) dx \\ &\quad - \sum_{\ell=1}^3 \int_{\partial\Omega} n_{\ell}(x)a_{\ell}^2(x) (\partial_{x_{\ell}}\overline{u(x)}) v(x) dS(x) + |s|^2\langle u, v \rangle_{L^2}, \end{aligned}$$

where $dS(x)$ is the infinitesimal surface area of $\partial\Omega$. If we use the boundary condition in (2.7), i.e., $\sum_{\ell=1}^3 a_\ell^2(x)n_\ell(x)u(x)\partial_{x_\ell} + a(x)u(x) = 0$, we get

$$\begin{aligned} \langle \text{Scal}(\mathcal{Q}_s(T))u, v \rangle_{L^2} &= \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} \left(a_\ell^2(x) \right) v(x) dx \\ &\quad + \sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x)\partial_{x_\ell}(u(x))} a_\ell(x)\partial_{x_\ell}(v(x)) dx \\ &\quad + \int_{\partial\Omega} a(x)\overline{u(x)}v(x) dS(x) + |s|^2 \langle u, v \rangle_{L^2}. \end{aligned}$$

Instead, if we use the boundary condition in (2.9), we obtain

$$\begin{aligned} \langle \text{Scal}(\mathcal{Q}_s(T))u, v \rangle_{L^2} &= \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} \left(a_\ell^2(x) \right) v(x) dx \\ &\quad + \sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x)\partial_{x_\ell}(u(x))} a_\ell(x)\partial_{x_\ell}(v(x)) dx. \end{aligned}$$

Relying on the above considerations we can give the following two definitions.

Definition 3.1 Let Ω be a bounded domain in \mathbb{R}^3 with the boundary $\partial\Omega$ of class \mathcal{C}^1 , let $a \in \mathcal{C}^0(\partial\Omega, \mathbb{R})$ and $a_1, a_2, a_3 \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$. We define the bilinear form:

$$\begin{aligned} b_s(u, v) &:= \sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x)\partial_{x_\ell}(u(x))} a_\ell(x)\partial_{x_\ell}(v(x)) dx + \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} \left(a_\ell^2(x) \right) v(x) dx \\ &\quad + \langle \text{Vect}(\mathcal{Q}_s(T))u, v \rangle_{L^2} + |s|^2 \langle u, v \rangle_{L^2} + \int_{\partial\Omega} a(x)\overline{u(x)}v(x) dS(x), \end{aligned} \tag{3.1}$$

for all functions $u, v \in H^1(\Omega, \mathbb{H})$.

Definition 3.2 Let Ω be either a bounded or an unbounded domain in \mathbb{R}^3 with the boundary $\partial\Omega$ of class \mathcal{C}^1 , let $a_1, a_2, a_3 \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$. We define the bilinear form:

$$\begin{aligned} b_s(u, v) &:= \sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x)\partial_{x_\ell}(u(x))} a_\ell(x)\partial_{x_\ell}(v(x)) dx + \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} \left(a_\ell^2(x) \right) v(x) dx \\ &\quad + \langle \text{Vect}(\mathcal{Q}_s(T))u, v \rangle_{L^2} + |s|^2 \langle u, v \rangle_{L^2}, \end{aligned} \tag{3.2}$$

for all functions $u, v \in H_0^1(\Omega, \mathbb{H})$.

Definition 3.3 Let \mathfrak{H} be the Hilbert space $H^1(\Omega, \mathbb{H})$ or some of its closed subspaces, where Ω is either a bounded or an unbounded domain in \mathbb{R}^3 . We say that $u \in \mathfrak{H}$ is the weak solution of the Problem 2.7 or of the Problem 2.9 for some $s \in \mathbb{H}$ if, given $F \in L^2(\Omega, \mathbb{H})$, we have

$$b_s(u, v) = \langle F, v \rangle_{L^2}, \quad \text{for all } v \in \mathfrak{H},$$

where b_s is the bilinear form defined in (3.1) or (3.2).

4 Weak solutions of the Problems 2.1 and 2.4

In this section we prove existence and uniqueness of the weak solutions of Problems 2.1 and 2.4 (Definition 3.3), using Lax-Milgram lemma. Moreover, we need crucial estimates on the S -resolvent operators in order to define the fractional powers of the operator T .

To prove existence and uniqueness of the weak solutions it will be sufficient to show that the bilinear forms $b_s(\cdot, \cdot)$, in Definitions 3.1 or 3.2, are continuous in $H^1(\Omega, \mathbb{H})$ and they are coercive in an appropriate closed subspace of $H^1(\Omega, \mathbb{H})$ where the choice of these subspaces depend on the boundary conditions of the problems.

First we prove the continuity while the coercivity will be proved in Sect. 4.1 for the first problem and in Sect. 4.2 for the second one. As a direct consequence of the coercivity, we will prove an L^2 estimate for the weak solution u that belongs to a subspace of $H^1(\Omega, \mathbb{H})$ and also we will prove an L^2 estimate for the term $T(u)$. These L^2 estimates will be crucial in order to prove the boundedness of the pseudo S -resolvent operator $\mathcal{Q}_s(T)$ and the estimates (2.6).

We recall that the bilinear form

$$b_s(\cdot, \cdot) : H^1(\Omega, \mathbb{H}) \times H^1(\Omega, \mathbb{H}) \rightarrow \mathbb{H},$$

for some $s \in \mathbb{H}$, is continuous if there exists a positive constant $C(s)$ such that

$$|b_s(u, v)| \leq C(s) \|u\|_{H^1} \|v\|_{H^1}, \quad \text{for all } u, v \in H^1(\Omega, \mathbb{H}).$$

We note that the constant $C(s)$ depends on $s \in \mathbb{H}$ but does not depend on u and $v \in H^1(\Omega, \mathbb{H})$.

