



Morse–Bott split symplectic homology

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Abstract. We associate a chain complex to a Liouville domain $(\overline{W}, d\lambda)$ whose boundary Y admits a Boothby–Wang contact form (i.e. is a pre-quantization space). The differential counts Floer cylinders with cascades in the completion W of \overline{W} , in the spirit of Morse–Bott homology (Bourgeois in *A Morse–Bott approach to contact homology*, Ph.D. Thesis. ProQuest LLC, Stanford University, Ann Arbor 2002; Frauenfelder in *Int Math Res Notices* 42:2179–2269, 2004; Bourgeois and Oancea in *Duke Math J* 146(1), 71–174, 2009). The homology of this complex is the symplectic homology of W (Diogo and Lisi in *J Topol* 12:966–1029, 2019). Let X be obtained from \overline{W} by collapsing the boundary Y along Reeb orbits, giving a codimension two symplectic submanifold Σ . Under monotonicity assumptions on X and Σ , we show that for generic data, the differential in our chain complex counts elements of moduli spaces of cascades that are transverse. Furthermore, by some index estimates, we show that very few combinatorial types of cascades can appear in the differential.

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1. Introduction and statement of main results

In this paper, we define Morse–Bott *split* symplectic homology theory for Liouville manifolds W of finite-type whose boundary $Y = \partial W$ is a prequantization space. This is inspired by the construction of Bourgeois and Oancea for positive symplectic homology, SH^+ [8]. Our main result is that we obtain transversality for all the relevant moduli spaces and thus have a well-defined theory. This is obtained by means of generic choice of the geometric data, as opposed to using abstract perturbations. This is an important preliminary step in the computation of this chain complex in [13]. Furthermore, in [13], we justify that the homology of this complex is indeed the symplectic homology of W .

The main purpose of this definition of a split version of symplectic homology is to enable computations in certain examples. For instance, we expect to be able to compute symplectic homology for completions W of $X \setminus \Sigma$, where both X and Σ are smooth projective complete intersections. Many of these examples fit in our framework and additionally enough is known about their Gromov–Witten invariants for the computation to be possible. In [13, Part 4], we illustrate our results by computing the well-known symplectic homology of T^*S^2 .

In the following, we consider a $2n$ -dimensional Liouville domain $(\overline{W}, d\lambda)$ with $\partial\overline{W} = Y$. Denoting by $\alpha = \lambda|_Y$ the contact form on the boundary induced by the Liouville form λ , we require that α is a Boothby–Wang contact form, i.e. its Reeb vector field induces a free S^1 action. Such a contact manifold is also called a prequantization space. We let Σ^{2n-2} denote the quotient by the Reeb vector field, and we note that $d\alpha$ descends to a symplectic form ω_Σ on the quotient. It follows that there exists a closed symplectic manifold (X, ω) for which Σ is a codimension 2 symplectic submanifold, Poincaré dual to a multiple of ω , with the property that $(X \setminus \Sigma, \omega)$ is symplectomorphic to $(\overline{W} \setminus Y, d\lambda)$. See Proposition 2.1 for a more detailed description of this, or see [21, Proposition 5]. We can think of X as the symplectic cut of \overline{W} along Y .

We let $(W, d\lambda)$ be the completion of \overline{W} , obtained by attaching a cylindrical end $\mathbb{R}^+ \times Y$, and take a Hamiltonian $H: \mathbb{R} \times Y \rightarrow \mathbb{R}$ with a growth condition (see Definition 3.1 for details). This Hamiltonian will have Morse–Bott families of 1-periodic orbits, and we then use the formalism of cascades

(similar in style to the Morse–Bott theories of [5, 19, 27]) to construct a Morse–Bott Floer homology associated with H . We also consider configurations that interact between $\mathbb{R} \times Y$ and W , as in [8].

In Sect. 3, we introduce the chain complex for split symplectic homology, and in Sect. 4 we describe the moduli spaces that are relevant to the differential. The chain complex will be generated by critical points of auxiliary Morse functions associated with our Morse–Bott manifolds of orbits. The differential will be obtained from moduli spaces of Floer cylinders with cascades together with asymptotic boundary conditions given in terms of the auxiliary Morse functions.

We then prove three main results. The first is a transversality theorem for “simple cascades”. These are elements of the relevant moduli spaces that are also somewhere injective when projected to Σ . This result holds without any monotonicity assumptions. A more precise formulation is provided in Proposition 5.9.

Theorem 1.1. *Simple Floer cylinders with cascades in W and $\mathbb{R} \times Y$ are transverse for a co-meagre set of compatible, Reeb-invariant and cylindrical almost complex structures on $(W, d\lambda)$.*

Theorem 1.1 builds on a transversality theorem for moduli spaces of spheres in X and in Σ , which may be of independent interest. This construction is somewhat analogous to the strings of pearls that Biran and Cornea study in the Lagrangian case [3]. Our result builds essentially on results from [29]. A more precise formulation is given in Proposition 5.26.

Theorem 1.2. *For a generic compatible almost complex structure, moduli spaces of somewhere injective spheres in Σ and of somewhere injective spheres in X with order of contact constraints at Σ , connected by gradient trajectories, are transverse assuming no two spheres have the same image.*

The third result builds on these two to describe the moduli spaces that are relevant under suitable monotonicity assumptions on X and Σ . In particular, the differential is computed from only four types of simple cascades. See Propositions 6.2 and 6.3 for more details.

Theorem 1.3. *Assume that (X, ω) is spherically monotone with monotonicity constant τ_X and assume that $\Sigma \subset X$ is Poincaré dual to $K\omega$ with $\tau_X > K > 0$.*

Then the split Floer homology of W is well defined, and does not depend on the choice of Hamiltonian or of compatible, Reeb-invariant and cylindrical almost complex structure on W (in a co-meagre set of such almost complex structures).

The only moduli spaces of split Floer cylinders with cascades that count in the differential are the following:

- (0) Morse trajectories in Y or in W ;
- (1) Floer cylinders in $\mathbb{R} \times Y$, projecting to non-trivial spheres in Σ , and with asymptotics constrained by descending/ascending manifolds of critical points in Y ;

- (2) holomorphic planes in W that converge to a generic Reeb orbit in Y ;
- (3) holomorphic planes in W constrained to have a marked point in the descending manifold of a Morse function in W and whose asymptotic limit is constrained by the auxiliary Morse function on the manifold of orbits in Y .

We remark that Cases (2) and (3) are non-trivial cascades, but their components in $\mathbb{R} \times Y$ lie in fibres of $\mathbb{R} \times Y \rightarrow \Sigma$. This is formulated more precisely in Propositions 6.2 and 6.3. See Figs. 4, 5 and 6 for a depiction of Cases (1)–(3).

The paper concludes with a discussion of orientations of the moduli spaces of cascades contributing to the differential, in Sect. 7.

2. Setup

We now provide details of the classes of Liouville manifolds for which we prove transversality in split Floer homology.

We begin by summarizing some constructions from [13], specifically

Proposition 2.1 [13, Lemma 2.2]. *Let $(\overline{W}, d\lambda)$ be a Liouville domain with boundary $Y = \partial\overline{W}$. Assume that $\alpha := \lambda|_Y$ has a Reeb vector field generating a free S^1 action.*

Then if Σ is the quotient of Y by the S^1 action, and ω_Σ is the symplectic form induced from $d\alpha$, there exists a symplectic manifold (X, ω) with $\Sigma \subset X$ so that $\omega|_\Sigma = \omega_\Sigma$, with the following properties:

- (i) $(X \setminus \Sigma, \omega)$ is symplectomorphic to $(\overline{W} \setminus \partial\overline{W}, d\lambda)$;
- (ii) $[\Sigma] \in H_{2n-2}(X; \mathbb{Q})$ is Poincaré dual to $[K\omega] \in H^2(X; \mathbb{Q})$ for some $K > 0$;
- (iii) if $N\Sigma$ denotes the symplectic normal bundle to Σ in X , equipped with a Hermitian structure (and hence symplectic structure), there exist a neighbourhood \mathcal{U} of the 0-section in $N\Sigma$ and a symplectic embedding $\varphi: \mathcal{U} \rightarrow X$. By shrinking \mathcal{U} as necessary, we may arrange that φ extends to an embedding of $\overline{\mathcal{U}}$. □

Definition 2.2. Let $X, \omega, \Sigma, \omega_\Sigma$ and $N\Sigma$ be as in Proposition 2.1.

Fix a Hermitian line bundle structure on the symplectic normal bundle $\pi: N\Sigma \rightarrow \Sigma$. A Hermitian connection on $N\Sigma$ can be encoded in terms of a connection 1-form $\Theta \in \Omega^1(N\Sigma \setminus \Sigma)$ with the property that $d\Theta = -K\pi^*d\omega_\Sigma$. Fix such a Hermitian connection 1-form Θ .

Since $N\Sigma$ is a Hermitian line bundle, we have the action of $U(1)$ on the bundle by rotation in the \mathbb{C} -fibres. The infinitesimal generator of this action is a vector field on which Θ evaluates to 1.

Let $\rho: N\Sigma \rightarrow [0, +\infty)$ denote the norm in $N\Sigma$ measured with respect to the Hermitian metric. Then, for each $x \in N\Sigma \setminus \Sigma$, let $\xi_x = (\ker \Theta_x \cap \ker d\rho) \subset T_x N\Sigma$. We may then extend the distribution ξ_x smoothly to the 0-section by defining $\xi_x = T_x \Sigma$ if $x \in \Sigma$.

Notice that this gives a splitting $T_x N\Sigma = \ker d\pi \oplus \xi_x$ and $d\pi|_{\xi_x}: \xi_x \rightarrow T_{\pi(x)}\Sigma$ is a symplectic isomorphism.

Any almost complex structure J_Σ on Σ can then be lifted to an almost complex structure on $N\Sigma$ by taking it to be the linearization of the bundle complex structure on $\ker d\pi$ and to be the pull-back of J_Σ to ξ_x by the isomorphism $d\pi: \xi_x \rightarrow T_{\pi(x)}\Sigma$.

We refer to any almost complex structure obtained in this way as a *bundle almost complex structure* on $N\Sigma$.

Define the open set $\mathcal{V} = X \setminus \overline{\varphi(\mathcal{U})}$. We will later perturb our almost complex structures in \mathcal{V} .

Proposition 2.3 [13, Lemma 2.6]. *Let \overline{W} , Y , λ , α , X , Σ and $\varphi: \mathcal{U} \rightarrow X$ be as in Proposition 2.1.*

Then there exists a diffeomorphism $\psi: W \rightarrow X \setminus \Sigma$ with the following properties:

- (i) *if J_X in a compatible almost complex structure on X that restricts to $\varphi(\mathcal{U})$ as the push-forward by φ of a bundle almost complex structure, then ψ^*J_X is a compatible almost complex structure on W that is cylindrical and Reeb-invariant on $W \setminus \overline{W}$;*
- (ii) *if J_W is a compatible almost complex structure on W , cylindrical and Reeb invariant on $W \setminus \overline{W}$, then the push-forward ψ_*J_W extends to an almost complex structure J_X on X and $J_\Sigma := J_X|_\Sigma$ is also given by restricting J_W to a parallel copy of $\{c\} \times Y$ for some $c > 0$, and taking the quotient by the Reeb S^1 action.*

Furthermore, ψ may be taken to be radial, in the sense that for $(r, y) \in [r_0, +\infty) \times Y$ in the cylindrical end of W , r_0 sufficiently large, then $\psi(r, y) = (\rho(r), y) \in N\Sigma \setminus \Sigma$. □

Note that the diffeomorphism ψ is not symplectic.

In Sect. 6.1, we impose additional conditions of monotonicity. These will also be relevant to the grading given in Definition 3.4.

Definition 2.4. Given a symplectic manifold (X^{2n}, ω) and a codimension-2 symplectic submanifold $\Sigma^{2n-2} \subset X$, say that (X, Σ, ω) is a *monotone triple* if

- (i) X is spherically monotone: there exists a constant $\tau_X > 0$ so that for each spherical homology class $A \in H_2^S(X)$, $\omega(A) = \tau_X \langle c_1(TX), A \rangle$;
- (ii) $[\Sigma] \in H_{2n-2}(X; \mathbb{Q})$ is Poincaré dual to $[K\omega]$ for some $K > 0$ with $\tau_X > K$.

In this case, we write $\omega_\Sigma = \omega|_\Sigma$.

Observe that Condition (ii) implies that (Σ, ω_Σ) is spherically monotone, monotonicity constant $\tau_\Sigma = \tau_X - K$.

We then obtain the following useful characterization of J_W -holomorphic planes in W (see also [23]):

Lemma 2.5 [13, Lemma 2.6]. *Under the diffeomorphism of Proposition 2.3, finite energy J_W -holomorphic planes in W correspond to J_X -holomorphic spheres in X with a single intersection with Σ . The order of contact gives the multiplicity of the Reeb orbit to which the plane converges.*

Proof. Under the map ψ , a finite energy J_W -plane in W gives a J_X -plane in $X \setminus \Sigma$. By the fact that ψ is radial, the restriction of $\psi^*\lambda$ to the cylindrical end $[r_0, +\infty) \times Y$ takes the form $g(r)\alpha$ for some function g with $g' > 0$ and $\lim_{r \rightarrow \infty} g(r) = 1$. It follows then that the integral of $\psi^*d\lambda$ over the plane is dominated by its Hofer energy as given in [10, Section 5.3]. Hence, the image of the plane under the map is a plane in $X \setminus \Sigma$ with finite ω -area.

It follows then that the singularity at ∞ is removable by Gromov’s removal of singularities theorem, and thus the plane admits an extension to a J_X -holomorphic sphere. The order of contact follows from considering the winding around Σ of a loop near the puncture. \square

We now formalize the class of almost complex structures that will be relevant to this paper.

Definition 2.6. An *admissible* almost complex structure J_X on X is compatible with ω and its restriction to $\varphi(\mathcal{U})$ is the push-forward by φ of a bundle almost complex structure on $N\Sigma$.

An almost complex structure J_W on $(W, d\lambda)$ is *admissible* if $J_W = \psi^*J_X$ for an admissible J_X . In particular, such almost complex structures are cylindrical and Reeb-invariant on $W \setminus \overline{W}$.

A compatible almost complex structure J_Y on the symplectization $\mathbb{R} \times Y$ is *admissible* if J_Y is cylindrical and Reeb-invariant.

In the following, we will identify W with $X \setminus \Sigma$ by means of the diffeomorphism ψ and identify the corresponding almost complex structures. By an abuse of notation, we will both write $\pi_\Sigma: Y \rightarrow \Sigma$ to denote the quotient map that collapses the Reeb fibres, and $\pi_\Sigma: \mathbb{R} \times Y \rightarrow \Sigma$ to denote the composition of this projection with the projection to Y .

Definition 2.7. Denote the space of almost complex structures in Σ that are compatible with ω_Σ by \mathcal{J}_Σ .

Let \mathcal{J}_Y denote the space of admissible almost complex structures on $\mathbb{R} \times Y$. Then the projection $d\pi_\Sigma$ induces a diffeomorphism between \mathcal{J}_Y and \mathcal{J}_Σ .

Let \mathcal{J}_W denote the space of admissible almost complex structures on W . By Proposition 2.3, for any $J_W \in \mathcal{J}_W$, we obtain an almost complex structure J_Σ .

Denote this map by $P: \mathcal{J}_W \rightarrow \mathcal{J}_\Sigma$. This map is surjective and open by Proposition 2.3. For any given $J_\Sigma \in \mathcal{J}_\Sigma$, $P^{-1}(J_\Sigma)$ consist of almost complex structures on W that differ in \overline{W} , or equivalently, can be identified (using ψ) with almost complex structures on X that differ in $\mathcal{V} = X \setminus \varphi(\mathcal{U})$.

3. The chain complex

We will now describe the chain complex for the split symplectic homology associated with W .

Definition 3.1. Let $h: (0, +\infty) \rightarrow \mathbb{R}$ be a smooth function with the following properties:

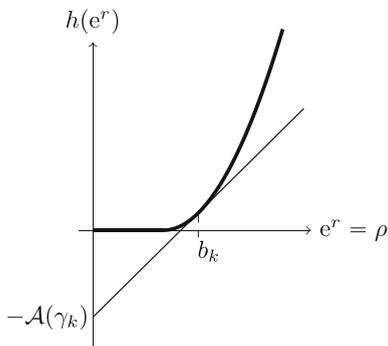


FIGURE 1. An admissible Hamiltonian and the graphical procedure for computing the action of a periodic orbit

- (i) $h(\rho) = 0$ for $\rho \leq 2$;
- (ii) $h'(\rho) > 0$ for $\rho > 2$;
- (iii) $h'(\rho) \rightarrow +\infty$ as $\rho \rightarrow \infty$;
- (iv) $h''(\rho) > 0$ for $\rho > 2$.

An *admissible Hamiltonian* $H: \mathbb{R} \times Y \rightarrow \mathbb{R}$ is given by $H(r, y) = h(e^r)$. Since this Hamiltonian $H(r, y) = 0$ for all $r \leq \ln 2$, we will also take $H = 0$ on W when considering cascades with components in W . See Fig. 1.

Since $\omega = d(e^r \alpha)$ on $\mathbb{R}_+ \times Y$, the Hamiltonian vector field associated with H is $h'(e^r)R$, where R is the Reeb vector field associated with α . The fibres of $Y \rightarrow \Sigma$ are periodic Reeb orbits. Their minimal periods are $T_0 := \int_{\pi_\Sigma^{-1}(p)} \alpha$, for $p \in \Sigma$. The 1-periodic orbits of H are thus of two types:

- (1) constant orbits: one for each point in W and at each point in $(-\infty, \log 2] \times Y \subset \mathbb{R} \times Y$;
- (2) non-constant orbits: for each $k \in \mathbb{Z}_+$, there is a Y -family of 1-periodic X_H -orbits, contained in the level set $Y_k := \{b_k\} \times Y$, for the unique $b_k > \log 2$ such that $h'(e^{b_k}) = kT_0$. Each point in Y_k is the starting point of one such orbit.

Remark 3.2. Notice that these Hamiltonians are Morse–Bott non-degenerate *except* at $\{\log 2\} \times Y$. These orbits will not play a role because they can never arise as boundaries of relevant moduli spaces (see [13, Lemma 4.8]).

Recall that a family of periodic Hamiltonian orbits for a time-dependent Hamiltonian vector field is said to be Morse–Bott non-degenerate if the connected components of the space of parametrized 1-periodic orbits form manifolds, and the tangent space of the family of orbits at a point is given by the eigenspace of 1 for the corresponding Poincaré return map. (Morse non-degeneracy requires the return map not to have 1 as an eigenvalue and hence such periodic orbits must be isolated.)

We also fix some auxiliary data, consisting of Morse functions and vector fields. Fix throughout a Morse function $f_\Sigma: \Sigma \rightarrow \mathbb{R}$ and a gradient-like vector field $Z_\Sigma \in \mathfrak{X}(\Sigma)$, which means that $\frac{1}{c}|df_\Sigma|^2 \leq df_\Sigma(Z_\Sigma) \leq c|df_\Sigma|^2$ for some

constant $c > 0$. Denote the time- t flow of Z_Σ by $\varphi_{Z_\Sigma}^t$. Given $p \in \text{Crit}(f_\Sigma)$, its stable and unstable manifolds (or ascending and descending manifolds, respectively) are

$$W_\Sigma^s(p) := \left\{ q \in \Sigma \mid \lim_{t \rightarrow -\infty} \varphi_{Z_\Sigma}^{-t}(q) = p \right\}, W_\Sigma^u(p) := \left\{ q \in \Sigma \mid \lim_{t \rightarrow -\infty} \varphi_{Z_\Sigma}^{-t}(q) = p \right\}. \tag{3.1}$$

Notice the sign of time in the flow, so that these are the stable/unstable manifolds for the flow of the negative gradient. We further require that (f_Σ, Z_Σ) be a Morse–Smale pair, i.e. that all stable and unstable manifolds of Z_Σ intersect transversally.

The contact distribution ξ defines an Ehresmann connection on the circle bundle $S^1 \hookrightarrow Y \rightarrow \Sigma$. Denote the horizontal lift of Z_Σ by $\pi_\Sigma^* Z_\Sigma \in \mathfrak{X}(Y)$. We fix a Morse function $f_Y: Y \rightarrow \mathbb{R}$ and a gradient-like vector field $Z_Y \in \mathfrak{X}(Y)$ such that (f_Y, Z_Y) is a Morse–Smale pair and the vector field $Z_Y - \pi_\Sigma^* Z_\Sigma$ is vertical (tangent to the S^1 -fibres). Under these assumptions, flow lines of Z_Y project under π_Σ to flow lines of Z_Σ .

Observe that critical points of f_Y must lie in the fibres above the critical points of f_Σ (and these are zeros of Z_Y and Z_Σ , respectively). For notational simplicity, we suppose that f_Y has two critical points in each fibre. In the following, given a critical point for f_Σ , $p \in \Sigma$, we denote the two critical points in the fibre above p by \hat{p} and \check{p} , the fibrewise maximum and fibrewise minimum of f_Y , respectively.

We will denote by $M(p)$ the Morse index of a critical point $p \in \Sigma$ of f_Σ , and by $\hat{M}(\hat{p}) = M(p) + i(\hat{p})$ the Morse index of the critical point $\hat{p} = \hat{p}$ or $\check{p} = \check{p}$ of f_Y . The fibrewise index has $i(\hat{p}) = 1$ and $i(\check{p}) = 0$.

Fix also a Morse function f_W and a gradient-like vector field Z_W on W , such that (f_W, Z_W) is a Morse–Smale pair and Z_W restricted to $[0, \infty) \times Y$ is the constant vector field ∂_r , where r is the coordinate function on the first factor. We denote also by (f_W, Z_W) the Morse–Smale pair that is induced on $X \setminus \Sigma$ by the diffeomorphism in Lemma 2.5. Denote by $M(x)$ the Morse index of $x \in \text{Crit}(f_W)$ with respect to f_W .

We now define the Morse–Bott symplectic chain complex of W and H . Recall that for every $k > 0$, each point in $Y_k := \{b_k\} \times Y \subset \mathbb{R}^+ \times Y$ is the starting point of a 1-periodic orbit of X_H , which covers k times its underlying Reeb orbit.

For each critical point $\tilde{p} = \hat{p}$ or $\tilde{p} = \check{p}$ of f_Y , there is a generator corresponding to the pair (k, \tilde{p}) . We will denote this generator by \tilde{p}_k . The complex is then given by

$$SC_*(W, H) = \left(\bigoplus_{k>0} \bigoplus_{p \in \text{Crit}(f_\Sigma)} \mathbb{Z}\langle \tilde{p}_k, \hat{p}_k \rangle \right) \oplus \left(\bigoplus_{x \in \text{Crit}(f_W)} \mathbb{Z}\langle x \rangle \right). \tag{3.2}$$

Definition 3.3. The *Hamiltonian action* of a loop $\gamma: S^1 \rightarrow \mathbb{R} \times Y$ is

$$\mathcal{A}(\gamma) = \int \gamma^*(\lambda - H dt).$$

In particular, $\mathcal{A}(\gamma) = 0$ for any constant orbit γ , and for any orbit $\gamma_k \in Y_k$, we have

$$\mathcal{A}(\gamma_k) = e^{b_k} h'(e^{b_k}) - h(e^{b_k}) > 0, \tag{3.3}$$

where b_k is the unique solution to the equation $h'(e^{b_k}) = kT_0$, as above. The action of γ_k is the negative of the y -intercept of the tangent line to the graph of h at e^{b_k} . See Fig. 1. The convexity of h implies that $\mathcal{A}(\gamma_k)$ is monotone increasing in k .

3.1. Gradings

We will now define the gradings of the generators. For this, we will assume that (X, Σ, ω) is a monotone triple as in Definition 2.4.

Definition 3.4. For a critical point \tilde{p} of f_Y , and a multiplicity k , we define

$$|\tilde{p}_k| = \tilde{M}(\tilde{p}) + 1 - n + 2 \frac{\tau_X - K}{K} k \in \mathbb{R}, \tag{3.4}$$

where we recall that τ_X is the monotonicity constant of X and $c_1(N\Sigma) = [K\omega_\Sigma]$.

For a critical point x of f_W , we define

$$|x| = n - M(x). \tag{3.5}$$

Finally, for convenience, we introduce an index similar to the SFT grading for the Reeb orbits to which a split Floer cylinder converges at augmentation punctures. If γ is such a Reeb orbit, it is a k -fold cover of a fibre of $Y \rightarrow \Sigma$ for some k . We then define its index to be

$$|\gamma|_0 = -2 + 2 \left(\frac{\tau_X - K}{K} \right) k. \tag{3.6}$$

The justifications of these gradings comes from analyzing the Conley–Zehnder indices of the 1-periodic Hamiltonian orbits. These are defined for Morse non-degenerate Hamiltonian/Reeb orbits, using a trivialization of TW or of $T(\mathbb{R} \times Y)$ over the orbit. See Definition 5.19 for the Morse–Bott analogue, and also [2, Section 3]; [22]. The first key observation is that the Conley–Zehnder index of an orbit only depends on the trivialization of the complex line bundle $L := \Lambda_{\mathbb{C}}^n TW$ over the orbit.

For a constant orbit, we may take a constant trivialization, and applying Definition 5.19, we obtain the Conley–Zehnder index of the constant orbit to be $-n + (2n - M(x)) = n - M(x)$.

A non-constant orbit γ in $\mathbb{R} \times Y$ projects to a point in Σ . From this, we may take a “constant” trivialization of $\gamma^*\xi$ by taking the horizontal lift of a constant trivialization of $T_{\pi(\gamma)}\Sigma$. Then by considering the linearized Hamiltonian flow in the vertical direction, we obtain the Conley–Zehnder index of the corresponding generator \tilde{p}_k to be

$$\text{CZ}_0(\tilde{p}_k) = \tilde{M}(\tilde{p}) + 1 - n. \tag{3.7}$$

The Conley–Zehnder index is computed using the splitting of $T(\mathbb{R} \times Y) = (\mathbb{R} \oplus \mathbb{R}R) \oplus \xi$. The contribution to the index is given by $i(\tilde{p})$ in the vertical $\mathbb{R} \oplus \mathbb{R}R$ direction, and by $M(p) + 1 - n$ in the horizontal ξ direction. Notice

that this index does not explicitly depend on the covering multiplicity k of the orbit.

Notice also that Y may be capped off by the normal disk bundle over Σ , and each orbit bounds a disk fibre. The trivialization induced by the fibre differs from the constant trivialization only in a k -fold winding in the $\mathbb{R}\partial_r \oplus \mathbb{R}R$ direction. The resulting Conley–Zehnder index of \tilde{p}_k for this trivialization induced by the disk fibre is then $\tilde{M}(\tilde{p}) + 1 - n - 2k$. We refer to this trivialization as the *normal bundle trivialization*.

Now, suppose that γ_k is the k -fold cover a simple Reeb orbit γ , and suppose it is contractible in W . Denote by \dot{B} a disk in W whose boundary is γ_k . As we pointed out, γ_k is also the boundary of a k -fold cover of a fibre of the normal bundle to Σ in X . This cover of a fibre can be concatenated with \dot{B} to produce a spherical homology class $B \in H_2^S(X)$ such that the intersection $B \bullet \Sigma = k > 0$. Conversely, note that any $B \in H_2^S(X)$ such that $B \bullet \Sigma = k$ gives rise to a disk \dot{B} bounding γ_k . The complex line bundle $L|_{\dot{B}}$ is trivial since \dot{B} is a disk. This induces a trivialization of L over γ_k , which can be identified with a trivialization of $L^{\otimes k}$ over γ . We refer to this as the *trivialization induced by \dot{B}* .

The relative winding of the trivialization of L over γ_k induced by \dot{B} and the normal bundle trivialization considered above is given by $\langle c_1(L), B \rangle$ since this represents the obstruction to extending the trivialization of L over \dot{B} to all of B . Recall that $c_1(L) = c_1(TX)$.

Putting this together, we obtain that the Conley–Zehnder index of the orbit with respect to the trivialization induced by the disk \dot{B} is given by

$$CZ_H^W(\tilde{p}_k) = \tilde{M}(\tilde{p}) + 1 - n - 2k + 2\langle c_1(TX), B \rangle. \tag{3.8}$$

Finally, we obtain the grading from Eq. (3.4) using the spherical monotonicity of X and the fact that $k = B \bullet [\Sigma] = K\omega(B)$.

$$\begin{aligned} CZ_H^W(\tilde{p}_k) &= \tilde{M}(\tilde{p}) + 1 - n + 2(\tau_X - K)\omega(B) \\ &= \tilde{M}(\tilde{p}) + 1 - n + 2\left(\frac{\tau_X - K}{K}\right)k. \end{aligned}$$

Note that this expression does not depend on the choice of spherical class B .

This formula holds when $k \in \omega(\pi_2(X))$, and we extend it as a fractional grading for all $k \in \mathbb{Z}$. (This corresponds to the fractional SFT grading from [16, Section 2.9.1].)

Finally, we compare our gradings with those described by Seidel [38] and generalized by McLean [30] (the latter considers Reeb orbits, but there is an analogous construction for Hamiltonian orbits) in the case that $c_1(TW) \in H_2(W; \mathbb{Z})$ is torsion, so $Nc_1(TW) = 0$ for a suitable choice of $N > 0$. Note that in our setting, this holds if X is monotone (and not just spherically monotone) and Σ is Poincaré dual to a multiple of ω .

First, we describe the Seidel–McLean approach. Recall that $L = \Lambda^n TW$. We choose a global trivialization of $L^{\otimes N}$ over W . This exists since $c_1(L)$ is N -torsion in $H^2(W; \mathbb{Z})$. Then for any orbit γ , we consider the (complex) rank nN bundle over γ given by $\gamma^*TW \oplus \dots \oplus \gamma^*TW$. Denote this by $\gamma^*(TW^N)$. Notice that the determinant of this bundle is precisely $\gamma^*(L^{\otimes N})$ and thus has

a trivialization already chosen. We now choose any trivialization of TW^N whose determinant matches the trivialization of $\gamma^*(L^{\otimes N})$. For any such trivialization, the linearized flow $d\phi_t \oplus d\phi_t \oplus \dots \oplus d\phi_t: TW^N \rightarrow TW^N$ gives a path of symplectic matrices. This gives a Conley–Zehnder index, which we denote by $CZ_{L^{\otimes N}}(\gamma)$. Notice that this Conley–Zehnder index depends only on the trivialization of $L^{\otimes N}$ and not on the further trivialization of TW^N . The Seidel–McLean grading is then defined to be

$$SM(\gamma) = \frac{1}{N} CZ_{L^{\otimes N}}(\gamma).$$

We now observe some immediate consequences of this construction. First of all, given a null-homologous orbit in W , a capping surface induces a trivialization of L over the orbit, unique up to homotopy, namely a trivialization that extends across the surface. This implies that there is a homotopically unique trivialization of $L^{\otimes N}$ over the orbit, hence the Seidel–McLean grading matches the Conley–Zehnder index induced by the trivialization coming from the capping surface, for null-homologous orbits. (Notice that because $c_1(L)$ is torsion, it does not matter which surface we use.) Thus, if γ is an orbit such that γ_k bounds a disk \dot{B} , (as discussed above, see notably Eq. (3.8)),

$$CZ_H^W(\tilde{p}_k) = SM(\tilde{p}_k).$$

We now consider an orbit γ_m that is an m -fold cover of a simple orbit γ . We may trivialize TW over γ by the constant trivialization discussed previously. This induces a trivialization of L over γ . By taking the N -fold tensor power, we obtain a trivialization of $L^{\otimes N}$ over γ . This has some winding $w \in \mathbb{Z}$ relative to the reference trivialization of $L^{\otimes N}$ over all of W (used above to define the Seidel–McLean grading). Under iteration of the orbit, we obtain a relative winding of mw between the constant trivialization and the reference trivialization. From this, we obtain

$$N CZ_0(\widehat{p}_m) = CZ_{L^{\otimes N}}(\gamma_m) + 2mw.$$

Now, it follows from this that

$$\begin{aligned} CZ_H^W(\tilde{p}_m) &= CZ_0(\tilde{p}_m) + 2 \left(\frac{\tau_X - K}{K} \right) m \\ &= SM(\tilde{p}_m) + 2 \left(\frac{\tau_X - K}{K} + \frac{w}{N} \right) m. \end{aligned}$$

As previously observed, CZ_H^W and SM are equal whenever m is in the image of $c_1(TX): \pi_2(X) \rightarrow \mathbb{Z}$, so it follows that

$$CZ_H^W(\tilde{p}_m) = SM(\tilde{p}_m)$$

for all m , as claimed.

The index (3.6) of the Reeb orbit was introduced for convenience in writing formulas for expected dimensions of moduli spaces. It comes from similar considerations for the Conley–Zehnder index of the Reeb orbit, together with the $n - 3$ shift coming from the grading of SFT. In particular, the Fredholm index for an unparametrized holomorphic plane in W asymptotic to the closed Reeb orbit γ_0 (free to move in its Morse–Bott family) will be given by $|\gamma_0|$.

Remark 3.5. Even though the idea of a fractional grading may seem unnatural at first, it can be thought of as a way of keeping track of some information about the homotopy classes of the Hamiltonian orbits.

Indeed, there can only be a Floer cylinder connecting two Hamiltonian orbits if the difference of their degrees is an integer. Hence, one could write the symplectic homology as a direct sum indexed by the fractional parts of the degrees. Alternatively, one could also decompose it as a direct sum over homotopy classes of Hamiltonian orbits, as done, for instance, in [9].

4. Split symplectic homology moduli spaces

In this section, we describe the moduli spaces of cascades that contribute to the differential in the Morse–Bott split symplectic homology of W .

We also define auxiliary moduli spaces of spherical “chains of pearls” in Σ and in X . (These are familiar objects, reminiscent of ones considered in the literature for Floer homology of compact symplectic manifolds [3, 32, 33].)

4.1. Split Floer cylinders with cascades

We now identify the moduli spaces of *split Floer cylinders with cascades* we use to define the differential on the chain complex (3.2).

First, we define the basic building blocks: split Floer cylinders. We consider two types of basic split Floer cylinders: one connecting two non-constant 1-periodic Hamiltonian orbits and one that connects a non-constant 1-periodic orbit to a constant one (in W).

Notice that we may identify a 1-periodic orbit of H with its starting point, and in this way, we have an identification between Y_k and the set of (parametrized) 1-periodic orbits of H that have covering multiplicity k over the underlying simple periodic orbit.

Definition 4.1. Let $x_{\pm} \in Y_{k_{\pm}}$ be 1-periodic orbits of X_H in $\mathbb{R} \times Y$. A *split Floer cylinder* from x_- to x_+ consists of a map $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$, where $\Gamma = \{z_1, \dots, z_k\} \subset \mathbb{R} \times S^1$ is a (possibly empty) finite subset, together with equivalence classes $[U_i]$ of J_W -holomorphic planes $U_i : \mathbb{C} \rightarrow W$ for each $z_i \in \Gamma$, such that

- \tilde{v} satisfies Floer’s equation

$$\partial_s \tilde{v} + J_Y(\partial_t \tilde{v} - X_H(\tilde{v})) = 0; \tag{4.1}$$

- $\lim_{s \rightarrow \pm\infty} \tilde{v}(s, \cdot) = x_{\pm}$;
- if $\Gamma \neq \emptyset$, then, for conformal parametrizations $\varphi_i : (-\infty, 0] \times S^1 \rightarrow \mathbb{R} \times S^1 \setminus \{z_1, \dots, z_k\}$ of neighbourhoods of the z_i , $\lim_{s \rightarrow -\infty} \tilde{v}(\varphi_i(s, \cdot)) = (-\infty, \gamma_i(\cdot))$, where the γ_i are periodic Reeb orbits in Y ;
- for each Reeb orbit γ_i above, $U_i : \mathbb{C} \rightarrow W$ is asymptotic to $(+\infty, \gamma_i)$.

We consider U_i up to the action of $\text{Aut}(\mathbb{C})$.

Call \tilde{v} the *upper level* of the split Floer cylinder.

See Figs. 4 and 5 for an illustration (ignore the horizontal segments in the figures, which represent gradient flow lines).

Definition 4.2. Let $x_+ \in Y_{k_+}$ for some k_+ , and let $x_- \in W$. A *split Floer cylinder* from x_- to x_+ consists of $\tilde{v}_1 = (b, v) : \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$ (where $\Gamma = \{z_1, \dots, z_k\} \subset \mathbb{R} \times S^1$ is a (possibly empty) finite subset), $\tilde{v}_0 : \mathbb{R} \times S^1 \rightarrow W$, and equivalence classes $[U_i]$ of J_W -holomorphic planes $U_i : \mathbb{C} \rightarrow W$ for each $z_i \in \Gamma$, such that

- \tilde{v}_1 solves Eq. (4.1);
 - \tilde{v}_0 is J_W -holomorphic;
 - $\lim_{s \rightarrow +\infty} \tilde{v}_1(s, \cdot) = x_+$;
 - $\lim_{s \rightarrow -\infty} \tilde{v}_1(s, \cdot) = (-\infty, \gamma(\cdot))$, for some Reeb orbit γ in Y ;
 - $\lim_{s \rightarrow +\infty} \tilde{v}_0(s, \cdot) = (+\infty, \gamma(\cdot))$, where γ is the same Reeb orbit;
 - $\lim_{s \rightarrow -\infty} \tilde{v}_0(s, \cdot) = x_-$;
 - if $\Gamma \neq \emptyset$, then, for conformal parametrizations $\varphi_i : (-\infty, 0] \times S^1 \rightarrow \mathbb{R} \times S^1 \setminus \{z_1, \dots, z_k\}$ of neighbourhoods of the z_i , $\lim_{s \rightarrow -\infty} \tilde{v}(\varphi_i(s, \cdot)) = (-\infty, \gamma_i(\cdot))$, where the γ_i are periodic Reeb orbits in Y ;
 - for each Reeb orbit γ_i above, $U_i : \mathbb{C} \rightarrow W$ is asymptotic to $(+\infty, \gamma_i)$.
- We consider U_i up to the action of $\text{Aut}(\mathbb{C})$.

Call \tilde{v}_1 the *upper level* of this split Floer cylinder.

See Fig. 6 for an illustration.

For the upper levels of split Floer cylinders in $\mathbb{R} \times Y$, we introduce a suitable form of energy, a hybrid between the standard Floer energy and the Hofer energy used in symplectizations. Recall that the Hofer energy of a punctured pseudoholomorphic curve \tilde{u} in the symplectization of Y with contact form α is given by

$$\sup \left\{ \int \tilde{u}^* d(\psi\alpha) \mid \psi : \mathbb{R} \rightarrow [0, 1] \text{ smooth and non-decreasing} \right\}.$$

In a symplectic manifold either compact or convex at infinity, the standard Floer energy of a cylinder $\tilde{v} : \mathbb{R} \times S^1 \rightarrow W$ is given by

$$\int \tilde{v}^* \omega - \tilde{v}^* dH \wedge dt.$$

In our situation, however, the target manifold is $\mathbb{R} \times Y$, which has a concave end. We, therefore, need to combine these two types of energy.

Definition 4.3. Consider a Hamiltonian $H : \mathbb{R} \times Y \rightarrow \mathbb{R}$ so that dH has support in $[R, \infty) \times Y$, for some $R \in \mathbb{R}$.

Let ∂_R be the set of all non-decreasing smooth functions $\psi : \mathbb{R} \rightarrow [0, \infty)$ such that $\psi(r) = e^r$ for $r \geq R$.

The *hybrid energy* E_R of $\tilde{v} : \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$ solving Floer’s equation (4.1) is then given by

$$E_R(\tilde{v}) = \sup_{\psi \in \partial} \int_{\mathbb{R} \times S^1} \tilde{v}^* (d(\psi\alpha) - dH \wedge dt). \tag{4.2}$$

Notice that this is equivalent to partitioning $\mathbb{R} \times S^1 \setminus \Gamma = S_0 \cup S_1$, so that $S_0 = \tilde{v}^{-1}([R, +\infty) \times Y)$ and $S_1 = S \setminus S_0$. Then \tilde{v} has finite hybrid energy if and only if $\tilde{v}|_{S_0}$ has finite Floer energy and $\tilde{v}|_{S_1}$ has finite Hofer energy.

Equivalently, \tilde{v} has finite hybrid energy if and only if the punctures $\{\pm\infty\} \cup \Gamma$ can be partitioned into Γ_F and Γ_C (with $+\infty \in \Gamma_F$, $\Gamma \subset \Gamma_C$ and $-\infty$ either in Γ_F or in Γ_C), such that in a neighbourhood of each puncture in Γ_F , the map \tilde{v} is asymptotic to a Hamiltonian trajectory and in a neighbourhood of each puncture in Γ_C , the map is proper and negatively asymptotic to an orbit cylinder in $\mathbb{R} \times Y$. This follows from a variation on the arguments in [10, Proposition 5.13, Lemma 5.15]. We will use the following notation to denote the Hamiltonian orbits to which such a punctured cylinder \tilde{v} is asymptotic:

$$\begin{aligned} \tilde{v}(+\infty, t) &= \lim_{s \rightarrow \infty} \tilde{v}(s, t) \\ \tilde{v}(-\infty, t) &= \lim_{s \rightarrow -\infty} \tilde{v}(s, t) \quad \text{if } -\infty \in \Gamma_F. \end{aligned}$$

Instead, if $-\infty \in \Gamma_C$, we will write $v(-\infty, t) = \lim_{s \rightarrow -\infty} v(s, t)$ for the Reeb orbit in Y that the curve converges to. Notice that since the cylinder is parametrized, the asymptotic limit is parametrized as well. Since there is an ambiguity of the S^1 parametrization of the Reeb orbits to which v is asymptotic at punctures $z \in \Gamma$, we will avoid using the analogous notation at punctures in Γ .

We now define a *split Floer cylinder with cascades* between two generators of the chain complex (3.2).

Definition 4.4. Fix $N \geq 0$. Let S_0, S_1, \dots, S_N be a collection of connected spaces of orbits, with $S_0 = Y_{k_0}$ or $S_0 = W$, and $S_i = Y_{k_i}$ for $1 \leq i \leq N$. Let (f_i, Z_i) , $i = 0, \dots, N$ be the pair of Morse function and gradient-like vector field of $f_i = f_Y$, $Z_i = Z_Y$ if $S_i = Y_{k_i}$ and $f_i = -f_W$, $Z_i = -Z_W$ if $S_i = W$.

Let x be a critical point of f_0 and y a critical point of f_N (so x and y are generators of the chain complex (3.2)).

A *Floer cylinder with 0 cascades* ($N = 0$), from x to y , consists of a positive gradient trajectory $\nu: \mathbb{R} \rightarrow S_0$, such that $\nu(-\infty) = x$, $\nu(+\infty) = y$ and $\dot{\nu} = Z_0(\nu)$.

A *Floer cylinder with N cascades*, $N \geq 1$, from x to y , consists of the following data:

- (i) $N - 1$ length parameters $l_i > 0, i = 1, \dots, N - 1$;
- (ii) two half-infinite gradient trajectories, $\nu_0: (-\infty, 0] \rightarrow S_0$ and $\nu_N: [0, +\infty) \rightarrow S_N$ with $\nu_0(-\infty) = x$, $\nu_N(+\infty) = y$ and $\dot{\nu}_i = Z_i(\nu_i)$ for $i = 0$ or N ;
- (iii) $N - 1$ gradient trajectories ν_i defined on intervals of length $l_i, \nu_i: [0, l_i] \rightarrow S_i$ for $i = 1, \dots, N - 1$ such that $\dot{\nu}_i = Z_i(\nu_i)$;
- (iv) N non-trivial split Floer cylinders from $\nu_{i-1}(l_{i-1}) \in S_{i-1}$ to $\nu_i(0) \in S_i$, where we take $l_0 = 0$.

In the case of a Floer cylinder with $N \geq 1$ cascades, we refer to the non-trivial Floer cylinders \tilde{v}_i as *sublevels*. Notice that if S_0 (and thus all S_i) is of the form Y_k , all the split Floer cylinders are as in Definition 4.1. If $S_0 = W$, then the bottom-most level is a split Floer cylinder as in Definition 4.2.

See Fig. 2 for a schematic illustration.

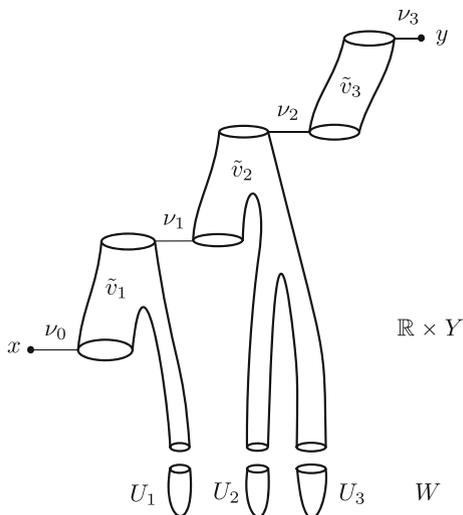


FIGURE 2. A split Floer cylinder with three cascades

Definition 4.5. We refer to the punctures Γ appearing in Definitions 4.1 and 4.2 as *augmentation punctures*. The corresponding J_W holomorphic planes, $U_i: \mathbb{C} \rightarrow W$ are referred to as *augmentation planes*. This terminology is by analogy to linearized contact homology, where rigid planes of this type give an (algebraic) augmentation of the full contact homology differential.

Remark 4.6. Notice that the hybrid energy of each sublevel must be non-negative. Since we require that the sublevels are non-trivial, it follows that any such cascade with collections of orbits $S_i = Y_{k_i}$, $i = 1, \dots, N$ and $S_0 = Y_{k_0}$, or, if $S_0 = W$, with $k_0 = 0$, we must have that the sequence of multiplicities is monotone increasing $k_0 < k_1 < \dots < k_N$.

By a standard SFT-type compactness argument, the Floer–Gromov–Hofer compactification of a moduli space of split Floer cylinders with cascades will consist of several possible configurations. The length parameters can go to 0 or to ∞ (in the latter case, corresponding to a Morse-type breaking of the gradient trajectory). The split Floer cylinders can break at Hamiltonian orbits, thus increasing the number of cascades but with a length parameter of 0. They can also split off a holomorphic building with levels in $\mathbb{R} \times Y$ and in W . We will see that this latter degeneration cannot occur in low-dimensional moduli spaces, at least under our monotonicity assumptions. For energy reasons, these Floer cylinders with cascades will not break at constant Hamiltonian trajectories in $(-\infty, \log 2] \times Y$ (see [13, Lemma 4.8]).