The continuity of the bilinear forms $b_s(u, v)$ can be obtained in a similar way as described in [20] and in [18]. For the bilinear form (3.1), we need suitable estimates of the boundary term

Lemma 4.1 *Let $u \in H^1(\Omega, \mathbb{H})$ and let Ω be a bounded domain with $\partial\Omega$ is of class \mathcal{C}^1 . Furthermore let $a \in \mathcal{C}^0(\partial\Omega, \mathbb{R})$, then we have*

$$\left| \int_{\partial\Omega} a(x) |u(x)|^2 dS(x) \right| \leq \sup_{x \in \partial\Omega} |a(x)| C_{\partial\Omega}^2 \|u\|_{H^1(\Omega, \mathbb{H})}^2,$$

where $C_{\partial\Omega}$ is the constant in formula (4.1).

Proof It follows from the scalar valued case see [10, p.315], precisely, suppose that $u \in H^1(\Omega, \mathbb{R})$ and Ω is a bounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class \mathcal{C}^1 . Then $u|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, and there exists a positive constant $C_{\partial\Omega}$ such that

$$\|u\|_{H^{1/2}(\partial\Omega, \mathbb{R})} \leq C_{\partial\Omega} \|u\|_{H^1(\Omega, \mathbb{R})}. \tag{4.1}$$

From estimate (4.1) we get the statement. □

Proposition 4.2 (Continuity of b_s) *Let Ω be a bounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class C^1 . Assume that $a \in C^0(\partial\Omega, \mathbb{R})$ and $a_1, a_2, a_3 \in C^1(\overline{\Omega}, \mathbb{R})$. Then the terms in the bilinear form $b_s(\cdot, \cdot)$ defined in (3.1) satisfy the estimates:*

$$\begin{aligned} \left| \sum_{\ell=1}^3 \int_{\Omega} \overline{a_{\ell}(x) \partial_{x_{\ell}}(u(x))} a_{\ell}(x) \partial_{x_{\ell}}(v(x)) dx \right| &\leq \sup_{\ell=1,2,3, x \in \Omega} (a_{\ell}^2(x)) \|u\|_D \|v\|_D, \\ \left| \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_{\ell}}(u(x))} \partial_{x_{\ell}}(a_{\ell}^2(x)) v(x) dx \right| &\leq \frac{1}{2} \sup_{\ell=1,2,3, x \in \Omega} (\partial_{x_{\ell}}(a_{\ell}^2(x))) \|u\|_D \|v\|_{L^2} \end{aligned} \tag{4.2}$$

and

$$|\langle \text{Vect}(\mathcal{Q}_s(T))u, v \rangle_{L^2}| \leq \left(2 \sup_{i \neq \ell=1,2,3, x \in \Omega} (|a_i(x) \partial_{x_i} a_{\ell}(x)|) + 2|s_0| \sup_{\ell=1,2,3, x \in \Omega} (|a_{\ell}(x)|) \right) \|u\|_D \|v\|_{L^2}, \tag{4.3}$$

$$|s|^2 |\langle u, v \rangle_{L^2}| \leq |s|^2 \|u\|_{L^2} \|v\|_{L^2}, \tag{4.4}$$

while for the boundary term in (3.1) the following inequality holds:

$$\left| \int_{\partial\Omega} a(x) \overline{u(x)} v(x) dS(x) \right| \leq \sup_{x \in \partial\Omega} |a(x)| C_{\partial\Omega}^2 \|u\|_{H^1} \|v\|_{H^1}, \tag{4.5}$$

where $C_{\partial\Omega}$ is the constant in Theorem 4.1. Moreover, the bilinear forms $b_s(\cdot, \cdot)$ are continuous from $H^1(\Omega, \mathbb{H}) \times H^1(\Omega, \mathbb{H}) \rightarrow \mathbb{H}$, i.e., there exists a constant $C(s) > 0$ such that

$$|b_s(u, v)| \leq C(s) \|u\|_{H^1(\Omega, \mathbb{H})} \|v\|_{H^1(\Omega, \mathbb{H})}, \tag{4.6}$$

for all $s \in \mathbb{H}$.

Proof The above estimates are proved in [20] apart from (4.3) that follows by similar arguments. □

4.1 Weak solution of the Problem 2.1

Because of the Robin-type boundary conditions the natural space to obtain existence and uniqueness of the weak solution of the problem (2.7) is the closed subspace $\mathcal{H}(\Omega, \mathbb{H})$ of $H^1(\Omega, \mathbb{H})$ defined by

$$\mathcal{H}(\Omega, \mathbb{H}) := \left\{ u \in H^1(\Omega, \mathbb{H}) : \int_{\Omega} u(x) dx = 0 \right\},$$

with the norm

$$\|u\|_{\mathcal{H}}^2 := \|u\|_D^2 = \sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2}^2.$$

We adapt to the quaternionic setting the Poincaré–Wirtinger’s inequality (see for example [24, p.275]).

Corollary 4.3 *Let Ω be a bounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class C^1 and let $u \in \mathcal{H}(\Omega, \mathbb{H})$. Then we have*

$$\|u\|_{L^2(\Omega; \mathbb{H})}^2 \leq C_P^2 \|u\|_{\mathcal{H}}^2 \text{ for any } u \in \mathcal{H},$$

where C_P is the Poincaré–Wirtinger constant in (4.7).