We now define the split Floer differential ∂ on the chain complex (3.2). Given generators x, y , denote by

$$\mathcal{M}_{H,N}(x, y; J_W)$$

the space of split Floer cylinders with N cascades from x to y (with negative end at x and positive end at y).

For $N \geq 1$, this moduli space $\mathcal{M}_{H,N}(x, y; J_W)$ has an \mathbb{R}^N action by domain automorphisms corresponding to \mathbb{R} -translation of the domain cylinders $\mathbb{R} \times S^1$. When $N = 0$, this moduli space is of gradient trajectories, and also admits an \mathbb{R} reparametrization action.

When $|x| = |y| - 1$, these moduli spaces will be rigid modulo these actions. See Remark 5.10.

We now define

$$\begin{aligned} \partial y &= \sum_{|x|=|y|-1} \#(\mathcal{M}_{H,0}(x, y; J_W)/\mathbb{R}) x \\ &+ \sum_{|x|=|y|-1} \sum_{N=1}^{\infty} \#(\mathcal{M}_{H,N}(x, y; J_W/\mathbb{R}^N) x). \end{aligned} \tag{4.3}$$

We call ∂ the *split Floer differential* on (3.2).

5. Transversality for the Floer and holomorphic moduli spaces

In this section, we will build the transversality theory needed for the Floer cylinders with cascades that appear in the split Floer differential as in Eq. (4.3). In the process, we will also discuss transversality for pseudoholomorphic curves in X and in Σ , which will be necessary for the proof of our main result.

5.1. Statements of transversality results

Before stating the main result of this section, we will introduce some definitions allowing us to relate transversality for split Floer cylinders with cascades to transversality problems for spheres in Σ and in X with various constraints.

Lemma 5.1. *Let $\tilde{v}: \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$ be a finite hybrid energy Floer cylinder in $\mathbb{R} \times Y$ (as in Definition 4.3), converging to a Hamiltonian orbit in the manifold Y_+ at $+\infty$ and converging at $-\infty$ either to a Hamiltonian orbit in the manifold Y_- or to a Reeb orbit at $\{-\infty\} \times Y$, and with finitely many punctures at $\Gamma \subset \mathbb{R} \times S^1$ converging to Reeb orbits in $\{-\infty\} \times Y$. Then the projection $\pi_{\Sigma} \circ \tilde{v}$ extends to a smooth J_{Σ} -holomorphic sphere $\pi_{\Sigma} \circ \tilde{v}: \mathbb{C}P^1 \rightarrow \Sigma$.*

Proof. The projection $\pi_{\Sigma} \circ \tilde{v}$ is J_{Σ} -holomorphic on $\mathbb{R} \times S^1$ since H is admissible (as in Definition 3.1). The result now follows from Gromov’s removal of singularities theorem together with the finiteness of the energy of $\pi_{\Sigma} \circ \tilde{v}$. \square

To describe the projection to Σ of the levels of a split Floer cylinder with N cascades that map to $\mathbb{R} \times Y$, we introduce the following:

Definition 5.2. A *chain of pearls* from q to p , where p and q are critical points of f_{Σ} , consists of the following:

- $N \geq 0$ parametrized J_{Σ} -holomorphic spheres w_i in Σ with two distinguished marked points at 0 and ∞ and a possibly empty collection of additional marked points z_1, \dots, z_k on the union of the N domains (distinct from 0 or ∞ in each of the N spherical domains); the spheres are either non-constant or contain at least one additional marked point;

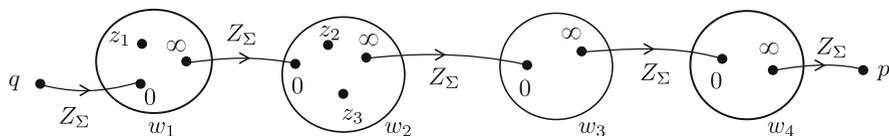


FIGURE 3. A chain of four pearls from q to p with three marked points

- if $N = 0$, an infinite positive flow trajectory of Z_Σ from q to p ; if $N \geq 1$, a half-infinite trajectory of Z_Σ from q to $w_1(0)$, a half-infinite trajectory of Z_Σ from $w_N(\infty)$ to p ;
- if $N \geq 1$, positive length parameters $l_i, i = 1, \dots, N - 1$, so that $\varphi^{l_i}_{Z_\Sigma}(w_i(\infty)) = w_{i+1}(0), i = 1, \dots, N - 1$.

See Fig. 3. If such a chain of pearls is the projection to Σ of the components in $\mathbb{R} \times Y$ is a split Floer cylinder, then the additional marked points in the pseudoholomorphic spheres correspond to augmentation punctures in the Floer cylinders, where they converge to cylinders over Reeb orbits that are capped by planes in W .

Notice that the geometric configuration of two spheres touching at a critical point of f_Σ admits an interpretation as a chain of pearls in Σ since the critical point is the image of any positive length flow line with that initial condition.

Definition 5.3. A *chain of pearls with a sphere in X* from x to p , where x is a critical point of f_W and p is a critical point of f_Σ , consists of the following:

- $N \geq 1$ parametrized J_Σ -holomorphic spheres w_i in Σ with two distinguished marked points at 0 and ∞ and a possibly empty collection of additional marked points z_1, \dots, z_k on the union of the N domains (distinct from 0 or ∞);
- a parametrized non-constant J_X -holomorphic sphere v in X ;
- a half-infinite trajectory of Z_Σ from $w_N(\infty)$ to p , a half-infinite trajectory of $-Z_X$ from x to $v(0)$ (where Z_X is the push-forward of Z_W by the inverse of the map from Lemma 2.5);
- positive length trajectories of Z_Σ from $w_i(\infty)$ to $w_{i+1}(0)$ for $i = 1, \dots, N - 1$;
- the sphere in X touches the first sphere in Σ : $w_1(0) = v(+\infty)$;
- the spheres w_2, \dots, w_N satisfy the stability condition that they are either non-constant or contain at least one of the additional marked points (v is automatically non-constant and w_1 is allowed to be constant and unstable).

Definition 5.4. An *augmented chain of pearls* [or an *augmented chain of pearls with a sphere in X*] is a chain of pearls [or chain of pearls with a sphere in X] together with k equivalence classes $[U_i]$ of J_X -holomorphic spheres $U_i: \mathbb{C}\mathbb{P}^1 \rightarrow X, i = 1, \dots, k$, with the following additional properties:

- for each $z \in \mathbb{C}\mathbb{P}^1, U_i(z) \in \Sigma$ if and only if $z = \infty$;

- if the puncture z_i is in the domain of the holomorphic sphere $w_{j_i} : \mathbb{CP}^1 \rightarrow \Sigma$, then $w_{j_i}(z_i) = U_i(\infty)$;
- each U_i is considered up to the action of $\text{Aut}(\mathbb{CP}^1, \infty) = \text{Aut}(\mathbb{C})$, that is, as an unparametrized sphere.

From Lemmas 5.1 and 2.5 and the fact that the trajectories of Z_Y cover trajectories of Z_Σ , it follows that a Floer cylinder with N cascades projects to a chain of pearls or a chain of pearls with a sphere in X . Additionally, again by Lemma 2.5, if any of the sublevels have augmentation planes, then those correspond to spheres in X passing through Σ at the images of the corresponding marked points in the chain of pearls.

Observe that we allow the sphere w_1 to be unstable in the definition of a chain of pearls in Σ with a sphere in X . The case in which w_1 is a constant curve without marked points corresponds to the situation in which the corresponding Floer cascade contains a non-trivial Floer cylinder \tilde{v}_1 contained in a single fibre of $\mathbb{R} \times Y \rightarrow \Sigma$, and has the asymptotic limits $\tilde{v}_1(+\infty, t)$ on a Hamiltonian orbit and $\tilde{v}_1(-\infty, t)$ on a closed Reeb orbit in $\{-\infty\} \times Y$. The Floer cylinder \tilde{v}_1 in $\mathbb{R} \times Y$ is non-trivial and hence stable, whereas the corresponding sphere w_1 in Σ is unstable. Since we do not quotient by automorphisms (yet), this does not pose a problem. (See Fig. 6 and Proposition 6.3 below, where this situation is analysed.)

Definition 5.5. A chain of pearls in Σ is *simple* if each sphere is either simple (i.e. not multiply covered, [29, Section 2.5]) or is constant, and if the image of no sphere is contained in the image of another. If the chain of pearls has a sphere v in X , we require v to be somewhere injective (but the first sphere in Σ is allowed to be constant, with image contained in the image of v).

An augmented chain of pearls is simple if the chain of pearls is simple and the augmentation spheres in X are somewhere injective, none has image contained in the fixed open neighbourhood $\varphi(\mathcal{U})$ and no sphere in X has image contained in the image of another sphere in X .

Remark 5.6. Recall that a chain of pearls with a sphere in X has a distinguished sphere v in X for which $v(0)$ is on the descending manifold of a critical point x of f_W . By the construction of f_W , this forces the image of v to intersect the complement of the tubular neighbourhood of Σ . As we revisit in Remark 6.7, Fredholm index considerations related to monotonicity will force the augmentation planes/spheres to leave the tubular neighbourhood.

Remark 5.7. Notice that our condition on a simple chain of pearls is slightly different than the condition imposed in [29, Section 6.1], with regard to constant spheres. For a chain of pearls to be simple by our definition, constant spheres may not be contained in another sphere, constant or not. In [29], there is no such condition on constant spheres.

Definition 5.8. Given a finite hybrid energy Floer cylinder with N cascades, we obtain an augmented chain of pearls (possibly with a sphere in X) by the following construction:

- (1) cylinders in $\mathbb{R} \times Y$ are projected to Σ : by Lemma 5.1 these form holomorphic spheres in Σ ;

- (2) planes in W are interpreted as spheres in X by Lemma 2.5;
- (3) flow lines of the gradient-like vector field Z_Y are projected to flow lines of Z_Σ .

We refer to this augmented chain of pearls in Σ (possibly with a sphere in X) as the projection of the Floer cylinder with N cascades.

A finite hybrid energy Floer cylinder with N cascades is *simple* if the projected chain of pearls is simple.

Given generators x, y of the chain complex (3.2), denote by

$$\mathcal{M}_{H,N}^*(x, y; J_W)$$

the space of *simple* split Floer cylinders with N cascades from x to y . Recall that if x or y is in $\mathbb{R} \times Y$, the corresponding generator is described by a critical point \tilde{p} of f_Y (which can be either \tilde{p} or \hat{p}), together with a multiplicity k . If instead, x or y is in W , it corresponds to a critical point of f_W . Note that when both x, y are generators in $\mathbb{R} \times Y$, the moduli space $\mathcal{M}_{H,N}^*(x, y; J_W)$ will depend on J_W only insofar as augmentation planes appear, otherwise it depends only on J_Y .

Proposition 5.9. *There exists a residual set $\mathcal{J}_W^{reg} \subset \mathcal{J}_W$ of almost complex structures such that for each $J_W \in \mathcal{J}_W^{reg}$, $\mathcal{M}_{H,N}^*(x, y; J_W)$ is a manifold.*

If $N = 0$, and thus x, y are generators in $\mathbb{R} \times Y$, then $x = \tilde{q}_k, y = \tilde{p}_k$ for the same multiplicity k , and

$$\dim_{\mathbb{R}} \mathcal{M}_{H,0}^*(\tilde{q}_k, \tilde{p}_k; J_W) = |\tilde{p}_k| - |\tilde{q}_k|.$$

If $N \geq 1$, and x, y are generators in $\mathbb{R} \times Y$, then $x = \tilde{q}_{k-}, y = \tilde{p}_{k+}$ and

$$\dim_{\mathbb{R}} \mathcal{M}_{H,N}^*(\tilde{q}_{k-}, \tilde{p}_{k+}; J_W) = |\tilde{p}_{k+}| - |\tilde{q}_{k-}| + N - 1.$$

Finally, if $x \in W$ and $y \in \mathbb{R} \times Y$, then $x \in \text{Crit}(f_W), y = \tilde{p}_k$ and

$$\dim_{\mathbb{R}} \mathcal{M}_{H,N}^*(x, \tilde{p}_k; J_W) = |\tilde{p}_k| - |x| + N.$$

Furthermore, the image $P(\mathcal{J}_W^{reg}) \subset \mathcal{J}_\Sigma$ (recall Definition 2.7) is residual and consists of almost complex structures that are regular for simple pseudo-holomorphic spheres in Σ .

The two different formulas involving N reflect the fact that N counts the number of cylinders in $\mathbb{R} \times Y$. In the case of a Floer cascade that descends to W , there are, therefore, $N + 1$ cylinders in the cascade.

Remark 5.10. These index formulas justify that the moduli spaces are rigid (modulo their \mathbb{R}, \mathbb{R}^N and \mathbb{R}^{N+1} actions) when the index difference is 1, which then justifies the definition of the differential given in Eq. (4.3). Indeed, observe that the case $N = 0$ corresponds to a pure Morse configuration and does not depend on any almost complex structure. We count rigid flow lines modulo the \mathbb{R} action, and thus require $|y| - |x| = 1$. For generators x, y in $\mathbb{R} \times Y$, we consider these N cylinders modulo the \mathbb{R} action on each one, giving an \mathbb{R}^N action. From this, a rigid configuration has $|y| - |x| + N - 1 = N$. For the case with $x \in W$, we have $N + 1$ cylinders in the Floer cascade, so we have a rigid configuration modulo the \mathbb{R}^{N+1} action when $N + 1 = |y| - |x| + N$.

The split Floer differential ∂ , introduced in Eq. (4.3) was defined by counting elements in $\mathcal{M}_{H,N}(x, y; J_W)$. We will see in Propositions 6.2 and 6.3 that our monotonicity assumptions imply that this is equivalent to counting simple configurations in $\mathcal{M}_{H,N}^*(x, y; J_W)$.

The rest of this section will be devoted to the proof of Proposition 5.9. It will proceed in the following steps:

- Section 5.2 describes the Fredholm setup for Floer cylinders with cascades. On a first reading, it can be skipped and used as a reference. In Sect. 5.2.1, we discuss the necessary function spaces and linear theory for the Morse–Bott problems. Then Sect. 5.2.2 splits the linearization of the Floer operator in such a way as to split the transversality problem into two problems. The first is a Cauchy–Riemann operator acting on sections of a complex line bundle, and it is transverse for topological reasons (automatic transversality). The second is a transversality problem for a Cauchy–Riemann operator in Σ .
- Section 5.3 adapts the transversality arguments from [29] to obtain transversality for chains of pearls in Σ .
- Section 5.4 shows transversality for the components of the cascades contained in W . This problem is translated into the equivalent problem of obtaining transversality for spheres in X with order of contact conditions at Σ , together with evaluation maps. The main technical point is an extension of the transversality results from [12].
- Finally, Sect. 5.5 uses the splitting from Sect. 5.2.2 to lift the transversality results in Σ to obtain transversality for Floer cylinders with cascades, and to finish the proof of Proposition 5.9.

5.2. A Fredholm theory for Floer cylinders with cascades

5.2.1. A Fredholm theory for Morse–Bott asymptotics. In this section, we collect some facts about Cauchy–Riemann operators on Hermitian vector bundles over punctured Riemann surfaces, specifically in the context of *degenerate* asymptotic operators. These facts can mostly be found in the literature, but not in a unified way. The main reference for these results is [37]. Additional references include [1, Sections 2.1–2.3], [7, 25, 37, 41].

We begin by introducing some Sobolev spaces of sections of appropriate bundles. Let $\Gamma \subset \mathbb{R} \times S^1$ be a finite set of punctures and denote $\mathbb{R} \times S^1 \setminus \Gamma$ by \dot{S} . Write $\Gamma_+ = \{+\infty\}$ and $\Gamma_- = \{-\infty\} \cup \Gamma$. Consider, for each puncture $z \in \Gamma$, exponential cylindrical polar coordinates of the form $(-\infty, -1] \times S^1 \rightarrow \mathbb{R} \times S^1 \setminus \Gamma: \rho + i\eta \mapsto z_0 + \epsilon e^{2\pi(\rho+i\eta)}$. Choose $\epsilon > 0$ sufficiently small; these are embeddings and that the image of these embeddings for any two different punctures are disjoint.

Let $E \rightarrow \dot{S}$ be a (complex) rank n Hermitian vector bundle over \dot{S} together with a preferred set of trivializations in a small neighbourhood of $\Gamma \cup \{\pm\infty\}$. While the bundle E over \dot{S} is trivial if there is at least one puncture, this is no longer the case once we specify these preferred trivializations near $\Gamma \cup \{\pm\infty\}$. We, therefore, associate a first Chern number to this bundle relative to the asymptotic trivializations. There are several equivalent definitions. One approach is to consider the complex determinant bundle $\Lambda_{\mathbb{C}}^n E$. The

trivialization of E at infinity gives a trivialization of this determinant bundle at infinity, and we can now count zeros of a generic section of $\Lambda_{\mathbb{C}}^n E$ that is constant (with respect to the prescribed trivializations) near the punctures. We denote this Chern number by $c_1(E)$, but emphasize that it depends on the choice of these trivializations near the punctures.

Since we cannot specify where an augmentation puncture appears when we stretch the neck on a Floer cylinder, we should have the punctures in Γ free to move on the domain $\mathbb{R} \times S^1$. This creates a problem when we try to linearize the Floer operator in a family of domains where the positions of the punctures are not fixed. We will instead consider a $2\#\Gamma$ parameter family of almost complex structures on $\mathbb{R} \times S^1$, but fix the location of the punctures. Specify a fixed collection Γ of punctures on $\mathbb{R} \times S^1$ and, for any other collection of augmentation punctures, choose an isotopy with compact support from the new punctures to the fixed ones. We take the push-forward of the standard complex structure in $\mathbb{R} \times S^1$ by the final map of the isotopy, to produce a family of complex structures on $\mathbb{R} \times S^1$, which can be assumed standard near Γ and outside of a compact set.

For each $z \in \Gamma$, let $\beta_z : \dot{S} \rightarrow [0, +\infty)$ be a function supported in a small neighbourhood of z , with $\beta_z(\rho, \eta) = -\rho$ near the puncture (where (ρ, η) are cylindrical polar coordinates near z , as above). Similarly, let $\beta_+ : \mathbb{R} \times S^1 \rightarrow [0, +\infty)$ be supported in a region where s is sufficiently large and $\beta_+(s, t) = s$ for s large enough. Let $\beta_- : \mathbb{R} \times S^1 \rightarrow [0, +\infty)$ have support near $-\infty$, and $\beta_-(s, t) = -s$ for s sufficiently small.

In many situations, it will be convenient to consider the function

$$\beta := \sum_{z \in \Gamma} \beta_z + \beta_- + \beta_+. \tag{5.1}$$

Finally, on the punctured cylinder \dot{S} , we take the measure induced by an area form on \dot{S} that has the form $ds \wedge dt$ for $|s|$ large and that has the form $d\rho \wedge d\eta$ in the cylinder polar coordinates near each puncture in Γ . Notice that pairing this with the domain complex structure induces a metric on \dot{S} for which the vector field ∂_η , defined near a puncture in Γ by the exponential cylindrical polar coordinates, has norm comparable to 1.

Given a vector of weights $\delta : \Gamma \cup \{\pm\infty\} \rightarrow \mathbb{R}$, we define $W^{1,p,\delta}(\dot{S}, E)$ to be the space of sections u of E for which

$$u_{e^{\sum \delta_z \beta_z + \delta_- \beta_- + \delta_+ \beta_+}} \in W^{1,p}(\dot{S}, E)$$

(with respect to the measure and metric described above). Note that these sections decay exponentially fast at the punctures where $\delta > 0$ and are allowed to have exponential growth at punctures where $\delta < 0$. We can similarly define $L^{p,\delta}(\dot{S}, E)$. While these definitions involve making various choices, the resulting metrics are strongly equivalent. In practice, we will typically take $p > 2$ to obtain continuity of the sections. By a similar construction, we may define $W^{m,p,\delta}$ as well.

We will say that a differential operator $D : \Gamma(E) \rightarrow \Gamma(\Lambda^{0,1} T^* \dot{S} \otimes E)$ is a Cauchy–Riemann operator if it is a real linear Cauchy–Riemann operator [29, Definition C.1.5] such that, near $\pm\infty$, it takes the form:

$$(D\sigma) \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \sigma + J(s, t) \frac{\partial}{\partial t} \sigma + A(s, t) \sigma, \tag{5.2}$$

where $J(s, t)$ is a smooth function on $\mathbb{R}^\pm \times S^1$ with values in almost complex structures on \mathbb{C}^n compatible with the standard symplectic form, and $A(s, t)$ takes values in real matrices on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We further impose that these functions converge uniformly as $s \rightarrow \pm\infty$, $J(s, t) \rightarrow J_z(t)$ and $A(s, t) \rightarrow A_z(t)$, where $A_z(t)$ is a loop of self-adjoint matrices. We impose the same conditions near punctures $z \in \Gamma$, using the local coordinates (ρ, η) instead of (s, t) in (5.2).

A Cauchy–Riemann operator $D: \Gamma(E) \rightarrow \Gamma(\Lambda^{0,1}T^*\dot{S} \otimes E)$ acting on smooth sections induces an operator on various Sobolev spaces of sections. Of particular importance for us will be that for any vector of weights $\delta: \Gamma \cup \{\pm\infty\} \rightarrow \mathbb{R}$, D induces an operator

$$D: W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

as well as operators $D: W^{k,p,\delta} \rightarrow W^{k-1,p,\delta}$. In the following, we will not emphasize the distinction and refer to the operators $W^{1,p,\delta} \rightarrow L^{p,\delta}$ as Cauchy–Riemann operators. For generic weight vectors, these operators will be Fredholm as formulated precisely in Theorem 5.16 below.

Associated with a Cauchy–Riemann operator D , we obtain *asymptotic operators* at each puncture in $\Gamma \cup \{\pm\infty\}$ by

$$\mathbf{A}_z := -J_z(t) \frac{d}{dt} - A_z(t).$$

This is a densely defined unbounded self-adjoint operator on $L^2(S^1, \mathbb{R}^{2n})$. Let $\sigma(\mathbf{A}_z) \subset \mathbb{R}$ denote its spectrum. This will consist of a discrete set of eigenvalues. If an asymptotic operator \mathbf{A}_z does not have 0 in its spectrum, we say the asymptotic operator is non-degenerate. If all the asymptotic operators are non-degenerate, we say D itself is non-degenerate.

Note that we obtain a path of symplectic matrices associated with the asymptotic operator \mathbf{A}_z by finding the fundamental matrix Φ to the ODE $\frac{d}{dt}x = J_z(t)A_z(t)x$. The asymptotic operator is non-degenerate if and only if the time-1 flow of the ODE does not have 1 in the spectrum. We will consider a description of the Conley–Zehnder index in terms of properties of the asymptotic operator itself [24, Lemmas 3.4, 3.5, 3.6, 3.9].

Remark 5.11. An asymptotic operator induces a path of symplectic matrices, and this identification (understood correctly) is a homotopy equivalence. This will allow us to associate a Conley–Zehnder index to a periodic orbit of a Hamiltonian vector field, given a trivialization of the tangent bundle along the orbit. To do so, we take the linearized flow map, which defines a path $\Phi: [0, 1] \rightarrow \text{Sp}(2n)$ with respect to the fixed trivialization. If we fix a path of almost complex structures, this path of symplectic matrices satisfies an ODE as in the previous paragraph, which in turn specifies an asymptotic operator. The Conley–Zehnder index of the Hamiltonian orbit is by definition the Conley–Zehnder index of this asymptotic operator. This is homotopic to the asymptotic operator coming from the linearized Floer operator.