Proof Under the above hypotheses on the bounded domain Ω in \mathbb{R}^3 the Poincaré–Wirtinger inequality claims that for all $u \in H^1(\Omega, \mathbb{R})$ the following inequality holds:

$$\left\| u - |\Omega|^{-1} \int_{\Omega} u(x) dx \right\|_{L^2(\Omega, \mathbb{R})} \leq C_P \|\nabla u\|_{L^2(\Omega, \mathbb{R})}, \tag{4.7}$$

where C_P does not depend on u . The quaternionic case follows from estimate (4.7). \square

Theorem 4.4 *Let Ω be a bounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class C^1 . Assume that $a \in C^0(\partial\Omega, \mathbb{R})$ and let T be the operator defined in (1.1) with coefficients $a_1, a_2, a_3 \in C^1(\overline{\Omega}, \mathbb{R})$. Define the following constants:*

$$C_T := \min_{\ell=1,2,3} \inf_{x \in \Omega} (a_\ell^2(x)), \quad C'_T := \sum_{i,\ell=1}^3 \|a_\ell \partial_{x_\ell} a_i\|_\infty, \quad K_{a,\Omega} := C_{\partial\Omega}^2 \|a\|_\infty, \tag{4.8}$$

where $\|\cdot\|_\infty$ denotes the sup norm and $C_{\partial\Omega}$ is the constant in Theorem 4.1. Moreover, assume that

$$C_T - C'_T C_P - K_{a,\Omega} (1 + C_P^2) > 0 \text{ and } C_T > 0, \tag{4.9}$$

where C_P is the constant in (4.7). Then:

(I) *The boundary value Problem (2.7) has a unique weak solution $u \in \mathcal{H}(\Omega, \mathbb{H})$, for $s \in \mathbb{H} \setminus \{0\}$ with $Re(s) = 0$, and*

$$\|u\|_{L^2}^2 \leq \frac{1}{s^2} Re(b_s(u, u)). \tag{4.10}$$

(II) *Moreover, we have the following estimate*

$$\|T(u)\|_{L^2}^2 \leq C Re(b_s(u, u)), \tag{4.11}$$

for every $u \in \mathcal{H}(\Omega, \mathbb{H})$, and $s \in \mathbb{H} \setminus \{0\}$ with $Re(s) = 0$, where

$$C := 1 - \frac{C'_T C_P}{C_T} - \frac{K_{a,\Omega}(1 + C_P^2)}{C_T}.$$

Proof Step (I). To prove the existence and uniqueness of the weak solution using the Lax-Milgram Lemma, it is sufficient to prove the coercivity of the bilinear form $b_s(\cdot, \cdot)$, made explicit in the Definition 3.1, since its continuity is proved in Proposition 4.2. First we write explicitly $Re b_{j_{s_1}}(u, u)$, where we have set $s = j_{s_1}$, for $s_1 \in \mathbb{R}$ and $j \in \mathbb{S}$:

$$\begin{aligned} Re b_{j_{s_1}}(u, u) &= s_1^2 \|u\|_{L^2}^2 + \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 \\ &\quad + Re \left(\frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} (a_\ell^2(x_\ell)) u(x) dx + \langle \text{Vect}(\mathcal{Q}_{j_{s_1}}(T))u, u \rangle_{L^2} \right) \\ &\quad + \int_{\partial\Omega} a|u(x)|^2 dS(x). \end{aligned}$$

By the Cauchy-Schwartz inequality and Lemma 4.1, we have

$$Re b_{j_{s_1}}(u, u) \geq s_1^2 \|u\|_{L^2}^2 + C_T \sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2}^2 - C'_T \sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2} \|u\|_{L^2} - K_{a,\Omega} \|u\|_{H^1}^2.$$

Since

$$\|u\|_{H^1}^2 \leq (1 + C_P^2) \|u\|_{\mathcal{H}}^2,$$

we obtain

$$Re b_s(u, u) \geq s_1^2 \|u\|_{L^2}^2 + \left(C_T - C'_T C_P - K_{a,\Omega} (1 + C_P^2) \right) \|u\|_{\mathcal{H}}^2. \tag{4.12}$$

By the hypothesis, we know that

$$\mathcal{K}_\Omega := C_T - C'_T C_P - K_{a,\Omega} (1 + C_P^2) > 0,$$

thus the following estimates hold:

$$Re b_{j_{s_1}}(u, u) \geq \mathcal{K}_\Omega \|u\|_{\mathcal{H}}^2 \tag{4.13}$$

and

$$Re b_{j_{s_1}}(u, u) \geq s_1^2 \|u\|_{L^2}^2. \tag{4.14}$$

In particular the inequality (4.14) implies the inequality (4.10), while the inequality (4.13) implies the coercivity of $b_{j_{s_1}}(\cdot, \cdot)$ and, by the Lax-Milgram Lemma, we have

that for any $w \in L^2(\Omega, \mathbb{H})$ there exists a unique $u_w \in \mathcal{H}$ such that

$$b_{j_{s_1}}(u_w, v) = \langle w, v \rangle_{L^2}, \quad \text{for all } v \in \mathcal{H} \text{ and for all } s_1 \in \mathbb{R}.$$

Step (II). What remains to prove is the inequality (4.11). Starting from (3.1) and applying the Cauchy-Schwartz inequality, Lemma 4.1 for the boundary term, Corollary 4.3 for the term $\|u\|_{L^2}$ and the inequality

$$\sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2}^2 \leq \frac{1}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2$$

we have:

$$\begin{aligned} \operatorname{Re} b_{j_{s_1}}(u, u) &\geq \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 + \int_{\partial\Omega} a|u(x)|^2 dS(x) \\ &\quad + \operatorname{Re} \left(\sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x) \partial_{x_\ell}(u(x))} \partial_{x_\ell}(a_\ell(x)) u(x) dx + \langle \operatorname{Vect}(\mathcal{Q}_{j_{s_1}}(T))u, u \rangle_{L^2} \right) \\ &\geq \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - \frac{K'_{a, \Omega}(1 + C_P^2)}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - C'_T \sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2} \|u\|_{L^2} \\ &\geq \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - \frac{K'_{a, \Omega}(1 + C_P^2)}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - C'_T C_P \sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2}^2 \\ &\geq \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - \frac{K'_{a, \Omega}(1 + C_P^2)}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - \frac{C'_T C_P}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2. \end{aligned}$$

Collecting the term $\sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2$, observing that, with some computations, we have:

$$\sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 = \|Tu\|_{L^2}^2 \tag{4.15}$$

and since the condition (4.9) holds, we get the desired inequality (4.11):

$$\operatorname{Re} b_{j_{s_1}}(u, u) \geq \left(1 - \frac{C'_T C_P}{C_T} - \frac{K_{a, \Omega}(1 + C_P^2)}{C_T} \right) \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 = C \|Tu\|_{L^2}^2,$$

where we have set

$$C := 1 - \frac{C'_T C_P}{C_T} - \frac{K_{a, \Omega}(1 + C_P^2)}{C_T}.$$

□

Although the technique for proving Theorem 4.4 is different from the technique used in Theorem 4.1 of [14], we note that the condition (4.9) differs from the condition

in Theorem 4.1 of [14] for some terms that arise since in this article we supposed the components of T are non commutative and a Robin-type condition on the boundary of Ω instead of a Dirichlet condition.

4.2 Weak solution of the Problem 2.4

In Theorem 4.4 we proved the invertibility of the operator $\mathcal{Q}_s(T)$ in the case where Ω is a bounded domain. To invert the operator $\mathcal{Q}_s(T)$ in the case where Ω is an unbounded domain, we adapt the strategy explained in Theorem 4.4 but, due to the unboundedness of Ω , we will need more restrictive assumptions on the coefficients of T . In fact, the coefficients of T are such that the first derivatives of the coefficients of the operator $\mathcal{Q}_s(T)$ belong to the space $L^3(\Omega, \mathbb{H})$. We need a couple of lemmas, that are well known to adapt the Sobolev–Gagliardo–Nirenberg inequality to the quaternions.

We recall formula (5) in Theorem 8.8 p.212 in [10] and we give a sketch of the proof for the sake of completeness.

Lemma 4.5 *For any $u \in W^{1,1}(\mathbb{R})$, we have*

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|u'\|_{L^1(\mathbb{R})}. \tag{4.16}$$

Proof We prove the statement for $v \in C_0^1(\mathbb{R})$, the general case will follow from the fact that $C_0^1(\mathbb{R})$ is dense in $W^{1,1}(\mathbb{R})$. We have:

$$v(x) = \int_{-\infty}^x v'_i(x) dx$$

thus we can conclude that

$$\sup_{x \in \mathbb{R}} |v(x)| \leq \int_{-\infty}^{+\infty} |v'(x)| dx. \tag{4.17}$$

If $v \in W^{1,1}(\mathbb{R})$ then there exists a sequence $v_j \in C_0^1(\mathbb{R})$ such that $v_j \xrightarrow{W^{1,1}} v$. Inequality (4.17) implies the convergence of the sequence to v in $L^\infty(\mathbb{R})$. Thus the estimate (4.17) holds true for any $v \in W^{1,1}(\mathbb{R})$. \square

The following lemma can be proved for \mathbb{R}^n even though we will consider the case \mathbb{R}^3 . It is Lemma 9.4 p.278 in [10] and we give a sketch of the proof.

Lemma 4.6 *Let $F_i \in L^{n-1}(\mathbb{R}^n, \mathbb{R})$ for $i = 1, \dots, n$ such that F_i does not depend on x_i . Then*

$$\int_{\mathbb{R}^n} |F_1 \cdots F_n| dV \leq \prod_{i=1}^n \left(\int_{\mathbb{R}^{n-1}} |F_i|^{n-1} dV_i \right)^{\frac{1}{n-1}},$$

where $dV_i := dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$.

Proof The proof follows by an induction argument. The case $n = 2$ is a consequence of the following fact:

$$\begin{aligned} \int_{\mathbb{R}^2} |F_1(x_2) \cdot F_2(x_1)| dx_1 \wedge dx_2 &= \int_{\mathbb{R}} |F_1(x_2)| \int_{\mathbb{R}} |F_2(x_1)| dx_1 dx_2 \\ &= \int_{\mathbb{R}} |F_1(x_2)| dx_2 \cdot \int_{\mathbb{R}} |F_2(x_1)| dx_1. \end{aligned}$$

Now we suppose that we have proved the statement in the case $n = k - 1$ when $k > 2$ is an integer. By the Hölder inequality we have

$$\int_{\mathbb{R}^{n-1}} |F_1 \cdots F_k| dV_1 \leq \left(\int_{\mathbb{R}^{n-1}} |F_1|^{n-1} dV_1 \right)^{\frac{1}{n-1}} \cdot \left(\int_{\mathbb{R}^{n-1}} |F_2 \cdots F_k|^{\frac{n-1}{n-2}} dV_1 \right)^{\frac{n-2}{n-1}}. \tag{4.18}$$

By induction we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^{n-1}} |F_2 \cdots F_k|^{\frac{n-1}{n-2}} dV_1 \right)^{\frac{n-2}{n-1}} &\leq \left[\prod_{j=2}^n \left(\int_{\mathbb{R}^{n-2}} (|F_j|^{\frac{n-1}{n-2}})^{n-2} d(V_1)_j \right)^{\frac{1}{n-2}} \right]^{\frac{n-2}{n-1}} \\ &= \prod_{j=2}^n \left[\int_{\mathbb{R}^{n-2}} |F_j|^{n-1} d(V_1)_j \right]^{\frac{1}{n-1}}. \end{aligned} \tag{4.19}$$

Integrating over x_1 the inequality (4.18) and using the inequality (4.19), we have

$$\begin{aligned} \int_{\mathbb{R}^n} |F_1 \cdots F_k| dV &\leq \left(\int_{\mathbb{R}^{n-1}} |F_1|^{n-1} dV_1 \right)^{\frac{1}{n-1}} \cdot \int_{\mathbb{R}} \prod_{j=2}^n \left[\int_{\mathbb{R}^{n-2}} |F_j|^{n-1} d(V_1)_j \right]^{\frac{1}{n-1}} dx_1 \\ &\stackrel{\text{Hölder inequality}}{\leq} \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} |F_j|^{n-1} dV_j \right)^{\frac{1}{n-1}}, \end{aligned}$$

which concludes the proof. □

So we finally have the Sobolev–Gagliardo–Nirenberg inequality for the quaternions obtained by adapting Theorem 9.9, p.278 in [10] and using the above lemmas.