Proposition 5.12. *Suppose \mathbf{A}_z is non-degenerate and E is a rank 1 vector bundle.*

If $u: S^1 \rightarrow \mathbb{C}$ is an eigenfunction of \mathbf{A}_z corresponding to the eigenvalue λ , it must be nowhere vanishing. The winding number of u is then the degree of the map $\frac{u}{|u|}: S^1 \rightarrow S^1$. Any two eigenfunctions corresponding to the same eigenvalue λ have the same winding number. This is then referred to as the winding number of the eigenvalue, and is denoted by $w(\lambda)$.

The function $w: \sigma(\mathbf{A}_z) \rightarrow \mathbb{Z}$ is non-decreasing in λ and is surjective. If λ_{\pm} are eigenvalues so that $\lambda_- < 0 < \lambda_+$ and there are no eigenvalues in the interval (λ_-, λ_+) , then

$$\text{CZ}(\mathbf{A}_z) = w(\lambda_-) + w(\lambda_+).$$

□

This formulation will be the most useful for our calculations. Furthermore, in the case of a higher rank bundle, we use the axiomatic description, see, for instance, [24, Theorem 3.1] to observe that $\text{CZ}(\mathbf{A}_z)$ is invariant under deformations for which 0 is never in the spectrum, and that if the operator can be decomposed as the direct sum of operators, then the Conley–Zehnder index is additive.

The following computation is useful at several points in the paper. It can often be combined with Proposition 5.12 to compute Conley–Zehnder indices of interest.

Lemma 5.13. *Given a constant $C \geq 0$, the spectrum $\sigma(A_C)$ of the operator*

$$\mathbf{A}_C := -i \frac{d}{dt} - \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} : W^{1,p}(S^1, \mathbb{C}) \rightarrow L^p(S^1, \mathbb{C})$$

is the set

$$\left\{ \frac{1}{2} \left(-C - \sqrt{C^2 + 16\pi^2 k^2} \right) \mid k \in \mathbb{Z} \right\} \cup \left\{ \frac{1}{2} \left(-C + \sqrt{C^2 + 16\pi^2 k^2} \right) \mid k \in \mathbb{Z} \right\}.$$

If λ is an eigenvalue associated with $k \in \mathbb{Z}$, then the winding number of the corresponding eigenvector is $|k|$ if $\lambda \geq 0$ and $-|k|$ if $\lambda \leq 0$. If $C = 0$, then all eigenvalues have multiplicity 2 (see Table 1). If $C > 0$, then the same is true except for the eigenvalues $-C$ and 0, corresponding to $k = 0$ above, both of which have multiplicity 1 (see Table 2).

In particular, the $\sigma(\mathbf{A}_0) = 2\pi\mathbb{Z}$ and the winding number of $2\pi k$ is k .

Proof. An eigenvector $v: S^1 \rightarrow \mathbb{C}$ of \mathbf{A}_C with eigenvalue λ solves the equation

$$-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{v} - \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} v = \lambda v \Leftrightarrow \dot{v} = \begin{pmatrix} 0 & -\lambda \\ C + \lambda & 0 \end{pmatrix} v.$$

Computing the eigenvalues of the matrix on the right, and requiring that they be of the form $2\pi ik$, $k \in \mathbb{Z}$ (since $v(t+1) = v(t)$), yields the result. □

TABLE 1. Eigenvalues of \mathbf{A}_0

eigenvalues	...	-4π	-2π	0	2π	4π	...
multiplicities	...	2	2	2	2	2	...
winding numbers	...	-2	-1	0	1	2	...

TABLE 2. Eigenvalues of \mathbf{A}_C in increasing order, if $C > 0$

eigenvalues	...	$\frac{1}{2}(-C - \sqrt{C^2 + 16\pi^2})$	$-C$	0	$\frac{1}{2}(-C + \sqrt{C^2 + 16\pi^2})$...
multiplicities	...	2	1	1	2	...
winding #s	...	-1	0	0	1	...

Corollary 5.14. *Take $C \geq 0$ and $\delta > 0$ such that $[-\delta, \delta] \cap \sigma(\mathbf{A}_C) = \{0\}$. Then*

$$\text{CZ}(\mathbf{A} + \delta) = \begin{cases} 0 & \text{if } C > 0 \\ -1 & \text{if } C = 0 \end{cases} \quad \text{and} \quad \text{CZ}(\mathbf{A}_C - \delta) = 1.$$

For any $n \geq 0$, taking

$$-i \frac{d}{dt} : W^{1,p}(S^1, \mathbb{C}^n) \rightarrow L^p(S^1, \mathbb{C}^n),$$

we have

$$\text{CZ} \left(-i \frac{d}{dt} \pm \delta \right) = \mp n.$$

Proof. The case $n = 1$ follows from Proposition 5.12 and Lemma 5.13. The case of general n uses the additivity of CZ under direct sums. \square

Definition 5.15. A key observation for our computations of Fredholm indices (as noted, for instance, in [25]) is that a Cauchy–Riemann operator

$$D : W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1} T^* \dot{S} \otimes E)$$

with asymptotic operators \mathbf{A}_z is conjugate to the Cauchy–Riemann operator

$$D^\delta : W^{1,p}(\dot{S}, E) \rightarrow L^p(\dot{S}, \Lambda^{0,1} T^* \dot{S} \otimes E)$$

$$D^\delta = e^{\sum \delta_z \beta_z(s)} D e^{-\sum \delta_z \beta_z(s)}.$$

This has asymptotic operators $\mathbf{A}_z^\delta = \mathbf{A}_z \pm \delta_z$, which are non-degenerate and where the sign is positive at positive punctures and negative at negative punctures. We refer to these as the δ -perturbed asymptotic operators.

Notice that the operator D^δ depends on the choice of cut-off functions $\beta_z, z \in \Gamma \cup \{\pm\infty\}$. A different choice of cut-off function will give an operator that differs only by a compact operator. This is thus of secondary importance for what we discuss here.

Note that with the sign conventions that we have chosen, a positive weight $\delta_z > 0$ always corresponds to the constraint of exponential decay at the puncture. A negative weight $\delta_z < 0$ always corresponds to allowing exponential growth.

Theorem 5.16. *Let $\delta : \Gamma \cup \{\pm\infty\} \rightarrow \mathbb{R}$ such that $-\delta_z \notin \sigma(\mathbf{A}_z)$ for positive punctures $z \in \Gamma_+$ and such that $+\delta_z \notin \sigma(\mathbf{A}_z)$ for negative punctures $z \in \Gamma_-$.*

Then the Cauchy–Riemann operator

$$D : W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

with asymptotic operators \mathbf{A}_z , $z \in \Gamma \cup \{\pm\infty\}$ is Fredholm and its index is given by

$$\text{Ind}(D, \delta) = n\chi_{\dot{S}} + 2c_1(E) + \sum_{z \in \Gamma_+} \text{CZ}(\mathbf{A}_z + \delta_z) - \sum_{z \in \Gamma_-} \text{CZ}(\mathbf{A}_z - \delta_z).$$

This observation about the conjugation of the weighted operator to the non-degenerate case, combined with Riemann–Roch for punctured domains (see, for instance, [25, Theorem 2.8], [37, Theorem 3.3.11], [42, Theorem 5.4]) gives the following.

Now, a useful fact for us is a description of how the Conley–Zehnder index changes as a weight crosses the spectrum of the operator:

Lemma 5.17. *Let $\delta > 0$ with $[-\delta, +\delta] \cap \sigma(\mathbf{A}_z) = \{0\}$. Then*

$$\text{CZ}(\mathbf{A}_z - \delta) - \text{CZ}(\mathbf{A}_z + \delta) = \dim(\ker \mathbf{A}_z).$$

□

For a proof (using the spectral flow idea of [35]), see, for instance, [40, Proposition 4.5.22].

To obtain a result that is useful for our moduli spaces of cascades asymptotic to Morse–Bott families of orbits, we consider the following modification of our function spaces.

To each puncture, we associate a subspace of the kernel of the corresponding asymptotic operator, which we denote by $V_z, z \in \Gamma, V_-, V_+$ and write \mathbf{V} for this collection. Then for each puncture $z \in \Gamma$ and also $\pm\infty$, we associate a smooth bump function μ_z, μ_{\pm} , supported near and identically 1 even nearer to its puncture. We then define

$$\begin{aligned} W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E) &= \{u \in W_{\text{loc}}^{1,p}(\dot{S}, E) \mid \exists c_z \in V_z, z \in \Gamma, c_- \in V_-, c_+ \in V_+ \\ &\text{such that } u - \sum c_z \mu_z - c_- \mu_- - c_+ \mu_+ \in W^{1,p,\delta}(\dot{S}, E)\}. \end{aligned} \tag{5.3}$$

We remark that we are using the asymptotic cylindrical coordinates near Γ and the asymptotic trivialization of E to define the local sections $c_z \mu_z$.

In this paper, we are primarily concerned with Cauchy–Riemann operators defined on $\dot{S} = \mathbb{R} \times S^1$ and on $\dot{S} = \mathbb{R} \times S^1 \setminus \{P\}$ (a cylinder with one additional negative puncture). In the case of $\mathbb{R} \times S^1$, we will write $\mathbf{V} = (V_-; V_+)$, and in the case of $\mathbb{R} \times S^1 \setminus \{P\}$, we will write $\mathbf{V} = (V_-, V_P; V_+)$. (The negative punctures are enumerated first, and separated from the positive puncture by a semicolon.)

Observe that since the vector spaces V_z are in the kernel of the corresponding asymptotic operators, for any choice of \mathbf{V} and any vector of weights δ , we have that the Cauchy–Riemann operator D can be extended to

$$D : W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E).$$

Let $\dim V_z$ denote the dimension of the vector space V_z and let $\text{codim } V_z = \dim(\ker \mathbf{A}_z / V_z)$. Combining Theorem 5.16 with Lemma 5.17, we have

Theorem 5.18. *Let $\delta > 0$ be sufficiently small that, for each puncture $z \in \Gamma \cup \{\pm\infty\}$, $[-\delta, \delta] \cap \sigma(\mathbf{A}_z) \subset \{0\}$. For each $z \in \Gamma$, fix a subspace $V_z \subset \ker \mathbf{A}_z$. Denote the collection of all V_z by \mathbf{V} .*

Then the operator

$$D: W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1} T^* \dot{S} \otimes E)$$

is Fredholm, and its Fredholm index is given by

$$\begin{aligned} \text{Ind}(D) &= n\chi_{\dot{S}} + 2c_1(E) + \sum_{z \in \Gamma_+} (\text{CZ}(\mathbf{A}_z + \delta) + \dim(V_z)) \\ &\quad - \sum_{z \in \Gamma_-} (\text{CZ}(\mathbf{A}_z + \delta) + \text{codim}(V_z)). \end{aligned}$$

In applications where there are Morse–Bott manifolds of orbits, we will typically take V_z to be the tangent space to the descending manifold of a critical point p_z on the manifold of orbits at a positive puncture, and V_z will be the tangent space to ascending manifold of a critical point p_z at a negative puncture. In either case, the contribution to $\text{Ind}(D)$ of $\dim V_z$ or of $\text{codim } V_z$ will be the Morse index of the appropriate critical point. This motivates the following definition.

Definition 5.19. Let $\delta > 0$ be sufficiently small. If p_z is a critical point of an auxiliary Morse function on the manifold of orbits associated with z , then the Conley–Zehnder index of the pair (\mathbf{A}_z, p_z) is

$$\text{CZ}(\mathbf{A}_z, p_z) = \text{CZ}(\mathbf{A}_z + \delta) + M(p_z),$$

where $M(p_z)$ is the Morse index of p_z .

In this case, we can write the Fredholm index as

$$\text{Ind}(D) = n\chi_{\dot{S}} + 2c_1(E) + \sum_{z \in \Gamma_+} \text{CZ}(\mathbf{A}_z, p_z) - \sum_{z \in \Gamma_-} \text{CZ}(\mathbf{A}_z, p_z).$$

We conclude with a lemma that is particularly useful when applying the automatic transversality result [41, Proposition 4.22]. The lemma states that the Fredholm index of an operator with a small negative weight at a puncture is the same as that of the corresponding operator with a small positive weight at that puncture, if the puncture is decorated with the kernel of the corresponding asymptotic operator. The former indices are used in [41, Proposition 4.22], whereas the latter can be computed using Theorem 5.18. Additionally, the latter arises naturally in the linearization of the nonlinear problem.

We first learned this result from Wendl [40]. We give a proof of this formulation since it is slightly stronger than what we have found in the literature (and is still not as strong as can be proved.)

Lemma 5.20. *Let D be a Cauchy–Riemann operator. Fix a puncture $z_0 \in \Gamma \cup \{\pm\infty\}$.*

Let δ and δ' be vectors of sufficiently small weights so that the differential operator induces a Fredholm operator on $W^{1,p,\delta}$ and on $W^{1,p,\delta'}$, and $\delta_{z_0} > 0$ and $\delta'_{z_0} < 0$, the interval $[\delta'_{z_0}, \delta_{z_0}] \cap \sigma(\mathbf{A}_{z_0}) = \{0\}$, and for each $z \in \Gamma \cup \{\pm\infty\}$ with $z \neq z_0$, the weights $\delta_z = \delta'_z$.

Let \mathbf{V} be the trivial vector space at each puncture other than z_0 and let V_{z_0} be the kernel of the asymptotic operator \mathbf{A}_{z_0} at z_0 .

Then the induced operators

$$D_\delta : W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

$$D_{\delta'} : W^{1,p,\delta'}(\dot{S}, E) \rightarrow L^{p,\delta'}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

have the same Fredholm index and their kernels and cokernels are isomorphic.

Proof. The main idea of the lemma is contained in [40, Proposition 4.5.22], which contains a proof of the equality of Fredholm indices. See also the very closely related [43, Proposition 3.15].

Note that $W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E)$ is a subspace of $W^{1,p,\delta'}(\dot{S}, E)$, and thus the kernel of D_δ is contained in the kernel of $D_{\delta'}$.

Now, by a linear version of the analysis done in [25, 39], any element of the kernel of $D_{\delta'}$ converges exponentially fast at z_0 to an eigenfunction of the asymptotic operator, with exponential rate governed by the eigenvalue (in this case 0). Therefore, any element of the kernel of $D_{\delta'}$ must converge exponentially fast to an element of the kernel of the asymptotic operator at z_0 . Hence, the kernel of $D_{\delta'}$ is contained in the kernel of D_δ .

We conclude that the kernels of the two operators may be identified. Since their Fredholm indices are the same, their cokernels are also isomorphic. \square

5.2.2. The linearization at a Floer solution. The first step in the proof of Proposition 5.9 is to set up the appropriate Fredholm problem. Given a Floer solution $\tilde{v} : \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$, we consider exponentially weighted Sobolev spaces of sections of the pull-back bundle $\tilde{v}^*T(\mathbb{R} \times Y)$ since the asymptotic limits are (Morse–Bott) degenerate. For $\delta > 0$, we denote by $W^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, v^*T(\mathbb{R} \times Y))$ the space of sections that decay exponentially like $e^{-\delta|s|}$ near the punctures (also in cylindrical coordinates near the punctures Γ), as in the previous section.

We similarly define $W^{m,p,\delta}$ sections with exponential decay/growth. The following results will not depend on m except in the case of jet conditions considered in Sect. 5.4, where m will need to be sufficiently large that the order of contact condition can be defined.

To consider a parametric family of punctured cylinders in which the asymptotic limits move in their Morse–Bott families, we let \mathbf{V} be a collection of vector spaces, associating with each puncture $z \in \Gamma \cup \{\pm\infty\}$ a vector subspace V_z of the tangent space to the corresponding Morse–Bott family of orbits. For $\delta > 0$, we then consider the space of sections $W_{\mathbf{V}}^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, v^*T(\mathbb{R} \times Y))$ that converge exponentially at each puncture z to a vector in the corresponding vector space V_z .

Remark 5.21. In this paper, we will not always be careful to specify how small δ has to be. It is worth pointing out that there is no value of δ that works for all moduli spaces. The reason is that we need $|\delta|$ to be smaller than the absolute value of all eigenvalues in the spectra of the relevant linearized operators. Lemma 5.13 computes the spectrum of a number of these relevant asymptotic operators, and as we see in Table 2, the smallest positive eigenvalue $\frac{1}{2}(-C + \sqrt{C^2 + 16\pi^2})$ becomes arbitrarily small as $C \rightarrow \infty$. As will become clear from Lemma 5.22 and Eq. (5.5), the relevant value for C here is $h''(e^{bk})e^{bk}$, which can become arbitrarily large as the multiplicity $k \rightarrow \infty$. Since the relevant moduli spaces in the differential involve connecting orbits of bounded multiplicities, for any given moduli space, we may choose δ sufficiently small.

We now adapt an observation first used in [6, 15] to show that the linearization of the Floer operator is upper triangular with respect to the splitting of $T(\mathbb{R} \times Y)$ as $(\mathbb{R} \oplus \mathbb{R}R) \oplus \xi$. We then describe the non-zero blocks in this upper triangular presentation of the operator. The two diagonal terms are of special importance: one will be a Cauchy–Riemann operator acting on sections of a complex line bundle, and the other can be identified with the linearization of the Cauchy–Riemann operator for spheres in Σ .

We now explain this construction in more detail. Let $\tilde{v}: \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$ be a Floer solution with punctures Γ . The Hamiltonian need not be admissible, but needs to be radial (i.e. depending only on r , the symplectization variable). The almost complex structure J_Y is assumed to be admissible. We consider three possible cases for the asymptotics of such a curve.

In the first case, \tilde{v} is asymptotic to a closed Hamiltonian orbit at $\tilde{v}(+\infty, t)$, to a closed Hamiltonian orbit at $\tilde{v}(-\infty, t)$, and with negative ends converging to Reeb orbits at the punctures in Γ . The second case has \tilde{v} asymptotic to a closed Hamiltonian orbit at $\tilde{v}(+\infty, t)$, but with negative ends converging to Reeb orbits in $\{-\infty\} \times Y$ at $\{-\infty\} \cup \Gamma$. These two cases correspond to an upper level of a split Floer cylinder as in Definitions 4.1 and 4.2, respectively.

The third case we consider is most directly applicable to studying holomorphic curves in $\mathbb{R} \times Y$: \tilde{v} has a positive cylindrical end at $+\infty$ converging to a Reeb orbit in $\{+\infty\} \times Y$, and has negative cylindrical ends at the punctures $\{-\infty\} \cup \Gamma$. For such a curve, we may assume that H is identically 0, and thus this example includes J_Y -holomorphic curves. This is of independent interest, and is useful in [13]. Part of this was sketched in [16, Section 2.9.2].

Let $w = \pi_\Sigma \circ \tilde{v}: \mathbb{C}\mathbb{P}^1 \rightarrow \Sigma$ be the smooth extension of the projection of \tilde{v} to the divisor (as given by Lemma 2.5). The linearized projection $d\pi_\Sigma$ induces an isomorphism of complex vector bundles

$$\tilde{v}^*(T(\mathbb{R} \times Y)) \cong (\mathbb{R} \oplus \mathbb{R}R) \oplus w^*T\Sigma.$$

To see this, note that for each point $p \in Y$, $d\pi_\Sigma$ induces a symplectic isomorphism $(\xi_p, d\alpha) \cong (T_{\pi_\Sigma(p)}\Sigma, K\omega_\Sigma)$. By the Reeb invariance of the almost complex structure (and thus S^1 -invariance under rotation in the fibre), this then gives a complex vector bundle isomorphism.

Let \mathbf{V} associate with each puncture $z \in \Gamma \cup \{\pm\infty\}$ the tangent space to Y if the corresponding limit of \tilde{v} is a closed Hamiltonian orbit and the tangent space to $\mathbb{R} \times Y$ if the corresponding limit of \tilde{v} is a closed Reeb orbit. As will be clearer shortly, this is associating with each puncture the entirety of the kernel of the corresponding asymptotic operator.

Let

$$D_{\tilde{v}}: W_{\mathbf{V}}^{1,p,\delta}(\tilde{v}^*T(\mathbb{R} \times Y)) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), \tilde{v}^*T(\mathbb{R} \times Y))) \tag{5.4}$$

be the linearization of the nonlinear Floer operator at the solution \tilde{v} , for $\delta > 0$ sufficiently small. The vector spaces \mathbf{V} correspond to allowing the asymptotic limits to move in their Morse–Bott families. We have then a linearized evaluation map at the punctures with values in $\bigoplus_{z \in \{\pm\infty\} \cup \Gamma} V_z$. Let

$$D_w^\Sigma: W^{1,p}(w^*T\Sigma) \rightarrow L^p(\text{Hom}^{0,1}(T\mathbb{C}\mathbb{P}^1, w^*T\Sigma))$$

be the linearized Cauchy–Riemann operator in Σ at the holomorphic sphere w . We also have the linearized Cauchy–Riemann operator \dot{D}_w^Σ at the holomorphic cylinder $s + it \mapsto w(e^{2\pi(s+it)}) = \pi_\Sigma(\tilde{v}(s, t))$. Then $(\pi_\Sigma \circ \tilde{v})^*T\Sigma = w^*T\Sigma|_{\mathbb{R} \times S^1 \setminus \Gamma}$ is a Hermitian vector bundle over $\mathbb{R} \times S^1 \setminus \Gamma$. Let \mathbf{V}_Σ be the kernels of the asymptotic operators of \dot{D}_w^Σ at each of the punctures, $\pm\infty$ and Γ . (These are explicitly given by $V_\Sigma(-\infty) = T_{w(0)}\Sigma$, $V_\Sigma(+\infty) = T_{w(\infty)}\Sigma$, $V_\Sigma(z) = T_{w(z)}\Sigma$ for each marked point $z \in \Gamma$.) We consider this operator acting on the space of sections

$$\dot{D}_w^\Sigma: W_{\mathbf{V}_\Sigma}^{1,p,\delta}(w^*T\Sigma|_{\mathbb{R} \times S^1 \setminus \Gamma}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), w^*T\Sigma|_{\mathbb{R} \times S^1 \setminus \Gamma})).$$

The operator D_w^Σ is Fredholm independently of the weight, but \dot{D}_w^Σ is only Fredholm when the weight δ is not an integer multiple of 2π . Furthermore, by combining [43, Proposition 3.15] with Lemma 5.20, for $0 < \delta < 2\pi$, these operators have the same Fredholm index and their kernels and cokernels are isomorphic by the map induced by restricting a section of $w^*T\Sigma$ to the punctured cylinder.

Finally, define $D_{\tilde{v}}^{\mathbb{C}}$ by

$$D_{\tilde{v}}^{\mathbb{C}}: W_{\mathbf{V}_0}^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, \mathbb{C}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), \mathbb{C}))$$

$$(D_{\tilde{v}}^{\mathbb{C}}F)(\partial_s) = F_s + iF_t + \begin{pmatrix} h''(e^b)e^b & 0 \\ 0 & 0 \end{pmatrix} F, \tag{5.5}$$

where \mathbf{V}_0 associates the vector space $i\mathbb{R}$ with the punctures at which \tilde{v} converges to a closed Hamiltonian orbit and associates the vector space \mathbb{C} at punctures at which \tilde{v} converges with a closed Reeb orbit. Notice that again these are chosen so that they precisely give the kernels of the corresponding asymptotic operators of $D_{\tilde{v}}^{\mathbb{C}}$.