Lemma 4.7 *For any $u \in H^1(\mathbb{R}^n, \mathbb{H})$, we have $u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n, \mathbb{H})$ and the following estimate holds true*

$$\|u\|_{L^{2n/(n-2)}(\mathbb{R}^n, \mathbb{H})} \leq K_n \sum_{i=1}^n \|\partial_{x_i} u\|_{L^2(\mathbb{R}^n, \mathbb{H})},$$

where

$$K_n := \frac{2n - 2}{n - 2}.$$

Proof We can suppose $u \in C_0^1(\mathbb{R}^n, \mathbb{H})$. First we observe that:

$$\begin{aligned} \left| \partial_{x_i} \left(|u|^{\frac{2n-2}{n-2}} \right) \right| &= \left| \partial_{x_i} \left[\left(|u|^2 \right)^{\frac{n-1}{n-2}} \right] \right| \\ &= \frac{2n-2}{n-2} \left(|u|^2 \right)^{\frac{n-1}{n-2}-1} \left| \sum_{j=0}^3 u_j \partial_{x_i} u_j \right| \\ &\leq \frac{2n-2}{n-2} |\partial_{x_i} u| |u|^{\frac{n}{n-2}}, \end{aligned} \tag{4.20}$$

so we have

$$\begin{aligned} \left[\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dV \right]^{\frac{n-2}{2n}} &\leq \left[\int_{\mathbb{R}^n} \prod_{i=1}^n \sup_{x_i \in \mathbb{R}} |u(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n)|^{\frac{2}{n-2}} dV \right]^{\frac{n-2}{2n}} \\ &\stackrel{\text{Lemma 4.6}}{\leq} \left[\prod_{i=1}^n \left(\int_{\mathbb{R}^{n-1}} \sup_{x_i \in \mathbb{R}} |u(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n)|^{\frac{2n-2}{n-2}} dV_i \right)^{\frac{1}{n-1}} \right]^{\frac{n-2}{2n}} \\ &\stackrel{\text{Lemma 4.5+(4.20)}}{\leq} \left[\prod_{i=1}^n \left(\frac{2n-2}{n-2} \int_{\mathbb{R}^n} |u|^{\frac{n}{n-2}} |\partial_{x_i} u| dV \right)^{\frac{1}{n-1}} \right]^{\frac{n-2}{2n}} \\ &\stackrel{\text{H\"older inequality}}{\leq} \left[\left(\frac{2n-2}{n-2} \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dV \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \int_{\mathbb{R}^n} |\partial_{x_j} u|^2 dV \right)^{\frac{1}{2}} \right)^{\frac{n}{n-1}} \right]^{\frac{n-2}{2n}}. \end{aligned}$$

The above chain of inequalities can be summarized by the following inequality

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n, \mathbb{H})} \leq \left(\frac{2n-2}{n-2} \right)^{\frac{n-2}{2n-2}} \|u\|_{L^{\frac{2n-2}{n-2}}(\mathbb{R}^n, \mathbb{H})} \left[\sum_{i=1}^n \|\partial_{x_i} u\|_{L^2(\mathbb{R}^n, \mathbb{H})} \right]^{\frac{n-2}{2n}}.$$

Thus we conclude that

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n, \mathbb{H})} \leq \frac{2n-2}{n-2} \sum_{i=1}^n \|\partial_{x_i} u\|_{L^2(\mathbb{R}^n, \mathbb{H})}. \tag{4.21}$$

If $u \in H^1(\mathbb{R}^n, \mathbb{H})$ then there exists a sequence $u_j \in C_0^1(\mathbb{R}^n, \mathbb{H})$ such that $u_j \xrightarrow{H^1} u$. Inequality (4.21) implies the convergence of the sequence to u in $L^{\frac{2n}{n-2}}(\mathbb{R}^n, \mathbb{H})$. Thus the estimate (4.21) holds true for any $u \in H^1(\mathbb{R}^n, \mathbb{H})$. \square

We are now ready to prove the continuity of the bilinear forms $b_S(\cdot, \cdot)$ defined in (3.2)

Proposition 4.8 *Let Ω be an unbounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class C^1 . Let T be the operator defined in (1.1) with coefficients $a_1, a_2, a_3 \in C^1(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$. Suppose that*

$$M := \sum_{i,j=1}^3 \|a_i \partial_{x_i}(a_j)\|_{L^3(\Omega)} < +\infty \tag{4.22}$$

Then for any $u, v \in H_0^1(\Omega, \mathbb{H})$ we have

$$\left| \sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x) \partial_{x_\ell}(u(x))} a_\ell(x) \partial_{x_\ell}(v(x)) dx \right| \leq \sup_{\ell=1,2,3, x \in \Omega} (a_\ell^2(x)) \|u\|_D \|v\|_D \tag{4.23}$$

$$2s_0 |\langle Tu, v \rangle| \leq 2s_0 \sup_{\ell=1,2,3, x \in \Omega} (|a_\ell(x)|) \|u\|_D \|v\|_D \tag{4.24}$$

and

$$\left| \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} (a_\ell^2(x_\ell)) v(x) dx + \langle \text{Vect}(\mathcal{Q}_{j_{s_1}}(T))u, v \rangle_{L^2(\Omega)} \right| \leq MK_3 \|u\|_D \|v\|_D. \tag{4.25}$$

Moreover, the bilinear forms $b_s(\cdot, \cdot)$, defined in Definition 3.2, are continuous from $H_0^1(\Omega, \mathbb{H}) \times H_0^1(\Omega, \mathbb{H}) \rightarrow \mathbb{H}$, i.e., there exists a constant $C(s) > 0$ such that

$$|b_s(u, v)| \leq C(s) \|u\|_D \|v\|_D, \tag{4.26}$$

for all $s \in \mathbb{H}$.

Proof The estimates (4.23) and (4.24) follow by the boundedness of the coefficients a_ℓ 's and by the Hölder inequality. We prove the estimate (4.25). First we observe that since $u, v \in H_0^1(\Omega, \mathbb{H})$, we can extend u and v by 0 outside Ω and we still have $u, v \in H^1(\mathbb{R}^n, \mathbb{H})$. For a general function $u \in L^2(\Omega, \mathbb{H})$, we define

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c \end{cases}$$

Thus we have

$$\begin{aligned} & \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} (a_\ell^2(x)) v(x) dx + \langle \text{Vect}(\mathcal{Q}_s(T))u, v \rangle_{L^2(\Omega)} \\ &= \frac{1}{2} \sum_{\ell=1}^3 \int_{\mathbb{R}^n} \overline{\partial_{x_\ell}(\tilde{u}(x))} \partial_{x_\ell} (\widetilde{a_\ell^2(x)}) \tilde{v}(x) dx + \langle \text{Vect}(\widetilde{\mathcal{Q}_s(T)})\tilde{u}, \tilde{v} \rangle_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Eventually, by the Hölder inequality, Lemma 4.7 (for the case $n = 3$) and hypothesis (4.27), we have that:

$$\begin{aligned} & \left| \frac{1}{2} \sum_{\ell=1}^3 \int_{\mathbb{R}^3} \widetilde{\partial_{x_\ell}(u(x))} \partial_{x_\ell}(\widetilde{a_\ell^2(x)}) \widetilde{v}(x) dx + \langle \text{Vect}(\widetilde{\mathcal{Q}_{js_1}(\mathbb{T})}) \widetilde{u}, \widetilde{v} \rangle_{L^2(\mathbb{R}^3)} \right| \\ & \stackrel{\text{Hölder inequality}}{\leq} \left(\sum_{\ell=1}^3 \|\widetilde{\partial_{x_\ell} u}\|_{L^2(\mathbb{R}^3)} \right) \sum_{i,j=1}^3 \left(\int_{\mathbb{R}^3} |a_i \partial_{x_i} a_j \widetilde{v}|^2 dV \right)^{\frac{1}{2}} \\ & \stackrel{\text{Hölder inequality+(4.22)}}{\leq} \|u\|_D \sum_{i,j=1}^3 \left(\left(\int_{\mathbb{R}^3} |a_i \partial_{x_i} a_j|^{2 \cdot \frac{3}{2}} dV \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} |\widetilde{v}|^{2 \cdot 3} dV \right)^{\frac{1}{3}} \right)^{\frac{1}{2}} \\ & = M \|u\|_D \|\widetilde{v}\|_{L^6(\mathbb{R}^3)} \stackrel{\text{Lemma 4.7}}{\leq} K_3 M \|u\|_D \|v\|_D. \end{aligned}$$

The continuity of $b_s(\cdot, \cdot)$ for all $s \in \mathbb{H}$ is a direct consequence of the estimates (4.23), (4.24) and (4.25). □

Theorem 4.9 *Let Ω be an unbounded domain in \mathbb{R}^3 with boundary $\partial\Omega$ of class \mathcal{C}^1 . Let T be the operator defined in (1.1) with coefficients $a_1, a_2, a_3 \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$. Suppose that*

$$M := \sum_{i,j=1}^3 \|a_i \partial_{x_i}(a_j)\|_{L^3(\Omega)} < +\infty \tag{4.27}$$

and

$$C_T := \min_{\ell=1,2,3} \inf_{x \in \Omega} (a_\ell^2(x)) > 0, \quad C_T - MK_3 > 0 \tag{4.28}$$

where $K_3 = 4$ is the constant in Lemma 4.7 for $n = 3$. Then:

(I) *The boundary value Problem (2.9) has a unique weak solution $u \in H_0^1(\Omega, \mathbb{H})$, for $s \in \mathbb{H} \setminus \{0\}$ with $Re(s) = 0$, and*

$$\|u\|_{L^2}^2 \leq \frac{1}{s^2} Re(b_s(u, u)). \tag{4.29}$$

(II) *Moreover, we have the following estimate*

$$\|T(u)\|_{L^2}^2 \leq C Re(b_s(u, u)), \tag{4.30}$$

for every $u \in H_0^1(\Omega, \mathbb{H})$, and $s \in \mathbb{H} \setminus \{0\}$ with $Re(s) = 0$, where

$$C := \frac{C_T - MK_3}{C_T}.$$

Proof In order to use the Lax-Milgram Lemma to prove the existence and the uniqueness of the solution for the weak formulation of the problem, it is sufficient to prove the coercivity of the bilinear form $b_s(\cdot, \cdot)$ in Definition 3.2 since the continuity is proved

in Proposition 4.8. First we write explicitly $\operatorname{Re} b_{js_1}(u, u)$, where we have set $s = js_1$, for $s_1 \in \mathbb{R}$ and $j \in \mathbb{S}$:

$$\begin{aligned} \operatorname{Re} b_{js_1}(u, u) &= s_1^2 \|u\|_{L^2}^2 + \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 \\ &\quad + \operatorname{Re} \left(\frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} (a_\ell^2(x)) u(x) dx + \langle \operatorname{Vect}(\mathcal{Q}_{js_1}(T))u, u \rangle_{L^2} \right). \end{aligned} \tag{4.31}$$

By the estimate (4.25) applied in the case $u = v$ we have

$$\begin{aligned} &\left| \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_\ell}(u(x))} \partial_{x_\ell} (a_\ell^2(x_\ell)) u(x) dx + \langle \operatorname{Vect}(\mathcal{Q}_{js_1}(T))u, u \rangle_{L^2(\Omega)} \right| \\ &\leq MK_3 \|u\|_D^2. \end{aligned} \tag{4.32}$$

Finally using the inequality (4.32) in (4.31), we obtain

$$\operatorname{Re} b_{js_1}(u, u) \geq s_1^2 \|u\|_{L^2}^2 + (C_T - MK_3) \|u\|_D^2.$$

By the hypothesis (4.28) we know that

$$\mathcal{K}_\Omega := C_T - MK_3 > 0$$

thus the quadratic form $b_{js_1}(\cdot, \cdot)$ is coercive for every $s_1 \in \mathbb{R}$ and the following estimates hold:

$$\operatorname{Re} b_{js_1}(u, u) \geq \min(\mathcal{K}_\Omega, s_1^2) \|u\|_{H^1}. \tag{4.33}$$