Lemma 5.22. *The isomorphism $\tilde{v}^*T(\mathbb{R} \times Y) \cong (\mathbb{R} \oplus \mathbb{R}\mathbb{R}) \oplus w^*T\Sigma$ induces a decomposition:*

$$D_{\tilde{v}} = \begin{pmatrix} D_{\tilde{v}}^{\mathbb{C}} & M \\ 0 & \dot{D}_w^\Sigma \end{pmatrix}$$

where M is a multiplication operator that evaluates on ∂_s to a fibrewise linear map $M: w^*T\Sigma \rightarrow \mathbb{R} \oplus \mathbb{R}R$, decaying at the punctures. (In particular, M is compact.) Furthermore, if $w = \pi_\Sigma \circ \tilde{v}$ is non-constant, then M is pointwise surjective except at finitely many points.

Proof. In our setting, the nonlinear Floer operator takes the form of the left-hand side of the equation:

$$d\tilde{v} + J_Y(\tilde{v})d\tilde{v} \circ i - h'(e^r)R \otimes dt + h'(e^r)\partial_r \otimes ds = 0.$$

Write $\tilde{v} = (b, v): \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$. If we apply dr to the previous equation, and use the fact that $dr \circ J_Y = -\alpha$, we get

$$db - v^*\alpha \circ i + h'(e^b)ds = 0.$$

Denoting by $\pi_\xi: TY \rightarrow \xi$ the projection along the Reeb vector field, we get

$$\pi_\xi d\tilde{v} + J_Y(\tilde{v})\pi_\xi d\tilde{v} \circ i = 0. \tag{5.6}$$

Let g be the metric on $\mathbb{R} \times Y$ given by $g = dr^2 + \alpha^2 + d\alpha(\cdot, J_Y\cdot)$. This metric is J_Y -invariant. Let $\tilde{\nabla}$ be the Levi-Civita connection for g . Let ∇ be the Levi-Civita connection on $T\Sigma$ for the metric $\omega_\Sigma(\cdot, J_\Sigma\cdot)$.

Then it follows that the linearization $D_{\tilde{v}}$ applied to a section ζ of $\tilde{v}^*T(\mathbb{R} \times Y)$ satisfies

$$\begin{aligned} D_{\tilde{v}}\zeta(\partial_s) &= \tilde{\nabla}_s\zeta + J_Y(\tilde{v})\tilde{\nabla}_t\zeta + \left(\tilde{\nabla}_\zeta J_Y(\tilde{v})\right)\partial_t\tilde{v} - \tilde{\nabla}_\zeta(J_Y X_H)(\tilde{v}) \\ &= \tilde{\nabla}_s\zeta + J_Y(\tilde{v})\tilde{\nabla}_t\zeta + \left(\tilde{\nabla}_\zeta J_Y(\tilde{v})\right)\partial_t\tilde{v} + \tilde{\nabla}_\zeta(h'(e^r)\partial_r)|_{r=b}. \end{aligned} \tag{5.7}$$

Notice that $\tilde{\nabla}\partial_r = 0$ since g is a product metric. We have then

$$\tilde{\nabla}_\zeta(h'(e^r)\partial_r)|_{r=b} = h''(e^b)e^b dr(\zeta)\partial_r.$$

Observe also that for any vector field V in $T\Sigma$, there is a unique horizontal lift \tilde{V} to Y with the property $\alpha(\tilde{V}) = 0$. For any two vector fields V and W in $T\Sigma$ since $d\alpha(\tilde{V}, \tilde{W}) = K\omega_\Sigma(V, W)$, we have the following:

$$[\tilde{V}, \tilde{W}] = \widetilde{[V, W]} - K\omega_\Sigma(V, W)R.$$

From this, it follows that the Levi-Civita connection $\tilde{\nabla}$ satisfies the following identities:

$$\begin{aligned} \tilde{\nabla}_{\tilde{V}}\tilde{W} &= \widetilde{\nabla_V W} - \frac{K}{2}\omega_\Sigma(V, W)R \\ \tilde{\nabla}_R R &= 0 \\ \tilde{\nabla}_R \tilde{V} &= -\frac{1}{2}J_Y \tilde{V}. \end{aligned}$$

A simple computation using the Reeb-flow invariance of J_Y and the torsion-free property of the connection gives

$$\tilde{\nabla}_{\partial_r} J_Y = 0 = \tilde{\nabla}_R J_Y.$$

We will now compute $D_{\tilde{v}}\zeta(\partial_s)$, first when $\zeta = \zeta_1\partial_r + \zeta_2R = (\zeta_1 + i\zeta_2)\partial_r$, and then when ζ is a section of $\tilde{v}^*\xi$.

For the first computation, it suffices to notice the following two identities:

$$\begin{aligned} D_{\tilde{v}}\partial_r(\partial_s) &= h''(e^b)e^b\partial_r \\ D_{\tilde{v}}R(\partial_s) &= 0. \end{aligned}$$

It follows then from the Leibniz rule that we have

$$D_{\tilde{v}}(\zeta_1 + i\zeta_2)\partial_r(\partial_s) = (\zeta_s + i\zeta_t) + (h''(e^b)e^b\zeta_1)\partial_r = D_{\tilde{v}}^{\mathbb{C}}(\zeta_1 + i\zeta_2)\partial_r(\partial_s).$$

Now consider the case when ζ is a section of $\tilde{v}^*\xi$, and is thus the lift $\zeta = \tilde{\eta}$ of a section η of $w^*T\Sigma$. We compute

$$\begin{aligned} \tilde{\nabla}_s R &= \tilde{\nabla}_{\pi_\xi v_s} R = -\frac{1}{2}J_Y \pi_\xi v_s \\ \tilde{\nabla}_s \zeta &= \widetilde{\nabla_{w_s} \eta} - \frac{K}{2}\omega_\Sigma(w_s, \eta)R - \frac{1}{2}\alpha(v_s)J_Y \zeta, \end{aligned}$$

and similarly for $\tilde{\nabla}_t$. We then obtain the following covariant derivatives of J_Y , where \tilde{W} is a section of $\tilde{v}^*\xi$:

$$\begin{aligned} (\tilde{\nabla}_\zeta J_Y)\partial_r &= \tilde{\nabla}_\zeta R - J_Y \tilde{\nabla}_\zeta \partial_r = -\frac{1}{2}J_Y \zeta \\ (\tilde{\nabla}_\zeta J_Y)R &= -\tilde{\nabla}_\zeta \partial_r - J_Y \tilde{\nabla}_\zeta R = -\frac{1}{2}\zeta \\ (\tilde{\nabla}_\zeta J_Y)\tilde{W} &= \tilde{\nabla}_\zeta(J_Y \tilde{W}) - J_Y \tilde{\nabla}_\zeta \tilde{W} \\ &= \widetilde{\nabla_\eta J_\Sigma W} - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma W)R - J_Y \left(\widetilde{\nabla_\eta W} - \frac{K}{2}\omega_\Sigma(\eta, W)R \right) \\ &= (\widetilde{\nabla_\eta J_\Sigma})W - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma W)R - \frac{K}{2}\omega_\Sigma(\eta, W)\partial_r. \end{aligned}$$

It follows then

$$\begin{aligned} D_{\tilde{v}}\zeta(\partial_s) &= \tilde{\nabla}_s \zeta + J_Y \tilde{\nabla}_t \zeta + (\tilde{\nabla}_\zeta J_Y)\tilde{v}_t \\ &= \widetilde{\nabla_s \eta} - \frac{1}{2}\alpha(v_s)J_Y \zeta - \frac{K}{2}\omega_\Sigma(w_s, \eta)R + J_Y \widetilde{\nabla_t \eta} + \frac{1}{2}\alpha(v_t)\zeta \\ &\quad + \frac{K}{2}\omega_\Sigma(w_t, \eta)\partial_r - \frac{1}{2}b_t J_Y \zeta - \frac{1}{2}\alpha(v_t)\zeta + (\widetilde{\nabla_\eta J_\Sigma})w_t \\ &\quad - \frac{K}{2}\omega_\Sigma(\eta, J_\Sigma w_t)R - \frac{K}{2}\omega_\Sigma(\eta, w_t)\partial_r \\ &= \widetilde{D_w^\Sigma \eta} + K\omega_\Sigma(w_t, \eta)\partial_r - K\omega_\Sigma(w_s, \eta)R. \end{aligned}$$

(Note that we use the fact that $\tilde{v}_s + J_Y \tilde{v}_t + h'(e^b)\partial_r = 0$ in the cancellations.)

Writing $\zeta = (\zeta_a, \zeta_b)$ under the isomorphism $\tilde{v}^*T(\mathbb{R} \times Y) \cong (\mathbb{R} \oplus \mathbb{R}R) \oplus w^*T\Sigma$, we obtain the decomposition:

$$D_{\tilde{v}}(\zeta_a, \zeta_b)(\partial_s) = \begin{pmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{pmatrix} \begin{pmatrix} \zeta_a \\ \zeta_b \end{pmatrix} (\partial_s).$$

Our calculations now establish that $D_{aa} = D_{\tilde{v}}^{\mathbb{C}}$ and $D_{ba} = 0$, $D_{bb} = \dot{D}_w^\Sigma$, and $D_{ab}\zeta(\partial_s) = K\omega_\Sigma(w_t, \pi_\Sigma \zeta)\partial_r - K\omega_\Sigma(w_s, \pi_\Sigma \zeta)R$. Observe that in particular, D_{ab} is a pointwise linear map from $\tilde{v}^*\xi|_p$ to $\mathbb{R}\partial_r \oplus \mathbb{R}R$. The map is surjective except at critical points of the pseudoholomorphic map w , of

which there are finitely many if w is non-constant. The decay claim follows since w converges to a point, and thus its derivatives decay exponentially fast. \square

Remark 5.23. Notice that for each puncture $z \in \{\pm\infty\} \cup \Gamma$, if $\gamma(t)$ denotes the corresponding asymptotic Hamiltonian or Reeb orbit, the previous result allows us to identify V_z with $T_{\gamma(0)}Y$ at a Hamiltonian orbit and with $\mathbb{R} \times T_{\gamma(0)}Y$ at a Reeb orbit.

Lemma 5.24. *Let $\tilde{v}: \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$ be a finite hybrid energy Floer cylinder with punctures Γ .*

Then the operator $D_{\tilde{v}}^{\mathbb{C}}$ defined in Eq. (5.5) is Fredholm for $\delta > 0$ sufficiently small.

The restriction

$$D_{\tilde{v}}^{\mathbb{C}}|_{W^{1,p,\delta}}: W^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, \mathbb{C}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), \mathbb{C}))$$

has Fredholm index $-1 - 2\#\Gamma$ if the positive puncture at $+\infty$ converges to a closed Hamiltonian orbit and has Fredholm index $-2 - 2\#\Gamma$ if the positive puncture converges to Reeb orbit at $+\infty \times Y$.

If \tilde{v} converges at both $\pm\infty$ to closed Hamiltonian orbits, then

$$D_{\tilde{v}}^{\mathbb{C}}: W_{\mathbf{V}_0}^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, \mathbb{C}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), \mathbb{C}))$$

has Fredholm index 1 and is surjective.

If, instead, \tilde{v} converges at $+\infty$ to a closed Hamiltonian orbit, and at $-\infty$ to a closed Reeb orbit in $\{-\infty\} \times Y$, then $D_{\tilde{v}}^{\mathbb{C}}$ has Fredholm index 2 and is surjective.

Finally, if \tilde{v} converges at $\pm\infty$ to closed Reeb orbits in $\{\pm\infty\} \times Y$, then $D_{\tilde{v}}^{\mathbb{C}}$ has Fredholm index 2 and is surjective.

In all three cases, the kernel of $D_{\tilde{v}}^{\mathbb{C}}$ contains the constant section i , which can be identified with the Reeb vector field.

Proof. We will apply the punctured Riemann–Roch Theorems 5.16 and 5.18. For this, we need to compute the Conley–Zehnder indices of the appropriately perturbed asymptotic operators. We will first identify the (Morse–Bott degenerate) asymptotic operators at each of the punctures, and then apply Corollary 5.14 to obtain the Conley–Zehnder indices of the $\pm\delta$ -perturbed operators.

Recall from Remark 5.21 that we have $|\delta| > 0$ smaller than the spectral gap for any of these punctures.

To consider the operator $D_{\tilde{v}}^{\mathbb{C}}: W_{\mathbf{V}_0}^{1,p,\delta} \rightarrow L^{p,\delta}$, it will be convenient to consider a related operator with the same formula, but on the much larger space of functions with exponential growth. By a slight abuse of notation, we will use the same name:

$$D_{\tilde{v}}^{\mathbb{C}}: W^{1,p,-\delta} \rightarrow L^{p,-\delta}$$

$$(D_{\tilde{v}}^{\mathbb{C}}F)(\partial_s) = F_s + iF_t + \begin{pmatrix} h''(e^b)e^b & 0 \\ 0 & 0 \end{pmatrix} F.$$

Then the kernel and cokernel of the operator acting on the spaces of sections with exponential growth can be identified with the kernel and cokernel of the operator acting on $W_{\mathbf{V}_0}^{1,p,\delta}$, by Lemma 5.20.

First, consider the case when \tilde{v} converges to a closed Hamiltonian orbit in $\{b_{\pm}\} \times Y$ as $s \rightarrow \pm\infty$. Then the asymptotic operator associated with $D_{\tilde{v}}^{\mathbb{C}}$ at $\pm\infty$ is given by

$$\mathbf{A}_{\pm} = -i \frac{d}{dt} - \begin{pmatrix} h''(e^{b_{\pm}}) e^{b_{\pm}} & 0 \\ 0 & 0 \end{pmatrix}.$$

In the case of δ -exponential decay, the relevant asymptotic operators are given by $\mathbf{A}_+ + \delta$ at the positive puncture $+\infty$ and by $\mathbf{A}_- - \delta$ at the negative puncture $-\infty$. In the case of δ -exponential growth, the relevant asymptotic operators are $\mathbf{A}_+ - \delta$ and $\mathbf{A}_- + \delta$, respectively.

For the case of exponential decay, Corollary 5.14 then gives the Conley–Zehnder index of 0 for $\mathbf{A}_+ + \delta$ and of 1 for $\mathbf{A}_- - \delta$.

In the case of exponential growth, Corollary 5.14 gives instead that the Conley–Zehnder index of $\mathbf{A}_+ - \delta$ is 1 and that of $\mathbf{A}_- + \delta$ is 0.

Associated with a Reeb puncture at $\pm\infty$ or at $P \in \Gamma$, we have the asymptotic operator

$$-i \frac{d}{dt}.$$

Writing $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \setminus \Gamma \rightarrow \mathbb{R} \times Y$, we have $b \rightarrow -\infty$ at both types of negative punctures and $b \rightarrow +\infty$ at the positive puncture.

As above, in the case of exponential decay, the relevant asymptotic operators are $-i \frac{d}{dt} + \delta$ at a positive puncture and $-i \frac{d}{dt} - \delta$ at a negative puncture. Again, by Corollary 5.14, we obtain a Conley–Zehnder index of -1 at $+\infty$ and a Conley–Zehnder indices of 1 at a negative puncture ($-\infty$ or $P \in \Gamma$).

If, instead, we consider exponential growth, we obtain Conley–Zehnder indices of $+1$ at positive punctures and -1 at negative punctures.

Applying now the punctured Riemann–Roch theorem 5.16, and using the fact that the Euler characteristic of the punctured cylinder is $-\#\Gamma$, we obtain that the Fredholm index of

$$D_{\tilde{v}}^{\mathbb{C}}|_{W^{1,p,\delta}} : W^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, \mathbb{C}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), \mathbb{C}))$$

is given by

$$-\#\Gamma - c - 1 - \#\Gamma = -c - 1 - 2\#\Gamma,$$

where $c = 0$ if the positive puncture converges to a Hamiltonian orbit, and $c = 1$ if the positive puncture converges to a Reeb orbit at $+\infty$, as claimed.

The injectivity of $D_{\tilde{v}}^{\mathbb{C}}$ restricted to $W^{1,p,\delta}$ follows from automatic transversality, applying [41, Proposition 2.2]. The criterion involves the adjusted Chern number [41, Equations (2.4) and (2.5)]. In our situation, there are $1 - c$ punctures with even Conley–Zehnder index. This adjusted Chern number then becomes

$$\begin{aligned}
 c_1(E, l, \mathbf{A}_\Gamma) &= \frac{1}{2}(\text{Ind}(D_{\tilde{v}}^{\mathbb{C}}|_{W^{1,p,\delta}}) - 2 + \#\Gamma_0) \\
 &= \frac{1}{2}(-c - 1 - 2\#\Gamma - 2 + (1 - c)) = -\#\Gamma - 1 - c < 0
 \end{aligned}$$

as necessary to apply [41, Proposition 2.2].

Now, applying Theorem 5.18, we compute that the Fredholm index of

$$D_{\tilde{v}}^{\mathbb{C}}: W_{\mathbf{V}_0}^{1,p,\delta}(\mathbb{R} \times S^1 \setminus \Gamma, \mathbb{C}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1 \setminus \Gamma), \mathbb{C}))$$

is given by

$$\begin{aligned}
 -\#\Gamma + 1 - (-\#\Gamma) - \begin{cases} 0 & \text{if } \tilde{v}(-\infty) \text{ converges to a Hamiltonian orbit} \\ -1 & \text{if } \tilde{v}(-\infty) \text{ converges to a Reeb orbit} \end{cases} \\
 = 1 \text{ or } 2, \text{ depending on the negative end of } \tilde{v}.
 \end{aligned}$$

Furthermore, the fact that the curve has genus 0 and one puncture with even Conley–Zehnder index precisely if $\lim_{s \rightarrow -\infty} \tilde{v}$ is a Hamiltonian orbit implies that

$$\begin{aligned}
 c_1(E, l, \mathbf{A}_\Gamma) &= \begin{cases} \frac{1}{2}(1 - 2 + 1) = 0 & \text{if } \tilde{v}(-\infty) \text{ converges to a Hamiltonian orbit} \\ \frac{1}{2}(2 - 2) = 0 & \text{if } \tilde{v}(-\infty) \text{ converges to a Reeb orbit.} \end{cases}
 \end{aligned}$$

In either case, the adjusted Chern number is less than the Fredholm index. Therefore, $D_{\tilde{v}}^{\mathbb{C}}$ satisfies the automatic transversality criterion and is thus surjective, as wanted.

It follows immediately from the expression for $D_{\tilde{v}}^{\mathbb{C}}$ that the constant i is in the kernel. Recalling that $\mathbb{C} = \tilde{v}^*(\mathbb{R} \oplus \mathbb{R}R)$ in the splitting given by Lemma 5.22, we then may identify this constant with the Reeb vector field R . □

To summarize the results of this section, by Lemma 5.22, a punctured Floer cylinder in $\mathbb{R} \times S^1$ is regular if the operators $D_{\tilde{v}}^{\mathbb{C}}$ and \dot{D}_w^Σ are surjective. Surjectivity of the latter is equivalent to surjectivity of D_w^Σ . Lemma 5.24 gives the surjectivity of $D_w^{\mathbb{C}}$. It thus remains to study transversality for D_w^Σ , specifically with respect to the evaluation maps that will allow us to define the moduli spaces of chains of pearls in Σ (see Sect. 5.3). Additionally, we need to consider transversality for moduli spaces of planes in W asymptotic to Reeb orbits in Y , or equivalently, the moduli spaces of spheres in X with an order of contact condition at Σ (see Sect. 5.4).

5.3. Transversality for chains of pearls in Σ

In this section and the next, we show that for generic almost complex structure (in a sense to be made precise), the moduli spaces of chains of pearls and moduli spaces of chains of pearls with spheres in X (possibly augmented as well) are transverse. We begin with the definition of several moduli spaces that will be useful.

Definition 5.25. Let $J_\Sigma \in \mathcal{J}_\Sigma$ be an almost complex structure compatible with ω_Σ . Given $p, q \in \text{Crit}(f_\Sigma)$ and a finite collection $A_1, \dots, A_N \in H_2(\Sigma; \mathbb{Z})$, let

$$\mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); q, p; J_\Sigma)$$

denote the space of simple chains of pearls in Σ from q to p (see Definition 5.5), such that $(w_i)_*[\mathbb{C}\mathbb{P}^1] = A_i$, with k marked points.

Let

$$\mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); J_\Sigma)$$

denote the moduli space of N parametrized J_Σ -holomorphic spheres in Σ , representing the classes $A_i, i = 1, \dots, N$, with k marked points, also satisfying the simplicity criterion of Definition 5.5, i.e. so each sphere is either somewhere injective or constant, each constant sphere has at least one augmentation marked point, and no sphere has image contained in the image of another.

For $J_W \in \mathcal{J}_W$, let $J_\Sigma = P(J_W)$ be the corresponding almost complex structure in \mathcal{J}_Σ and J_X the corresponding almost complex structure on X . Define

$$\mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); x, p, J_W)$$

to be the moduli space of simple chains of pearls in Σ with a sphere in X (as in Definitions 5.3 and 5.5), where x is a critical point of f_W and p is a critical point of f_Σ , and representing the spherical homology classes $[w_i] = A_i \in H_2(\Sigma; \mathbb{Z}), i = 1, \dots, N$ and $[v] = B \in H_2(X; \mathbb{Z}) \setminus 0$. In the following, we will write

$$l = B \bullet \Sigma = K\omega(B)$$

which is the order of contact of v with Σ .

Let

$$\mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); J_W)$$

denote the moduli space of N parametrized J_Σ -holomorphic spheres in Σ , representing the classes A_i , and of a J_X -holomorphic sphere in X representing the class B with order of contact $l = B \bullet \Sigma = K\omega(B)$, also satisfying the simplicity criterion of Definition 5.5, i.e. so each sphere in Σ is either somewhere injective or constant (if constant, it has at least one augmentation marked point), no image of a sphere in Σ is contained in the image of another and the image of the sphere in X is not contained in the tubular neighbourhood $\varphi(\mathcal{U})$ of Σ . Furthermore, the spheres in Σ have k marked points.

Let

$$\mathcal{M}_X^*((B_1, B_2, \dots, B_k); J_W)$$

denote the moduli space of k unparametrized J_X -holomorphic spheres in X , where each sphere is somewhere injective, no image of a sphere is contained in the image of another sphere, and so the image of each sphere is not contained in the tubular neighbourhood $\varphi(\mathcal{U})$ of Σ , and such that each sphere intersects Σ only at $\infty \in \mathbb{C}\mathbb{P}^1$ with order of contact $B_i \bullet \Sigma$. We can think of

an unparametrized sphere as an equivalence class of parametrized spheres, modulo the action of $\text{Aut}(\mathbb{C}\mathbb{P}^1, \infty) = \text{Aut}(\mathbb{C})$ on the domain.

Finally, let

$$\mathcal{M}_{k,\Sigma}^a((A_1, \dots, A_N), (B_1, \dots, B_k); q, p; J_W)$$

denote the moduli space of simple augmented chains of pearls in Σ with k unparametrized augmentation planes, and let

$$\mathcal{M}_{k,(X,\Sigma)}^a((B; A_1, \dots, A_N); (B_1, \dots, B_k); x, p; J_W)$$

denote the moduli space of simple augmented chains of pearls with a sphere in X . (See Definitions 5.4 and 5.5.)