In particular we have

$$\operatorname{Re} b_{js_1}(u, u) \geq s_1^2 \|u\|_{L^2}^2. \tag{4.34}$$

As a consequence the inequality (4.34) implies the inequality (4.10). The inequality (4.33) implies the coercivity of $b_{js_1}(\cdot, \cdot)$ and, by the Lax-Milgram Lemma, we have that for any $w \in L^2(\Omega, \mathbb{H})$ there exists $u_w \in H_0^1(\Omega, \mathbb{H})$, for $s_1 \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{S}$, such that

$$b_{js_1}(u_w, v) = \langle w, v \rangle_{L^2}, \quad \text{for all } v \in H_0^1(\Omega, \mathbb{H}).$$

What remains to prove is the inequality (4.11). Starting from (3.2), applying the inequality (4.32) and observing that

$$\sum_{\ell=1}^3 \|\partial_{x_\ell} u\|_{L^2}^2 \leq \frac{1}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2$$

we have:

$$\begin{aligned}
 \operatorname{Re} b_{j_{S_1}}(u, u) &\geq \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 + \operatorname{Re} \left(\sum_{\ell=1}^3 \int_{\Omega} \overline{a_\ell(x) \partial_{x_\ell}(u(x))} \partial_{x_\ell}(a_\ell(x)) u(x) dx \right. \\
 &\quad \left. + \langle \operatorname{Vect}(\mathcal{Q}_{j_{S_1}}(T))u, u \rangle_{L^2} \right) \\
 &\geq \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 - \frac{MK_3}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 \\
 &\geq \frac{C_T - MK_3}{C_T} \sum_{\ell=1}^3 \|a_\ell \partial_{x_\ell} u\|_{L^2}^2 \\
 &= C \|Tu\|_{L^2}^2,
 \end{aligned}$$

where we have set

$$C := \frac{C_T - MK_3}{C_T}$$

and this concludes the proof. □

Remark 4.10 As we have mentioned in the introduction the case Ω bounded with homogeneous Dirichlet boundary conditions has already been investigated in our previous work. Above we have treated the case when Ω is unbounded. In the case Ω is bounded the condition (4.27) is not required.

5 The estimates for the \mathcal{S} -resolvent operators and the fractional powers of T

After we prove existence and uniqueness results for the weak solutions of the problems we discussed in the previous sections we can give meaning to the boundary condition using classical results on regularity of elliptic equations up to the boundary. In the case of Robin boundary conditions this requires the assumptions that the boundary has to be more regular, in the case of second order operators the boundary has to be of class \mathcal{C}^2 if we want to have solutions in H^2 . In fact we can speak of the normal derivative $\partial_\nu u$ of a function $u \in H^2(\Omega, \mathbb{R})$ (more in general we can set this problem in $W^{2,p}$ for $1 \leq p < \infty$), we set $\partial_n u := (\nabla u)|_{\partial\Omega} \cdot n$, where n is the unit normal vector to $\partial\Omega$. This has meaning since $(\nabla u)|_{\partial\Omega} \in L^2(\partial\Omega)$ for $\Omega \subset \mathbb{R}^N$ bounded. For the regularity of the Neumann problem see p.299 in [10]. Using the estimates in Theorem 4.4 for the case of the Robin-type boundary conditions or estimate in Theorem 4.9, for the case of the Dirichlet boundary conditions in unbounded domains, we can now show in both cases that the S -resolvent operator of T decays fast enough along the set of purely imaginary quaternions.

Theorem 5.1 *Under the hypotheses of Theorem 4.4 or the hypotheses of Theorem 4.9, the operator $\mathcal{Q}_s(T)$ is invertible for any $s = js_1$, for $s_1 \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{S}$ and the following estimate*

$$\|\mathcal{Q}_s(T)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s_1^2} \tag{5.1}$$

holds. Moreover, the \mathcal{S} -resolvent operators satisfy the estimates

$$\|\mathcal{S}_L^{-1}(s, T)\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s|} \quad \text{and} \quad \|\mathcal{S}_R^{-1}(s, T)\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s|}, \tag{5.2}$$

for any $s = js_1$, for $s_1 \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{S}$, with a constant Θ that does not depend on s .

Proof We saw in Theorem 4.4 (respectively, Theorem 4.9) that for all $w \in L^2(\Omega, \mathbb{H})$ there exists $u_w \in \mathcal{H}$ (respectively $u_w \in H_0^1(\Omega, \mathbb{H})$), for $s_1 \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{S}$, such that

$$b_{js_1}(u_w, v) = \langle w, v \rangle_{L^2}, \quad \text{for all } v \in \mathcal{H}(\Omega, \mathbb{H}) \quad (\text{respectively, for all } v \in H_0^1(\Omega, \mathbb{H})).$$

Thus we can define the inverse operator $\mathcal{Q}_{js_1}(T)^{-1}(w) := u_w$ for any $w \in L^2(\Omega, \mathbb{H})$ (we note that the range of $\mathcal{Q}_{js_1}(T)^{-1}$ is in $\mathcal{H}(\Omega, \mathbb{H})$ (respectively in $H_0^1(\Omega, \mathbb{H})$)). The inequality (4.10) (respectively (4.29)), applied to $u := \mathcal{Q}_{js_1}(T)^{-1}(w)$, implies:

$$\begin{aligned} s_1^2 \|\mathcal{Q}_{js_1}(T)^{-1}(w)\|_{L^2}^2 &\stackrel{(4.10) \text{ (respectively (4.29))}}{\leq} \operatorname{Re} b_{js_1}(\mathcal{Q}_{js_1}(T)^{-1}(w), \mathcal{Q}_{js_1}(T)^{-1}(w)) \\ &\leq |b_{js_1}(\mathcal{Q}_{js_1}(T)^{-1}(w), \mathcal{Q}_{js_1}(T)^{-1}(w))| \\ &\leq |\langle w, \mathcal{Q}_{js_1}(T)^{-1}(w) \rangle_{L^2}| \\ &\leq \|w\|_{L^2} \|\mathcal{Q}_{js_1}(T)^{-1}(w)\|_{L^2}, \quad \text{for any } w \in L^2(\Omega, \mathbb{H}). \end{aligned} \tag{5.3}$$

Thus we have

$$\|\mathcal{Q}_{js_1}(T)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s_1^2}, \quad \text{for } s_1 \in \mathbb{R} \setminus \{0\} \text{ and } j \in \mathbb{S}.$$

The estimates (5.2) follow from the estimate (4.11) (respectively (4.30)). Indeed we have

$$\begin{aligned} C \|Tu_w\|^2 &\stackrel{(4.11) \text{ (respectively (4.30))}}{\leq} \operatorname{Re}(b_{js_1}(u_w, u_w)) \\ &\leq |b_{js_1}(u_w, u_w)| \\ &\leq |\langle w, u_w \rangle_{L^2}| \\ &\leq \|w\|_{L^2} \|u_w\|_{L^2} \\ &\stackrel{(5.1)}{\leq} \frac{1}{s_1^2} \|w\|_{L^2}^2, \end{aligned}$$

for $s_1 \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{S}$. This estimate implies

$$\|TQ_{js_1}(T)^{-1}w\|_{L^2} = \|Tu_w\|_{L^2} \leq \frac{1}{\sqrt{C}|s_1|} \|w\|_{L^2}$$

thus we obtain

$$\|TQ_{js_1}(T)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{\sqrt{C}|s_1|}. \tag{5.4}$$

In conclusion, if we set

$$\Theta := 2 \max \left\{ 1, \frac{1}{\sqrt{C}} \right\},$$

estimates (5.4) and (5.1) yield

$$\begin{aligned} \|S_R^{-1}(s, T)\|_{\mathcal{B}(L^2)} &= \|(T - \bar{s}\mathcal{I})Q_s(T)^{-1}\|_{\mathcal{B}(L^2)} \\ &\leq \|TQ_s(T)^{-1}\|_{\mathcal{B}(L^2)} + \|\bar{s}Q_s(T)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s_1|} \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \|S_L^{-1}(s, T)\|_{\mathcal{B}(L^2)} &= \|TQ_s(T)^{-1} - Q_s(T)^{-1}\bar{s}\|_{\mathcal{B}(L^2)} \\ &\leq \|TQ_s(T)^{-1}\|_{\mathcal{B}(L^2)} + \|Q_s(T)^{-1}\bar{s}\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s_1|}, \end{aligned} \tag{5.6}$$

for any $s = js_1 \in \mathbb{H} \setminus \{0\}$. □

Thanks to the above results, we are now ready to establish our main statement.

Theorem 5.2 *Under the hypotheses of Theorem 4.4 or the hypotheses of Theorem 4.9, for any $\alpha \in (0, 1)$ and $v \in \mathcal{D}(T)$, the integral*

$$P_\alpha(T)v := \frac{1}{2\pi} \int_{-j\mathbb{R}} s^{\alpha-1} ds_j S_R^{-1}(s, T)Tv$$

converges absolutely in L^2 .

Proof The right S -resolvent equation implies

$$S_R^{-1}(s, T)Tv = sS_R^{-1}(s, T)v - v, \quad \forall v \in \mathcal{D}(T)$$

and so

$$\begin{aligned} \frac{1}{2\pi} \int_{-j\mathbb{R}} \|s^{\alpha-1} ds_j S_R^{-1}(s, T)Tv\|_{L^2} &\leq \frac{1}{2\pi} \int_{-\infty}^{-1} |t|^{\alpha-1} \|S_R^{-1}(-jt, T)\|_{B(L^2)} \|Tv\|_{L^2} dt \\ &\quad + \frac{1}{2\pi} \int_{-1}^1 |t|^{\alpha-1} \|(-jt)S_R^{-1}(-jt, T)v - v\|_{L^2} dt \\ &\quad + \frac{1}{2\pi} \int_1^{+\infty} t^{\alpha-1} \|S_R^{-1}(jt, T)\|_{B(L^2)} \|Tv\|_{L^2} dt. \end{aligned}$$

As $\alpha \in (0, 1)$, the estimate (5.2) now yields

$$\begin{aligned} \frac{1}{2\pi} \int_{-j\mathbb{R}} \|s^{\alpha-1} ds_j S_R^{-1}(s, T)Tv\|_{L^2} &\leq \frac{1}{2\pi} \int_1^{+\infty} t^{\alpha-1} \frac{\Theta}{t} \|Tv\|_{L^2} dt + \frac{1}{2\pi} \int_{-1}^1 |t|^{\alpha-1} \left(|t| \frac{\Theta}{|t|} + 1 \right) \|v\|_{L^2} dt \\ &\quad + \frac{1}{2\pi} \int_1^{+\infty} t^{\alpha-1} \frac{\Theta}{t} \|Tv\|_{L^2} dt \\ &< +\infty. \end{aligned}$$

□

We conclude this paper with some comments.

- (I) In the literature there are several non linear models that involve the fractional Laplacian and even the fractional powers of more general elliptic operators, see for example, the books [11,31].
- (II) The S -spectrum approach to fractional diffusion problems used in this paper is a generalization of the method developed by Balakrishnan, see [5], to define the fractional powers of a real operator A . In the paper [15] following the book of M. Haase, see [27], has been developed the theory on fractional powers of quaternionic linear operators, see also [2,13].
- (III) The spectral theorem on the S -spectrum is also an other tool to define the fractional powers of vector operators, see [1] and for perturbation results see [12].
- (IV) An historical note on the discovery of the S -resolvent operators and of the S -spectrum can be found in the introduction of the book [17].

The most important results in quaternionic operators theory based on the S -spectrum and the associated theory of slice hyperholomorphic functions are contained in the books [3,4,16,17,21,22,25,26], for the case on n -tuples of operators see [23].

- (V) Our future research directions will consider the development of ideas from one and several complex variables, such as in [6–9,28–30] to the quaternionic setting.

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