To apply the Sard–Smale Theorem, we need to consider Banach spaces of almost complex structures, so we let $\mathcal{J}_\Sigma^r, \mathcal{J}_W^r$ be the space of C^r -regular almost complex structures otherwise satisfying the conditions of being in $\mathcal{J}_\Sigma, \mathcal{J}_W$. We impose $r \geq 2$ and in general will require r to be sufficiently large that the Sard–Smale theorem holds (this will depend on the Fredholm indices associated with the collection of homology classes and will also depend on the order of contact to Σ for the spheres in X).

For each of these moduli spaces, we also consider the corresponding universal moduli spaces as we vary the almost complex structure. For instance, we denote by $\mathcal{M}_{k,\Sigma}^*(A_1, \dots, A_N, \mathcal{J}_\Sigma^r)$ the moduli space of pairs $((w_i)_{i=1}^N, J_\Sigma)$ with $J_\Sigma \in \mathcal{J}_\Sigma^r$ and $(w_i)_{i=1}^N \in \mathcal{M}_{k,\Sigma}^*(A_1, \dots, A_N, J_\Sigma)$.

The main goal of this section and of the next is to prove that these moduli spaces of simple chains of pearls are transverse for generic almost complex structures. This is analogous to [29, Theorem 6.2.6], and indeed, the transversality theorem of McDuff–Salamon will be a key ingredient of our proof. Their Theorem 6.2.6 is about transversality of the universal evaluation map to a specific submanifold Δ^E of the target, whereas our work in this section establishes transversality to some other submanifolds. We will, furthermore, require an extension of the results from [12] (see Sect. 5.4), and an additional technical transversality point needed to be able to consider the lifted problem in $\mathbb{R} \times Y$.

Proposition 5.26. *There is a residual set $\mathcal{J}_W^{reg} \subset \mathcal{J}_W$ such that $\mathcal{J}_\Sigma^{reg} := P(\mathcal{J}_W^{reg})$ is a residual set in \mathcal{J}_Σ and such that for all $J_\Sigma \in \mathcal{J}_\Sigma^{reg}$ and $J_W \in \mathcal{J}_W^{reg}$, $p \in \text{Crit}(f_\Sigma)$, $q \in \text{Crit}(f_\Sigma)$ and $x \in \text{Crit}(f_W)$, the moduli spaces $\mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); q, p; J_\Sigma)$, $\mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); x, p, J_W)$, $\mathcal{M}_{k,\Sigma}^a((A_1, \dots, A_N), (B_1, \dots, B_k); q, p; J_W)$ and $\mathcal{M}_{k,(X,\Sigma)}^a((B; A_1, \dots, A_N); (B_1, \dots, B_k); x, p; J_W)$ are manifolds. Their dimensions are*

$$\begin{aligned} \dim \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); q, p; J_\Sigma) &= M(p) \\ &+ \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle - M(q) + N - 1 + 2k, \end{aligned}$$

$$\begin{aligned} \dim \mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); x, p, J_W) \\ = M(p) + \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle + 2(\langle c_1(TX), B \rangle - B \bullet \Sigma) + M(x) \end{aligned}$$

$$\begin{aligned}
 & -2(n-1) + N - 1 + 2k, \\
 \dim \mathcal{M}_{k,\Sigma}^a((A_1, \dots, A_N), (B_1, \dots, B_k); q, p; J_W) \\
 &= M(p) + \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle - M(q) + N - 1 \\
 & \quad + \sum_{i=1}^k (2 \langle c_1(TX), B_i \rangle - 2B_i \bullet \Sigma), \\
 \dim \mathcal{M}_{k,(X,\Sigma)}^a((B; A_1, \dots, A_N); (B_1, \dots, B_k); x, p; J_W) \\
 &= M(p) + \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle + 2(\langle c_1(TX), B \rangle - B \bullet \Sigma) + M(x) - 2(n-1) \\
 & \quad + N - 1 + \sum_{i=1}^k (2 \langle c_1(TX), B_i \rangle - 2B_i \bullet \Sigma),
 \end{aligned}$$

where $M(p)$ and $M(q)$ are the Morse indices of $p, q \in \text{Crit}(f_\Sigma)$ and $M(x)$ is the Morse index of $x \in \text{Crit}(f_W)$.

Proposition 5.27. [29, Proposition 6.2.7] $\mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r)$ is a Banach manifold.

We will also make use of the following definition and proposition, the latter of which we prove in the next section.

Definition 5.28. There is a universal evaluation map

$$\begin{aligned}
 \text{ev}_\Sigma: \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) &\rightarrow \Sigma^{2N} \\
 (w_1, \dots, w_N) &\mapsto (w_1(0), w_1(\infty), w_2(0), w_2(\infty), \dots, w_N(\infty)).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \text{ev}_{X,\Sigma}: \mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); \mathcal{J}_W) &\rightarrow X \times \Sigma^{2N+1} \\
 (v, w_1, \dots, w_N) &\mapsto (v(0), v(\infty), w_1(0), w_1(\infty), w_2(0), \dots, w_N(\infty)),
 \end{aligned}$$

where v is the holomorphic sphere in X and the w_i are the spheres in Σ .

We have an evaluation map coming from simple collections of spheres in X :

$$\begin{aligned}
 \text{ev}_\Sigma^a: \mathcal{M}_X^*((B_1, B_2, \dots, B_k); \mathcal{J}_W^r) &\rightarrow \Sigma^k \\
 (v_1, \dots, v_k) &\mapsto (v_1(\infty), v_2(\infty), \dots, v_k(\infty)).
 \end{aligned}$$

For spheres in Σ , we also obtain evaluation maps at the augmentation punctures

$$\text{ev}_\Sigma^a: \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \rightarrow \Sigma^k$$

and

$$\text{ev}_\Sigma^a: \mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); \mathcal{J}_W^r) \rightarrow \Sigma^k.$$

We refer to these three maps denoted ev^a as *augmentation evaluation maps*.

Proposition 5.29. *Let B_0, \dots, B_k be spherical classes in $H_2(X; \mathbb{Z})$. Let*

$$r \geq \max_i B_i \bullet \Sigma + 2.$$

The universal moduli space $\mathcal{M}_X^((B_1, \dots, B_k); \mathcal{J}_W^r)$ is a Banach manifold and the evaluation maps*

$$\begin{aligned} \text{ev}_\Sigma^a: \mathcal{M}_X^*((B_1, \dots, B_k); \mathcal{J}_W^r) &\rightarrow \Sigma^k : (f_1, f_2, \dots, f_k) \mapsto (f_1(\infty), \dots, f_k(\infty)) \\ \text{ev}_{X, \Sigma}: \mathcal{M}_X^*((B_0); \mathcal{J}_W^r) &\rightarrow X \times \Sigma : f \mapsto (f(0), f(\infty)) \end{aligned}$$

are submersions.

Recall that we have chosen a Morse function $f_\Sigma: \Sigma \rightarrow \mathbb{R}$ and a corresponding gradient-like vector field Z_Σ , such that (f_Σ, Z_Σ) is a Morse–Smale pair. The time- t flow of Z_Σ is denoted by $\varphi_{Z_\Sigma}^t$ and the stable (ascending) $W_\Sigma^s(q)$ and unstable (descending) manifolds $W_\Sigma^u(p)$ were defined in Eq. (3.1). (Note that these are the stable/unstable manifolds for the negative gradient flow.)

Definition 5.30. The *flow diagonal* in $\Sigma \times \Sigma$ associated with the pair (f_Σ, Z_Σ) is

$$\Delta_{f_\Sigma} := \left\{ (x, y) \in (\Sigma \setminus \text{Crit}(f_\Sigma))^2 \mid \exists t > 0 \text{ so } y = \varphi_{Z_\Sigma}^t(x) \right\},$$

where $\text{Crit}(f_\Sigma)$ is the set of critical points of f_Σ .

We will now establish transversality of the evaluation maps to appropriate products of stable/unstable manifolds, critical points, diagonals and flow diagonals. By [29, Proposition 6.2.8], the key difficulty will be to deal with constant spheres. For this, we will need the following lemma about evaluation maps intersecting with the flow diagonals.

Lemma 5.31. *Suppose $f_0: \mathcal{M}_0 \rightarrow \Sigma$ and $f_1: \mathcal{M}_1 \rightarrow \Sigma$ are submersions.*

Then

$$\begin{aligned} F: \mathcal{M}_0 \times \mathcal{M}_1 &\rightarrow \Sigma^3 \\ (m_0, m_1) &\mapsto (f_0(m_0), f_1(m_1), f_1(m_1)) \end{aligned}$$

is transverse to $\Delta_{f_\Sigma} \times \{p\}$ for each point $p \in \Sigma$.

Proof. Suppose $F(m_0, m_1) = (x, p, p) \in \Delta_{f_\Sigma} \times \{p\}$. Then there exists t so that $\varphi_{Z_\Sigma}^t(x) = \varphi_{Z_\Sigma}^t(f_0(m_0)) = f_1(m_1) = p$.

Notice that

$$E := \{ (d\varphi_{Z_\Sigma}^{-t}(p)v, v) \mid v \in T_p\Sigma \} \subset T_{(x,p)}\Delta_{f_\Sigma}.$$

For notational simplicity, we write $\Phi = d\varphi_{Z_\Sigma}^{-t}(p)$.

It follows then that

$$\begin{aligned} dF(m_0, m_1) \cdot T(\mathcal{M}_0 \times \mathcal{M}_1) &+ (E \oplus 0) \\ &= \{ (df_0|_{m_0}v_0 + \Phi w, df_1|_{m_1}v_1 + w, df_1|_{m_1}v_1) \mid v_0 \in T_{m_0}\mathcal{M}_0, \\ &\quad v_1 \in T_{m_1}\mathcal{M}_1, w \in T_p\Sigma \} \\ &= T\Sigma \oplus T\Sigma \oplus T\Sigma \end{aligned}$$

using the surjectivity of df_0, df_1 . This then establishes the result since $E \subset T_{(x,p)}\Delta_{f_\Sigma}$. □

From this, we now obtain the following:

Lemma 5.32. *Suppose \mathcal{M}_0 and B are manifolds and there is a map*

$$\text{ev} = (\text{ev}_-, \text{ev}_+): \mathcal{M}_0 \rightarrow B \times \Sigma$$

that is transverse to $A \times \text{pt}$, for a submanifold A of B and for all points $\text{pt} \in \Sigma$. Suppose also that \mathcal{M}_1 is a manifold with a submersion $e: \mathcal{M}_1 \rightarrow \Sigma$.

Then the map

$$\begin{aligned} \hat{\text{e}}\text{v}: \mathcal{M}_0 \times \mathcal{M}_1 &\rightarrow B \times \Sigma^3 \\ (m, n) &\mapsto (\text{ev}_-(m), \text{ev}_+(m), e(n), e(n)) \end{aligned}$$

is transverse to $A \times \Delta_{f_\Sigma} \times \text{pt}$, for all points $\text{pt} \in \Sigma$.

Proof. We apply the previous lemma, using $f_0 = \text{ev}_+$ and $f_1 = e$. Then $\hat{\text{e}}\text{v}(m, n) = (\text{ev}_-(m), F(m, n))$. The transversality to $A \times \Delta_{f_\Sigma} \times \text{pt}$ follows by the transversality of F to $\Delta_{f_\Sigma} \times \text{pt}$ together with the transversality of ev_- to A . □

Lemma 5.33. *Let $N \geq 1$, and let A_1, \dots, A_N be spherical homology classes in Σ and let B be a spherical homology class in X .*

Suppose that $S \subset \Sigma^{2N-2}$ is obtained by taking the product of some number of copies of $\Delta_{f_\Sigma} \subset \Sigma^2$ and of the complementary number of copies of $\{(p, p) \mid p \in \text{Crit}(f_\Sigma)\} \subset \Sigma^2$, in arbitrary order. Let $\Delta \subset \Sigma^2$ denote the diagonal.

Then if $\sum_{i=1}^N A_i \neq 0$, the universal evaluation map

$$\text{ev}_\Sigma: \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \rightarrow \Sigma^{2N}$$

is transverse to the submanifold $\{x\} \times S \times \{y\}$ for all $x, y \in \Sigma$.

If $B \neq 0$, the universal evaluation map

$$\text{ev}_{X,\Sigma}: \mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); \mathcal{J}_W^r) \rightarrow X \times \Sigma^{2N+1}$$

is transverse to the submanifold $\{x\} \times \Delta \times S \times \{y\}$ for any $x \in X, y \in \Sigma$.

Proof. We consider the case of $\mathcal{M}_{k,\Sigma}^*$ in detail since the argument is essentially the same for $\mathcal{M}_{k,(X,\Sigma)}^*$, though notationally more cumbersome.

Suppose that $((v_1, \dots, v_N), J) \in \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r)$ is in the pre-image of $\{x\} \times S \times \{y\}$. Write $S = S_1 \times S_2 \times \dots \times S_{N-1}$, where each $S_i \subset \Sigma^2$ is either the flow diagonal or the set of critical points.

Notice that the simplicity condition then requires that if some sphere v_i is constant, $1 < i < N$, we must have that S_{i-1} and S_i are flow diagonals. If v_1 is constant, then S_1 is a flow diagonal and if v_N is constant, S_{N-1} is a flow diagonal.

We will proceed by induction on N . The case $N = 1$ follows from [29, Proposition 3.4.2].

Now, for the inductive argument, we suppose the result holds for any $S \subset \Sigma^{2(N-1)-2}$ of the form specified, and for any $k \geq 0$, for any collection of $N - 1$ spherical classes, not all of which are zero.

Let now A_1, \dots, A_N be spherical homology classes, not all of which are zero. Notice that each of these homology classes is represented by a J_Σ -holomorphic sphere, and thus has $\omega_\Sigma(A_i) \geq 0$ for each i . In particular then, for such spherical classes, for any $1 \leq a \leq b \leq N$, A_a, A_{a+1}, \dots, A_b are not all zero if and only if $\sum_{i=a}^b A_i \neq 0$. Then at least one of A_1, \dots, A_{N-1} or A_2, \dots, A_N is a collection of spheres satisfying the hypotheses of the lemma. For simplicity of notation, let us assume that $A_1 + \dots + A_{N-1} \neq 0$. Let $S_0 = S_1 \times S_2 \times \dots \times S_{N-2}$. Let $k = k_0 + k_N$ where k_N is the number of marked points we consider on the last sphere. By the induction hypothesis, we have that the evaluation map

$$\mathcal{M}_{k_0, \Sigma}^*((A_1, \dots, A_{N-1}); \mathcal{J}_\Sigma^r) \rightarrow \Sigma^{2(N-1)}$$

is transverse to $\text{pt} \times S_0 \times \text{pt}$. Denote this map by ev_0 .

Notice that $\mathcal{M}_{k, \Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \subset \mathcal{M}_{k_0, \Sigma}^*((A_1, \dots, A_{N-1}); \mathcal{J}_\Sigma^r) \times \mathcal{M}_{k_N, \Sigma}^*(A_N; \mathcal{J}_\Sigma^r)$. Let then $\text{ev}_N: \mathcal{M}_{k, \Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \rightarrow \Sigma^2$ be the evaluation at 0 and ∞ in the N th sphere. We, therefore, have

$$\text{ev}_\Sigma: \mathcal{M}_{k, \Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \rightarrow \Sigma^{2N}$$

given by $\text{ev}_\Sigma = (\text{ev}_0, \text{ev}_N)$.

If $A_N \neq 0$, the result follows again from [29, Proposition 3.4.2].

If, instead, $A_N = 0$, we have from above that $S_{N-1} = \Delta_{f_\Sigma}$. Notice that the evaluation map of constant spheres on Σ has image on the diagonal in $\Sigma \times \Sigma$. The result now follows by applying Lemma 5.32.

The case with a sphere in X follows a nearly identical induction argument, though the base case consists of a single sphere in X . The required submersion to $X \times \Sigma$ now follows from Proposition 5.29, and the induction proceeds as before. \square

Proposition 5.34. *Let $N \geq 0$. Suppose that $S \subset \Sigma^{2N-2}$ is obtained by taking the product of some number of copies of $\Delta_{f_\Sigma} \subset \Sigma^2$ and of the complementary number of copies of $\{(p, p) \mid p \in \text{Crit}(f_\Sigma)\} \subset \Sigma^2$, in arbitrary order.*

Let $\Delta \subset \Sigma \times \Sigma$ denote the diagonal and let Δ_k denote the diagonal Σ^k in $\Sigma^k \times \Sigma^k$.

Let p, q be critical points of f_Σ and let x be a critical point of f_W .

Then the universal evaluation maps together with augmentation evaluation maps

$$\begin{aligned} \text{ev}_\Sigma \times \text{ev}_\Sigma^a \times \text{ev}_\Sigma^a: \mathcal{M}_{k, \Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \times \mathcal{M}_X^*((B_1, \dots, B_k); \mathcal{J}_W^r) \\ \rightarrow \Sigma^{2N} \times \Sigma^k \times \Sigma^k \end{aligned}$$

$$\begin{aligned} \text{ev}_{X, \Sigma} \times \text{ev}_\Sigma^a \times \text{ev}_\Sigma^a: \mathcal{M}_{k, (X, \Sigma)}^*((B; A_1, \dots, A_N); \mathcal{J}_W^r) \times \mathcal{M}_X^*((B_1, \dots, B_k); \mathcal{J}_W^r) \\ \rightarrow X \times \Sigma^{2N+1} \times \Sigma^k \times \Sigma^k \end{aligned}$$

are transverse to, respectively,

$$\begin{aligned} W_\Sigma^s(q) \times S \times W_\Sigma^u(p) \times \Delta_k \\ W_X^u(x) \times \Delta \times S \times W_\Sigma^u(p) \times \Delta_k. \end{aligned}$$

Proof. We will consider only the first case, the second being analogous. Notice first that by Proposition 5.29 the augmentation evaluation map

$$\text{ev}_\Sigma^a : \mathcal{M}_X^*((B_1, \dots, B_k); \mathcal{J}_W^r) \rightarrow \Sigma^k$$

is a submersion. It suffices, therefore, to prove that

$$\text{ev}_\Sigma : \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); \mathcal{J}_\Sigma^r) \rightarrow \Sigma^{2N}$$

is transverse to $W_\Sigma^s(q) \times S \times W_\Sigma^u(p)$.

The proposition follows immediately if at least one of the $A_i, i = 1, \dots, N$ is non-zero, or if we are considering the case of a chain of pearls with a sphere in X , by applying Lemma 5.33.

The only case then that must be examined is that of a chain of pearls entirely in Σ with all spheres constant. In this case, the evaluation map from the moduli space $\mathcal{M}_{k,\Sigma}^*((0, 0, \dots, 0), \mathcal{J}_W^r)$ factors through the evaluation map

$$\{(z_1, \dots, z_N) \in \Sigma^N \mid z_i = z_j \implies i = j\} \times \mathcal{J}_W^r \rightarrow \Sigma^{2N}.$$

Transversality follows from the Morse–Smale condition on the gradient-like vector field Z_Σ . This gives that the intersection of $W_\Sigma^s(q)$ and $W_\Sigma^u(p)$ is transverse, and hence that the diagonal in $\Sigma \times \Sigma$ is transverse to $W_\Sigma^s(q) \times W_\Sigma^u(p)$, which is what we need when $N = 1$. The case of $N \geq 2$ is similar, using the description of the tangent space to the flow diagonal at $(x, y) \in \Delta_{f_\Sigma}$, such that $\varphi_{Z_\Sigma}^t(x) = y$ for some $t > 0$, as

$$T_{(x,y)}\Delta_{f_\Sigma} = \{(v, d\varphi_{Z_\Sigma}^t(x)v + cZ_\Sigma(y)) \mid v \in T_x\Sigma, c \in \mathbb{R}\} \subset T_x\Sigma \oplus T_y\Sigma.$$

□

Proposition 5.34 can be combined with standard Sard–Smale arguments, the fact that $P: \mathcal{J}_W^r \rightarrow \mathcal{J}_\Sigma^r$ is an open and surjective map and Taubes’s method for passing to smooth almost complex structures (see for instance [29, Theorem 6.2.6]) to give the following proposition:

Proposition 5.35. *There exist residual sets of almost complex structures $\mathcal{J}_W^{reg} \subset \mathcal{J}_W$ and $\mathcal{J}_\Sigma^{reg} = P(\mathcal{J}_W^{reg})$, so that for fixed $J_W \in \mathcal{J}_W^{reg}$ and $J_\Sigma = P(J_W)$, the restrictions of the evaluation maps $\text{ev}_\Sigma \times \text{ev}_\Sigma^a \times \text{ev}_\Sigma^a$ and $\text{ev}_{X,\Sigma} \times \text{ev}_\Sigma^a \times \text{ev}_\Sigma^a$ to*

$$\begin{aligned} &\mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); J_\Sigma) \times \mathcal{M}_X^*((B_1, \dots, B_k); J_W) \quad \text{and} \\ &\mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); J_W) \times \mathcal{M}_X^*((B_1, \dots, B_k); J_W), \end{aligned}$$

respectively, are transverse to the submanifolds of Proposition 5.34.

The transversality statement of the main result of this section, Proposition 5.26 now follows. The dimension formulas follow from usual index arguments, combining Riemann–Roch with contributions from the constraints imposed by the evaluation maps.

5.4. Transversality for spheres in X with order of contact constraints in Σ

We will now consider transversality for a chain of pearls with a sphere in X . We will extend the results from Section 6 in [12]. In that paper, Cieliebak and Mohnke prove that the moduli space of simple curves not contained in Σ , with a condition on the order of contact with Σ , can be made transverse by a perturbation of the almost complex structure away from Σ . We will extend this result to show that additionally the evaluation map to Σ at the point of contact can be made transverse. This can be useful, for instance, to define relative Gromov–Witten invariants with constraints on homology classes in Σ .

Recall that Σ is a symplectic divisor and $N\Sigma$ is its symplectic normal bundle equipped with a Hermitian structure. Keeping in mind the discussion in Sect. 2 (in particular the identification of $X \setminus \Sigma$ with W in Lemma 2.5), we will by an abuse of notation identify an almost complex structure on W with the corresponding almost complex structure on X . We have fixed a symplectic neighbourhood $\varphi: \mathcal{U} \rightarrow X$ where $\varphi: \bar{\mathcal{U}} \rightarrow X$ is an embedding. From Definition 2.7, we require that all $J_X \in \mathcal{J}_W$ have that J_X is standard in the image $\varphi(\mathcal{U}) \subset X$ of this neighbourhood.

Fix an almost complex structure $J_0 \in \mathcal{J}_W$. We may suppose that $P(J_0) \in \mathcal{J}_\Sigma$ is an almost complex structure in the residual set \mathcal{J}_Σ^{reg} given by Proposition 5.26, though this is not strictly speaking necessary.

Let $\mathcal{V} := X \setminus \varphi(\bar{\mathcal{U}})$. Following Cieliebak–Mohnke [12], let $\mathcal{J}(\mathcal{V})$ be the set of all almost complex structures on X compatible with ω that are equal to J_0 on $\varphi(\mathcal{U})$. Similarly, we will let $\mathcal{J}^r(\mathcal{V})$ be the compatible almost complex structures of C^r regularity.

To define the order of contact, consider an almost complex structure $J_X \in \mathcal{J}_W$ and a J_X -holomorphic sphere $f: \mathbb{C}\mathbb{P}^1 \rightarrow X$ with $f(0) \in \Sigma$, an isolated intersection. Choose coordinates $s + it = z \in \mathbb{C}$ on the domain and local coordinates near $f(0) \in \Sigma$ on the target, such that $f(0) \in \Sigma \subset X$ corresponds to $0 \in \mathbb{C}^{n-1} = \mathbb{C}^{n-1} \times \{0\} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Write $\pi_{\mathbb{C}}: \mathbb{C}^n \rightarrow \mathbb{C}$ for projection onto the last coordinate (which is to be thought of as normal to Σ). Assume also that $J_X(0) = i$. Then f has contact of order l at 0 if the vector of all partial derivatives of orders 1 through l (denoted by $d^l f(0)$) has trivial projection to \mathbb{C} . We can write this condition as $d^l f(0) \in T_{f(0)}\Sigma$. We define then the order of contact at an arbitrary point in $\mathbb{C}\mathbb{P}^1$ by precomposing with a Möbius transformation. (This is well defined, by [12, Lemma 6.4].)

Define the space of simple pseudoholomorphic maps into X that have order of contact l at ∞ to a point in Σ to be

$$\begin{aligned} \mathcal{M}_{\infty, l, (X, \Sigma)}^*(\mathcal{J}_W) &:= \{(f, J_X) \in W^{m, p}(\mathbb{C}\mathbb{P}^1, X) \times \mathcal{J}_W \mid \bar{\partial}_{J_X} f = 0, \\ &\quad f(\infty) \in \Sigma, d^l f(\infty) \in T_{f(\infty)}\Sigma, \\ &\quad f \text{ simple, } f^{-1}(\mathcal{V}) \neq \emptyset\}, \end{aligned}$$

where we require $m \geq l + 2$. Note that our notation differs somewhat from the notation in [12].

In this section, we need to have a higher regularity on our Sobolev spaces to make sense of the order of contact condition. For the remaining moduli

spaces, for simplicity of notation, we have taken $m = 1$, where this is not a problem. Notice that by elliptic regularity, the moduli spaces themselves are manifolds of smooth maps, and are independent of the choice of m . This only affects the classes of deformations we consider in setting up the Fredholm theory.

In this section, we will prove Proposition 5.29, which was stated and used above:

Proposition 5.29. *Let B_0, \dots, B_k be spherical classes in $H_2(X; \mathbb{Z})$. Let*

$$r \geq \max_i B_i \bullet \Sigma + 2.$$

The universal moduli space $\mathcal{M}_X^((B_1, \dots, B_k); \mathcal{J}_W^r)$ is a Banach manifold and the evaluation maps*

$$\begin{aligned} \text{ev}_\Sigma^a : \mathcal{M}_X^*((B_1, \dots, B_k); \mathcal{J}_W^r) &\rightarrow \Sigma^k : (f_1, f_2, \dots, f_k) \mapsto (f_1(\infty), \dots, f_k(\infty)) \\ \text{ev}_{X, \Sigma} : \mathcal{M}_X^*((B_0); \mathcal{J}_W^r) &\rightarrow X \times \Sigma : f \mapsto (f(0), f(\infty)) \end{aligned}$$

are submersions.

Notice that it suffices to prove this when considering only pairs $(f, J_X) \in \mathcal{M}_X^*((B_0); \mathcal{J}_W^r)$ with the additional condition that $J_X \in \mathcal{J}^r(\mathcal{V})$.

We also observe that if $l = B_0 \bullet \Sigma$, we have that $\mathcal{M}_X^*((B_0); \mathcal{J}_W) \subset \mathcal{M}_{\infty, k, (X, \Sigma)}^*(\mathcal{J}_W)$ for each $k \leq l$. Furthermore, $\mathcal{M}_X^*((B_0); \mathcal{J}_W)$ is a connected component of $\mathcal{M}_{\infty, l, (X, \Sigma)}^*(\mathcal{J}_W)$. This observation will enable us to obtain the result by inducting on k .

The proposition will follow by a modification of the proof given in [12, Section 6]. Instead of reproducing their proof, we indicate the necessary modifications. To be as consistent as possible with their notation, we consider the point of contact with Σ to be at 0.

Consider a J_X -holomorphic map $f: \mathbb{C}\mathbb{P}^1 \rightarrow X$ such that $f(0) \in \Sigma$ with order of contact l . In the notation of [12], we are interested in the case of only one component $Z = \Sigma$. We will obtain transversality of the evaluation map at 0 by varying J_X freely in the complement of our chosen neighbourhood of the divisor, $\mathcal{V} = X \setminus \varphi(\bar{U})$.

The linearized Cauchy–Riemann operator at f with respect to a torsion-free connection is

$$(D_f \xi)(z) = \nabla_s \xi(z) + J_X(f(z)) \nabla_t \xi(z) + (\nabla_{\xi(z)} J_X(f(z))) f_t(z).$$

At a coordinate chart around $z = 0$, we can specialize to the standard Euclidean connection in $\mathbb{R}^{2n} = \mathbb{C}^n$ (which preserves \mathbb{C}^{n-1} along \mathbb{C}^{n-1}), we get

$$(D_f \xi)(z) = \xi_s(z) + J_X(f(z)) \xi_t(z) + A(z) \xi(z),$$

where

$$A(z) \xi(z) = (D_{\xi(z)} J_X(f(z))) f_t(z)$$

(see also page 317 in [12]).

We need the following adaptation of Corollary 6.2 in [12].

Lemma 5.36. *Suppose $(f, J_X) \in \mathcal{M}_{\infty, l, (X, \Sigma)}^*(\mathcal{J}_W^r)$ with $J_X \in \mathcal{J}^r(\mathcal{V})$, $r \geq m$.*

After choosing local coordinates, suppose $f(0) \in \Sigma$ and in coordinates around $f(0)$, Σ is mapped to \mathbb{C}^{n-1} and is thus preserved by J_X .

Denote the unit disk by D^2 and let $\xi: (D^2, 0) \rightarrow (\mathbb{C}^n, 0)$ be such that $D_f \xi = 0$. Given $0 < k \leq l$, if $\xi(0) \in \mathbb{C}^{n-1}$, $d^{k-1} \xi(0) \in \mathbb{C}^{n-1}$ and $\frac{\partial^k \xi}{\partial s^k}(0) \in \mathbb{C}^{n-1}$, then $d^k \xi(0) \in \mathbb{C}^{n-1}$.

Proof. We need to show that $\frac{\partial^k \xi}{\partial s^k \partial t^i}(0) \in \mathbb{C}^{n-1}$ for all $0 \leq i \leq k$. It will be convenient to use multi-index notation for partial derivatives, and denote the previous expression by $D^{(k-i, i)} \xi(0)$. The case $i = 0$ is part of the hypotheses of the lemma. For the induction step, note that $D_f \xi = 0$ combined with the product rule implies that

$$D^{(k-i, i)} \xi(z) = J_X(f(z)) \left(D^{(k-i+1, i-1)} \xi(z) + \sum_{\alpha, \beta} D^\alpha (J_X(f(z))) D^\beta \xi(z) + \sum_{\alpha', \beta'} D^{\alpha'} A(z) D^{\beta'} \xi(z) \right).$$

Here, α and β are multi-indices such that $\alpha = (a_1, a_2)$ for $0 \leq a_1 \leq k-i, 0 \leq a_2 \leq i-1, \alpha \neq (0, 0)$ and $\alpha + \beta = (k-i, i)$. Similarly, α' and β' are multi-indices such that $\alpha' = (a'_1, a'_2)$ for $0 \leq a'_1 \leq k-i, 0 \leq a'_2 \leq i-1$ and $\alpha' + \beta' = (k-i, i-1)$. The hypotheses of the lemma and the induction hypothesis imply that the derivatives of ξ on the right-hand side take values in $T_{f(0)}\Sigma$. The fact that J_X and ∇ preserve \mathbb{C}^{n-1} along \mathbb{C}^{n-1} , and that $d^l f(0) \in T_{f(0)}\Sigma$, implies the induction step. □

We now prove the key property of the linearized evaluation map:

Proposition 5.37. *For $m - 2/p > l, r \geq m$, the universal evaluation map*

$$\begin{aligned} \text{ev}_{X, \Sigma}: \mathcal{M}_{\infty, l, (X, \Sigma)}^*(\mathcal{J}_W^r) &\rightarrow \Sigma \\ (f, J_X) &\mapsto f(0) \end{aligned}$$

is a submersion.

Proof. We show that for every $0 \leq k \leq l$, and $(f, J_X) \in \mathcal{M}_{\infty, k, (X, \Sigma)}^*(\mathcal{J}^r(\mathcal{V}))$,

$$\begin{aligned} (d \text{ev}_{X, \Sigma})_{(f, J_X)}: T_{(f, J_X)} \mathcal{M}_{\infty, k, (X, \Sigma)}^*(\mathcal{J}^r(\mathcal{V})) &\rightarrow T_{f(0)} \Sigma \\ (\xi, Y) &\mapsto \xi(0) \end{aligned}$$

is surjective. By Lemma 6.5 in [12],

$$\begin{aligned} T_{(f, J_X)} \mathcal{M}_{\infty, k, (X, \Sigma)}^*(\mathcal{J}^r(\mathcal{V})) &= \{(\xi, Y) \in T_f W^{m, p}(\mathbb{C}P^1, X) \times T_{J_X} \mathcal{J}^r(\mathcal{V}) \mid \\ D_f \xi + \frac{1}{2} Y(f) \circ df \circ j &= 0, \\ \xi(0) \in T_{f(0)} \Sigma, d^k \xi(0) \in T_{f(0)} \Sigma\}. \end{aligned}$$

We argue by induction on k . The case $k = 0$ is a special case of Proposition 3.4.2 in [29]. We assume that the claim is true for $k - 1$ and prove it for k .

Take any $v \in T_{f(0)}\Sigma$. By induction, there is $(\xi_1, Y_1) \in T_{(f, J_X)}\mathcal{M}_{\infty, k-1, (X, \Sigma)}^*(\mathcal{J}^r(\mathcal{V}))$ such that $(\text{dev}_{X, \Sigma})_{(f, J_X)}(\xi_1, Y_1) = v$ and $d^{k-1}\xi_1(0) \in T_{f(0)}\Sigma$. Let now $\tilde{\xi} \in T_f W^{m, p}(\mathbb{C}\mathbb{P}^1, X)$ be given by

$$\tilde{\xi}(z) = -\frac{z^k}{k!}\beta(z)\pi_{\mathbb{C}}\left(\frac{\partial^k}{\partial s^k}\xi_1\right)(0),$$

where $\beta: \mathbb{C} \rightarrow [0, 1]$ is a smooth function that is identically 1 near 0 and has compact support contained in $\mathbb{C} \setminus f^{-1}(\mathcal{V})$. Writing

$$(D_f \tilde{\xi})(z) = \xi_s(z) + i\xi_t(z) + (J_X(f(z)) - i)\xi_t(z) + A(z)\xi(z),$$

we have $(D_f \tilde{\xi})(0) = 0$ and $d^{k-1}(D_f \tilde{\xi})(0) = 0$ (this follows the fact that $\tilde{\xi}_s + i\tilde{\xi}_t \equiv 0$ near 0). By Lemma 6.6 in [12], there is $(\hat{\xi}, \hat{Y}) \in T_f W^{m, p}(\mathbb{C}\mathbb{P}^1, X) \times T_{J_X} \mathcal{J}(\mathcal{V})$ such that $\hat{\xi}(0) = 0$, $d^k(\hat{\xi})(0) = 0$ and

$$D_f \hat{\xi} + \frac{1}{2}\hat{Y}(f) \circ df \circ j = -D_f \tilde{\xi}.$$

Let now $\xi_2 = \xi_1 + \tilde{\xi} + \hat{\xi}$ and $Y_2 = Y_1 + \hat{Y}$. We have

$$D_f \xi_2 + \frac{1}{2}Y_2(f) \circ df \circ j = 0$$

as well as $\xi_2(0) = v$, $d^{k-1}(\xi_2)(0) \in T_{f(0)}\Sigma$ and $\pi_{\mathbb{C}}\left(\frac{\partial^k}{\partial s^k}\xi_2\right)(0) = 0$.

Lemma 5.36 implies that $d^k(\xi_2)(0) \in T_{f(0)}\Sigma$; hence, $(\xi_2, Y_2) \in T_{(f, J_X)}\mathcal{M}_{\infty, l, (X, \Sigma)}^*(\mathcal{J}_W^r)$. This completes the proof. \square

Observe now that by combining this with standard arguments (see, for instance, [29, Proposition 3.4.2], which is also used in the proof of Proposition 5.26 above), we obtain the transversality for the evaluation at a point, taking values in X . This finishes the proof of Proposition 5.29.

5.5. Proof of Proposition 5.9

We are now ready to complete the proof of Proposition 5.9. To this end, we will show that the transversality problem for a cascade reduces to the already solved transversality problem for chains of pearls. The two key ingredients of this are the splitting of the linearized operator given by Lemma 5.22 and a careful study of the flow diagonal in $Y \times Y$.

Recall from Definition 2.7 that \mathcal{J}_Y denotes the space of compatible, cylindrical, Reeb–invariant almost complex structures on $\mathbb{R} \times Y$. These are obtained as lifts of the almost complex structures in \mathcal{J}_{Σ} . Let \mathcal{J}_Y^{reg} be the set of almost complex structures on $\mathbb{R} \times Y$ that are lifts of the almost complex structures in $\mathcal{J}_{\Sigma}^{reg}$ (see Proposition 5.35).

Recall from Definition 2.7 and from Proposition 2.3, if $J_W \in \mathcal{J}_W$ is an almost complex structure on W that is of the type we consider, it induces an almost complex structure $P(J_W) = J_{\Sigma} \in \mathcal{J}_{\Sigma}$. The restriction of J_W to the cylindrical end of W , J_Y , is then a translation and Reeb–flow invariant almost complex structure on $\mathbb{R} \times Y$ that has $d\pi_{\Sigma}J_Y = J_{\Sigma}d\pi_{\Sigma}$.

Recall that the biholomorphism $\psi: W \rightarrow X \setminus \Sigma$ given in Lemma 2.5 allows us to identify holomorphic planes in W with holomorphic spheres in X . In the following, we will suppress the distinction when convenient.

Recall also that by the definition of an admissible Hamiltonian (Definition 3.1), for each non-negative integer m , there exists a unique b_m so that $h'(e^{b_m}) = m$. Then $Y_m = \{b_m\} \times Y \subset \mathbb{R} \times Y$ is the corresponding Morse–Bott family of 1-periodic Hamiltonian orbits that wind m times around the fibre of $Y \rightarrow \Sigma$.

We now define moduli spaces of Floer cylinders, from which we will extract the moduli spaces of cascades by imposing the gradient flow line conditions. First, we define the moduli spaces relevant to the differential connecting two generators in $\mathbb{R} \times Y$. Then, we will define the moduli spaces relevant to the differential connecting to a critical point in W .

Definition 5.38. Let $N \geq 1$, let $A_1, \dots, A_N \in H_2(\Sigma; \mathbb{Z})$ be spherical homology classes. Let $J_Y \in \mathcal{J}_Y$.

Define $\mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^*((A_1, \dots, A_N); J_Y)$ to be a set of tuples of punctured cylinders $(\tilde{v}_1, \dots, \tilde{v}_N)$ with the following properties:

- (1) There is a partition of $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_N$ of k augmentation marked points with

$$\tilde{v}_i: \mathbb{R} \times S^1 \setminus \Gamma_i \rightarrow \mathbb{R} \times Y$$

so that \tilde{v}_i is a finite hybrid energy punctured Floer cylinder. For each $z_j \in \Gamma$, there is a positive integer multiplicity $k(z_j)$. Let v_i denote the projection to Y .

- (2) There is an increasing list of $N + 1$ multiplicities from k_- to k_+ :

$$k_- = k_0 < k_1 < k_2 < \dots < k_N = k_+$$

such that, for each i , the cylinder \tilde{v}_i has multiplicities k_i and k_{i-1} at $\pm\infty$, i.e. $\tilde{v}_i(+\infty, \cdot) \in Y_{k_i}, \tilde{v}_i(-\infty, \cdot) \in Y_{k_{i-1}}$.

- (3) The Floer cylinders \tilde{v}_i are simple in the sense that their projections to Σ are either somewhere injective or constant, if constant, they have at least one augmentation puncture, and their images are not contained one in the other.
- (4) For each i , and for every puncture $z_j \in \Gamma_i$, the augmentation puncture has a limit whose multiplicity is given by $k(z_j)$; i.e. $\lim_{\rho \rightarrow -\infty} v_i(z_j + e^{2\pi(\rho + i\theta)})$ is a Reeb orbit of multiplicity $k(z_j)$.
- (5) The projections of the Floer cylinders to Σ represent the homology classes $A_i, i = 1, \dots, N$; i.e. $(\pi_\Sigma(\tilde{v}_i))_{i=1}^N \in \mathcal{M}_k^*((A_1, \dots, A_N), J_\Sigma)$.

Let $B \in H_2(X; \mathbb{Z})$ be a spherical homology class, $B \neq 0$. Let J_W be an almost complex structure on W as given by Lemma 2.5, matching J_Y on the cylindrical end.

Definition 5.39. Define the moduli space

$$\mathcal{M}_{H,k,W; k_+}^*((B; A_1, \dots, A_N); J_W)$$

to consist of tuples

$$(\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_N)$$

with the properties

- (1) The map $\tilde{v}_0: \mathbb{R} \times S^1 \rightarrow W$ is a finite energy holomorphic cylinder with removable singularity at $-\infty$.
- (2) There is a partition of $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_N$ of k augmentation marked points with

$$\tilde{v}_i: \mathbb{R} \times S^1 \setminus \Gamma_i \rightarrow \mathbb{R} \times Y, \quad i \geq 1,$$

so that each \tilde{v}_i is a finite hybrid energy punctured Floer cylinder. For each $z_j \in \Gamma$, there is a positive integer multiplicity $k(z_j)$. Denote by v_i the projection of \tilde{v}_i to Y .

- (3) There is an increasing list of $N + 1$ multiplicities:

$$k_0 < k_1 < k_2 < \dots < k_N = k_+.$$

- (4) For each $i \geq 1$, and for every puncture $z_j \in \Gamma_i$, the augmentation puncture has a limit whose multiplicity is given by $k(z_j)$, i.e. $\lim_{\rho \rightarrow -\infty} v_i(z_j + e^{2\pi(\rho+i\theta)})$ is a Reeb orbit of multiplicity $k(z_j)$.
- (5) The Floer cylinders \tilde{v}_i for $i \geq 1$ are simple, in the strong sense that the projections to Σ are somewhere injective or constant, and have images not contained one in the other. The cylinder \tilde{v}_0 is somewhere injective in W .
- (6) The projections of the Floer cylinders to Σ represent the homology classes $A_i, i = 1, \dots, N$; i.e. $\pi_\Sigma(\tilde{v}_i)_{i=1}^N \in \mathcal{M}_k^*((B; A_1, \dots, A_N), J_W)$.
- (7) After identifying \tilde{v}_0 with a holomorphic sphere in X , \tilde{v}_0 represents the homology class $B \in H_2(X; \mathbb{Z})$.
- (8) The cylinder \tilde{v}_1 has multiplicity k_1 at $+\infty$ and $\tilde{v}_1(+\infty, \cdot) \in Y_{k_1}$. At $-\infty$, \tilde{v}_1 converges to a Reeb orbit in $\{-\infty\} \times Y$. This Reeb orbit has multiplicity k_0 .
- (9) For each $i \geq 2$, the cylinder \tilde{v}_i has multiplicities k_i and k_{i-1} at $\pm\infty$: $\tilde{v}_i(+\infty, \cdot) \in Y_{k_i}, \tilde{v}_i(-\infty, \cdot) \in Y_{k_{i-1}}$.
- (10) The cylinder \tilde{v}_0 converges at $+\infty$ to a Reeb orbit of multiplicity k_0 .

Observe that these moduli spaces are non-empty only if for each $i = 1, \dots, N$,

$$K\omega(A_i) = k_i - k_{i-1} - \sum_{z \in \Gamma_i} k(z).$$

Furthermore, for $\mathcal{M}_{H,k,W}^*$, we must also have

$$B \bullet \Sigma = K\omega(B) = k_0.$$

Note also that these moduli spaces have a large number of connected components, where different components have different partitions of Γ or different intermediate multiplicities.

Identifying holomorphic spheres in X with finite energy J_W -planes in W , we consider also the moduli space of holomorphic planes $\mathcal{M}_X^*((B_1, \dots, B_k); J_W)$ as in Definition 5.25.

The space $\mathcal{M}_{H,k,\mathbb{R} \times Y}^*((A_1, \dots, A_N); J_Y)$ consists of N -tuples of somewhere injective punctured Floer cylinders in $\mathbb{R} \times Y$. Similarly, $\mathcal{M}_{H,k,W}^*$ consist of N -tuples of punctured Floer cylinders in $\mathbb{R} \times Y$ together with a holomorphic plane in W (which we can, therefore, also interpret as a holomorphic

sphere in X). The cylinders and the eventual plane have asymptotics with matching multiplicities, but are otherwise unconstrained. These two moduli spaces, $\mathcal{M}_{H,k,\mathbb{R}\times Y}^*$ and $\mathcal{M}_{H,k,W}^*$ fail to be simple split Floer cylinders with cascades (as in Definition 5.8) in two ways: they are missing the gradient trajectory constraints on their asymptotic evaluation maps, and they are missing their augmentation planes. To impose these conditions, we will need to study these evaluation maps and establish their transversality.

Proposition 5.40. *For $J_Y \in \mathcal{J}_Y^{reg}$, $\mathcal{M}_{H,k,\mathbb{R}\times Y}^*((A_1, \dots, A_N); J_Y)$ is a manifold of dimension*

$$N(2n - 1) + \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle + 2k.$$

For $J_W \in \mathcal{J}_W^{reg}$, $\mathcal{M}_{H,k,W}^((B; A_1, \dots, A_N); J_W)$ is a manifold of dimension*

$$N(2n - 1) + 2n + 1 + \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle + 2(\langle c_1(TX), B \rangle - B \bullet \Sigma) + 2k.$$

Proof. Consider first the case of cylinders in $\mathbb{R} \times Y$. Let

$$(\tilde{v}_1, \dots, \tilde{v}_N) \in \mathcal{M}_{H,k,\mathbb{R}\times Y}^*((A_1, \dots, A_N); J_Y).$$

Recall from Proposition 5.26 that for $J_\Sigma \in \mathcal{J}_\Sigma^{reg}$, we have transversality for $D_{w_i}^\Sigma$ for each sphere $w_i = \pi_\Sigma(v_i)$.

Let $\delta > 0$ be sufficiently small. For each $i = 1, \dots, N$, by Lemma 5.24, $D_{\tilde{v}_i}^\mathbb{C}$ is surjective when considered on $W^{1,p,-\delta}$ (with exponential growth), and has Fredholm index 1. The operator considered instead on the space $W_{\mathbf{V}}^{1,p,\delta}$, with $V_{-\infty} = V_{+\infty} = i\mathbb{R}$ and $V_P = \mathbb{C}$ for any puncture P on the domain of \tilde{v}_i , has the same kernel and cokernel by Lemma 5.20. Thus, the operator, acting on sections free to move in the Morse–Bott family of orbits, is surjective and has index 1.

Since the operator $D_{\tilde{v}_i}$ is upper triangular from Lemma 5.22, and its diagonal components are both surjective, the operator is surjective. Since the Fredholm index is the sum of these, each component \tilde{v}_i contributes an index of $1 + 2n - 2 + 2 \langle c_1(T\Sigma), A_i \rangle + 2k_i = 2n - 1 + 2 \langle c_1(T\Sigma), A_i \rangle + 2k_i$, where k_i is the number of punctures.

We now consider the case of a collection

$$(\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_N) \in \mathcal{M}_{H,k,W}^*((B; A_1, \dots, A_N); J_W).$$

The same consideration as previously gives that $\tilde{v}_2, \dots, \tilde{v}_N$ are transverse and each contributes an index of $2n - 1 + 2 \langle c_1(T\Sigma), A_i \rangle + 2k_i$, where k_i is the number of punctures. For the component \tilde{v}_1 , again applying Lemma 5.22 and applying Lemma 5.24 in the case where the $-\infty$ end of the cylinder converges to a Reeb orbit at $\{-\infty\} \times Y$, we obtain that the vertical Fredholm operator is surjective and has index 2. The linearized Floer operator at \tilde{v}_1 is then surjective and has index $2n + 2 \langle c_1(T\Sigma), A_1 \rangle + 2k_1$. By Lemma 2.5, the plane \tilde{v}_0 can be identified with a sphere in X with an order of contact $l = B \bullet \Sigma$

with Σ . Its Fredholm index is $2n + 2(\langle c_1(TX), B \rangle - l)$. The total Fredholm index is, therefore,

$$(N - 1)(2n - 1) + 2n + 2n + \sum_{i=1}^N 2 \langle c_1(T\Sigma), A_i \rangle + 2(\langle c_1(TX), B \rangle - B \bullet \Sigma) + 2k.$$

For both cases, the result now follows from the implicit function theorem. □

It now suffices to prove the transversality of evaluation maps to the products of stable/unstable manifolds and flow diagonals, and also transversality of the augmentation evaluation maps, to obtain the constraints coming from pseudo-gradient flow lines. Indeed, let $(\tilde{v}_1, \dots, \tilde{v}_N)$ be a collection of N cylinders in $\mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^*((A_1, \dots, A_N); J_Y)$. Write each of the $\tilde{v}_i : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times Y$ as a pair $\tilde{v}_i = (b_i, v_i)$. We then have asymptotic evaluation maps

$$\begin{aligned} \tilde{e\tilde{v}}_Y & : \mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^*((A_1, \dots, A_N); J_Y) \rightarrow Y^{2N} \\ (\tilde{v}_1, \dots, \tilde{v}_N) & \mapsto (\lim_{s \rightarrow -\infty} v_1(s, 1), \lim_{s \rightarrow +\infty} v_1(s, 1), \dots, \\ & \lim_{s \rightarrow -\infty} v_N(s, 1), \lim_{s \rightarrow +\infty} v_N(s, 1)). \end{aligned} \tag{5.8}$$

If $(\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_N) \in \mathcal{M}_{H,k,W}^*((B; A_1, \dots, A_N); J_W)$, we have

$$\begin{aligned} \tilde{e\tilde{v}}_{W,Y} : \mathcal{M}_{H,k,W; k_+}^*((B; A_1, \dots, A_N); J_W) & \rightarrow W \times Y^{2N+1} \\ (\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_N) & \mapsto \left(\tilde{v}_0(0), \lim_{r \rightarrow +\infty} \pi_Y \tilde{v}_0(r + i0), \lim_{s \rightarrow -\infty} v_1(s, 1), \dots, \right. \\ & \left. \lim_{s \rightarrow +\infty} v_N(s, 1) \right). \end{aligned} \tag{5.9}$$

These maps are C^1 smooth, which follows from exploiting the asymptotic expansion of a Floer cylinder near its asymptotic limit, as described by [39]. Details for this are given in [17].

We also have augmentation evaluation maps. For each puncture $z_0 \in \Gamma$, there exists an index $i \in \{1, \dots, N\}$ so that the augmentation puncture z_0 is a puncture in the domain of v_i . For this augmentation puncture, we have the asymptotic evaluation map $v_i \mapsto \lim_{z \rightarrow z_0} \pi_\Sigma(v_i(z)) \in \Sigma$. Combining all of these evaluation maps over all punctures in Γ , we obtain

$$\begin{aligned} \tilde{e\tilde{v}}_\Sigma^a & : \mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^*((A_1, \dots, A_N); J_Y) \rightarrow \Sigma^k \\ \tilde{e\tilde{v}}_\Sigma^a & : \mathcal{M}_{H,k,W; k_+}^*((B; A_1, \dots, A_N); J_W) \rightarrow \Sigma^k. \end{aligned}$$

Note that these maps are C^1 smooth, either by [17] or by combining [43, Proposition 3.15] with the smoothness for the evaluation map for closed spheres.

Define the *flow diagonal* in $Y \times Y$ to be

$$\tilde{\Delta}_{f_Y} := \{(x, y) \in (Y \setminus \text{Crit}(f_Y))^2 : \exists t > 0 \text{ s.t. } \varphi_{Z_Y}^t(x) = y\},$$

where $\text{Crit}(f_Y)$ is the set of critical points of f_Y .

Let $\tilde{p}, \tilde{q} \in Y$ be critical points of f_Y and let $W_Y^u(\tilde{p}), W_Y^s(\tilde{q})$ be the unstable/stable manifolds of \tilde{p}, \tilde{q} , as in (3.1).

We may now describe the moduli space of simple split Floer cylinders from \tilde{q}_{k-} to \tilde{p}_{k+} as the unions of the fibre products of these moduli spaces under the asymptotic evaluation maps and augmentation evaluation maps. For notational convenience, we write

$$\begin{aligned} \tilde{\text{ev}}: \mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^* ((A_1, \dots, A_N); J_Y) \times \mathcal{M}_X^*((B_1, \dots, B_k); J_W) \\ \rightarrow Y^{2N} \times \Sigma^k \times \Sigma^k \\ (\tilde{\mathbf{v}}, \mathbf{v}) \mapsto (\tilde{\text{ev}}_Y(\tilde{\mathbf{v}}), \tilde{\text{ev}}_\Sigma^a(\tilde{\mathbf{v}}), \text{ev}_\Sigma^a(\mathbf{v})). \end{aligned} \tag{5.10}$$

Write $\Delta_{\Sigma^k} \subset \Sigma^k \times \Sigma^k$ to denote the diagonal Σ^k . Then define

$$\begin{aligned} \mathcal{M}_H^*(\tilde{q}_{k-}, \tilde{p}_{k+}; (A_1, \dots, A_N), (B_1, \dots, B_k); J_W) \\ = \tilde{\text{ev}}^{-1} \left(W_Y^s(\tilde{q}) \times \left(\tilde{\Delta}_{f_Y} \right)^{N-1} \times W_Y^u(\tilde{p}) \times \Delta_{\Sigma^k} \right). \end{aligned}$$

From this, we have

$$\begin{aligned} \mathcal{M}_{H,N}^*(\tilde{q}_{k-}, \tilde{p}_{k+}; J_W) \\ = \bigcup_{(A_1, \dots, A_N)} \bigcup_{k \geq 0} \bigcup_{(B_1, \dots, B_k)} \mathcal{M}_H^*(\tilde{q}_{k-}, \tilde{p}_{k+}; (A_1, \dots, A_N), (B_1, \dots, B_k); J_W). \end{aligned} \tag{5.11}$$

Similarly, if $x \in W$ is a critical point of f_W , and letting $W_W^u(x)$ be the descending manifold of x in W for the gradient-like vector field $-Z_W$, we define

$$\begin{aligned} \tilde{\text{ev}}: \mathcal{M}_{H,k,W; k_+}^* ((B; A_1, \dots, A_N); J_W) \times \mathcal{M}_X^*((B_1, \dots, B_k); J_W) \\ \rightarrow W \times Y^{2N+1} \times \Sigma^k \times \Sigma^k \\ ((\tilde{v}_0, \tilde{\mathbf{v}}), \mathbf{v}) \mapsto (\tilde{\text{ev}}_{W,Y}(\tilde{v}_0, \tilde{\mathbf{v}}), \tilde{\text{ev}}_\Sigma^a(\tilde{\mathbf{v}}), \text{ev}_\Sigma^a(\mathbf{v})). \end{aligned}$$

Then define

$$\begin{aligned} \mathcal{M}_H^*(x, \tilde{p}_{k+}; (B; A_1, \dots, A_N), (B_1, \dots, B_k); J_W) \\ = \tilde{\text{ev}}^{-1} \left(W_W^u(x) \times \tilde{\Delta} \times \left(\tilde{\Delta}_{f_Y} \right)^{N-1} \times W_Y^u(\tilde{p}) \times \Delta_{\Sigma^k} \right). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \mathcal{M}_{H,N}^*(x, \tilde{p}_{k+}; J_W) \\ = \bigcup_{(B; A_1, \dots, A_N)} \bigcup_{k \geq 0} \bigcup_{(B_1, \dots, B_k)} \mathcal{M}_H^*(x, \tilde{p}_{k+}; (B; A_1, \dots, A_N), (B_1, \dots, B_k); J_W). \end{aligned} \tag{5.12}$$

To establish transversality for our moduli spaces, it then becomes necessary to show transversality of the evaluation maps to these products of descending/ascending manifolds, diagonals and flow diagonals. Recall the space of almost complex structures \mathcal{J}_W^{reg} given in Proposition 5.35. We denote by \mathcal{J}_Y^{reg} the space of cylindrical almost complex structures on $\mathbb{R} \times Y$ obtained from restrictions of elements in \mathcal{J}_W^{reg} . The following result will provide the final step in the proof of Proposition 5.9.

Proposition 5.41. *Let $J_W \in \mathcal{J}_W^{reg}$ and let $J_Y \in \mathcal{J}_Y^{reg}$ be the induced almost complex structure on $\mathbb{R} \times Y$.*

Let \tilde{q}, \tilde{p} denote critical points of f_Y , and let x be a critical point of f_W in W . Let k_+ and k_- be non-negative multiplicities, $k_+ > k_-$.

Let A_1, \dots, A_N be spherical homology classes in Σ , let B, B_1, \dots, B_k be spherical homology classes in X , $k \geq 0$.

Let $\tilde{\Delta} \subset Y \times Y$ and $\Delta_{\Sigma^k} \subset \Sigma^k \times \Sigma^k$ be the diagonals.

Then

(1) *the evaluation map*

$$\begin{aligned} \tilde{ev}_Y \times \tilde{ev}_\Sigma^a \times ev_\Sigma^a : \mathcal{M}_{H,k,\mathbb{R} \times Y;k_-,k_+}^*((A_1, \dots, A_N); J_Y) \\ \times \mathcal{M}_X^*((B_1, \dots, B_k); J_W) \rightarrow Y^{2N} \times \Sigma^k \times \Sigma^k \end{aligned}$$

is transverse to the submanifold

$$W_Y^s(\tilde{q}) \times \left(\tilde{\Delta}_{f_Y}\right)^{N-1} \times W_Y^u(\tilde{p}) \times \Delta_{\Sigma^k};$$

(2) *the evaluation map*

$$\begin{aligned} \tilde{ev}_{W,Y} \times \tilde{ev}_\Sigma^a \times ev_\Sigma^a : \mathcal{M}_{H,k,W;k_+}^*((B; A_1, \dots, A_N); J_W) \\ \times \mathcal{M}_X^*((B_1, \dots, B_k); J_W) \rightarrow W \times Y^{2N+1} \times \Sigma^k \times \Sigma^k \end{aligned}$$

is transverse to the submanifold

$$W_W^u(x) \times \tilde{\Delta} \times \left(\tilde{\Delta}_{f_Y}\right)^{N-1} \times W_Y^u(\tilde{p}) \times \Delta_{\Sigma^k}.$$

To prove this proposition, we will need a better description of the relationship between the moduli spaces of spheres in Σ , and the moduli spaces of Floer cylinders in $\mathbb{R} \times Y$ (or in W).

Lemma 5.42. *The maps*

$$\pi_\Sigma^{\mathcal{M}} : \mathcal{M}_{H,k,\mathbb{R} \times Y;k_-,k_+}^*((A_1, \dots, A_N); J_Y) \rightarrow \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); J_\Sigma)$$

$$\pi_\Sigma^{\mathcal{M}} : \mathcal{M}_{H,k,W;k_+}^*((B; A_1, \dots, A_N); J_W) \rightarrow \mathcal{M}_{k,X,\Sigma}^*((B; A_1, \dots, A_N); J_W)$$

induced by $\pi_\Sigma : \mathbb{R} \times Y \rightarrow \Sigma$ are submersions. The fibres have a locally free $(S^1)^N$ torus action by constant rotation by the action of the Reeb vector field.

Proof. We will study the case of

$$\pi_\Sigma^{\mathcal{M}} : \mathcal{M}_{H,k,\mathbb{R} \times Y}^*((A_1, \dots, A_N); J_Y) \rightarrow \mathcal{M}_{k,\Sigma}^*((A_1, \dots, A_N); J_\Sigma)$$

in detail. The case with a sphere in X follows by the same argument with a small notational change. It also suffices to consider the case with $N = 1$ since moduli spaces with more spheres are open subsets of products of these.

Suppose $\pi_\Sigma(\tilde{v}) = w$ with $\tilde{v} \in \mathcal{M}_{H,k,\mathbb{R} \times Y;k_-,k_+}^*(A; J_Y)$ and $w \in \mathcal{M}_{k,\Sigma}^*(A; J_\Sigma)$. Recall the splitting of the linearized Floer operator at \tilde{v} , given in Lemma 5.22 as

$$D_{\tilde{v}} = \begin{pmatrix} D_{\tilde{v}}^{\mathbb{C}} & M \\ 0 & D_w^\Sigma \end{pmatrix}.$$

By definition, $T_w \mathcal{M}_{k,\Sigma}^*(A) = \ker D_w^\Sigma$ and $T_{\tilde{v}} \mathcal{M}_{H,k,\mathbb{R} \times Y;k_-,k_+}^*(A; J_Y) = \ker D_{\tilde{v}}$. By Lemma 5.24, $D_{\tilde{v}}^{\mathbb{C}}$ is surjective. It follows then that any section

ζ_0 of $w^*T\Sigma$ that is in the kernel of D_w^Σ can be lifted to a section (ζ_1, ζ_0) of $\tilde{v}^*T\Sigma \cong (\mathbb{R} \oplus \mathbb{R}R) \oplus w^*T\Sigma$ that is in the kernel of $D_{\tilde{v}}$.

Notice now that $d\pi_\Sigma(\zeta_1, \zeta_0) = \zeta_0$, establishing that the evaluation map is a submersion.

Also observe that S^1 acts on the curve \tilde{v} by the Reeb flow. By the Reeb invariance of J_Y and of the admissible Hamiltonian H , the rotated curve is in the same fibre of π_Σ^M . Furthermore, for small rotation parameter, the curve will be distinct (as a parametrized curve) from \tilde{v} . \square

The next result justifies why it was reasonable to assume $k_+ > k_-$ in Proposition 5.41. The fact that $k_+ \neq k_-$ will also be used below.

Lemma 5.43. *Let $A := [w] \in H_2(\Sigma; \mathbb{Z})$, where $w: \mathbb{C}P^1 \rightarrow \Sigma$ is the continuous extension of $\pi_\Sigma \circ \tilde{v}$. Assume that either $A \neq 0$ or $\Gamma \neq \emptyset$. Then $k_+ > k_-$.*

Proof. Denote by w^*Y the pull-back under w of the S^1 -bundle $Y \rightarrow \Sigma$. The map \tilde{v} gives a section s of w^*Y , defined in the complement of $\Gamma \cup \{0, \infty\}$. By [11, Theorem 11.16], the Euler number $\int_{\mathbb{C}P^1} e(w^*Y)$ (where e is the Euler class) is the sum of the local degrees of the section s at the points in $\Gamma \cup \{0, \infty\}$.

Denote the multiplicities of the periodic X_H -orbits $x_\pm(t) = \lim_{s \rightarrow \pm\infty} v(s, t)$ by k_\pm , respectively, and denote the multiplicities of the asymptotic Reeb orbits at the punctures $z_1, \dots, z_m \in \Gamma$ by k_1, \dots, k_m , respectively. The positive integers k_\pm and k_i are the absolute values of the degrees of s at the respective points. Taking signs into account, we get

$$\int_{\mathbb{C}P^1} e(w^*Y) = k_+ - k_- - k_1 - \dots - k_m.$$

We will show that this quantity is non-negative. We have

$$\int_{\mathbb{C}P^1} e(w^*Y) = \int_{\mathbb{C}P^1} w^*e(Y \rightarrow \Sigma) = \int_{\mathbb{C}P^1} w^*e(N\Sigma),$$

where $N\Sigma$ is the normal bundle to Σ in Y . Now, $e(N\Sigma) = s^* \text{Th}(N\Sigma)$, where $s: \Sigma \rightarrow N\Sigma$ is the zero section and $\text{Th}(N\Sigma)$ is the Thom class of $N\Sigma$ [11, Proposition 6.41]. If $j: N\Sigma \rightarrow X$ is a tubular neighbourhood, then $j_* \text{Th}(N\Sigma) = PD([\Sigma]) = [K\omega] \in H^2(X; \mathbb{R})$ [11, Equation (6.23)]. If $\iota: \Sigma \hookrightarrow X$ is the inclusion, then

$$\begin{aligned} \int_{\mathbb{C}P^1} w^*e(N\Sigma) &= \int_{\mathbb{C}P^1} w^*s^* \text{Th}(N\Sigma) = \int_{\mathbb{C}P^1} w^*\iota^*j_* \text{Th}(N\Sigma) \\ &= \int_{\mathbb{C}P^1} w^*\iota^*K\omega = K\omega(A) \geq 0 \end{aligned}$$

since $K > 0$ and w is a J_Σ -holomorphic sphere. We conclude that

$$k_+ - k_- - k_1 - \dots - k_m = K\omega(A) \geq 0.$$

If $A \neq 0$, we get a strict inequality. If $A = 0$, we get an equality, but the assumptions of the lemma imply that $\sum_{i=1}^m k_i > 0$. In either case, we get $k_+ > k_-$, as wanted. \square

Recall that the gradient-like vector field Z_Y has the property that $d\pi_\Sigma Z_Y = Z_\Sigma$. Also recall that we may use the contact form α as a connection to lift vector fields from Σ to vector fields on Y , tangent to ξ . If V is a vector field on Σ , we write $\pi_\Sigma^* V := \tilde{V}$ to be the vector field on Y uniquely determined by the conditions $\alpha(\tilde{V}) = 0$, $d\pi_\Sigma \tilde{V} = V$. This extends as well to lifting vector fields on $\Sigma \times \Sigma$ to vector fields on $Y \times Y$.

Lemma 5.44. *The flow diagonal in Y satisfies*

$$\pi_\Sigma(\tilde{\Delta}_{f_Y}) \subset \Delta_{f_\Sigma} \cup \{(p, p) \mid p \in \text{Crit}(f_\Sigma)\}.$$

Let $(\tilde{x}, \tilde{y}) \in \tilde{\Delta}_{f_Y}$ and $x = \pi_\Sigma(\tilde{x})$, $y = \pi_\Sigma(\tilde{y})$. Let $t > 0$ be so that $\tilde{y} = \varphi_{Z_Y}^t(\tilde{x})$. Then if $x = y$, we have $x \in \text{Crit}(f_\Sigma)$ and

$$\begin{aligned} T_{(\tilde{x}, \tilde{y})} \tilde{\Delta}_{f_Y} \\ = \{(aR + v, bR + \pi_\Sigma^* d\varphi_{Z_\Sigma}^t d\pi_\Sigma v) \in TY \oplus TY \mid a, b \in \mathbb{R} \text{ and } \alpha(v) = 0\}. \end{aligned} \tag{5.13}$$

If $x \neq y$, then $(x, y) \in \Delta_{f_\Sigma}$. Then there exists a positive $g = g(\tilde{x}, \tilde{y}) > 0$ so that

$$T_{(\tilde{x}, \tilde{y})} \tilde{\Delta}_{f_Y} = \mathbb{R}(R, gR) \oplus H, \tag{5.14}$$

where the subspace H is such that $d\pi_\Sigma|_H : H \rightarrow T\Delta_{f_\Sigma}$ induces a linear isomorphism.

Proof. Observe first that if $x = \pi_\Sigma \tilde{x}$, we have

$$\pi_\Sigma \varphi_{Z_Y}^t(\tilde{x}) = \varphi_{Z_\Sigma}^t(x).$$

This gives $d\pi_\Sigma d\varphi_{Z_Y}^t(\tilde{x}) = d\varphi_{Z_\Sigma}^t d\pi_\Sigma(\tilde{x})$. From this, it follows that $d\varphi_{Z_Y}^t(\tilde{x})R$ is a multiple of the Reeb vector field. Observe also that $\varphi_{Z_\Sigma}^t$ and $\varphi_{Z_Y}^t$ are both orientation-preserving diffeomorphisms for all t . We, therefore, obtain that if $y = \varphi_{Z_\Sigma}^t(x)$, $\varphi_{Z_Y}^t$ induces a diffeomorphism between the fibres $\pi_\Sigma^{-1}(x) \rightarrow \pi_\Sigma^{-1}(y)$. Additionally, we must have then that $d\varphi_{Z_Y}^t(\tilde{x})R$ is a positive multiple of the Reeb vector field. Let $g(\tilde{x}, \tilde{y}) > 0$ such that $d\varphi_{Z_Y}^t(\tilde{x})R = g(\tilde{x}, \tilde{y})R$.

In general, if $\tilde{y} = \varphi_{Z_Y}^t(\tilde{x})$, we have

$$T_{(\tilde{x}, \tilde{y})} \tilde{\Delta}_{f_Y} = \{(v, d\varphi_{Z_Y}^t(\tilde{x})v + cZ_Y(\tilde{y})) \mid v \in T_x Y, c \in \mathbb{R}\}. \tag{5.15}$$

Consider first the case of $x = y$. Then both \tilde{x} and \tilde{y} are in the same fibre of $Y \rightarrow \Sigma$. By definition of the flow diagonal, there exists $t > 0$ so that $\varphi_{Z_Y}^t(\tilde{x}) = \tilde{y}$, and hence Z_Y is vertical, $Z_\Sigma(x) = 0$. It follows that $x \in \text{Crit}_{f_\Sigma}$. From this, it now follows that $\pi_\Sigma(\tilde{\Delta}_{f_Y}) \subset \Delta_{f_\Sigma} \cup \{(p, p) \mid p \in \text{Crit}(f_\Sigma)\}$.

We now consider the consequences of Eq. (5.15) in this case of $x = y$. Any $v \in T_x Y$ may be written as $v_0 + aR$ where $\alpha(v_0) = 0$. Furthermore, since $x = y \in \text{Crit}(f_\Sigma)$, and by definition, neither \tilde{x} nor \tilde{y} are critical points of f_Y , we obtain that $Z_Y(\tilde{y})$ is a non-zero multiple of the Reeb vector field. Equation (5.13) now follows from the fact that $d\pi_\Sigma \varphi_{Z_Y}^t(\tilde{x}) = d\varphi_{Z_\Sigma}^t(x) d\pi_\Sigma$.

We now consider when $x \neq y$. Let $H = \{(v, d\varphi_{Z_Y}^t(\tilde{x})v + cZ_Y(\tilde{y})) \mid \alpha(v) = 0\}$. Then

$$d\pi_\Sigma(H) = \{(v, d\varphi_{Z_\Sigma}^t(x)v + cZ_\Sigma) \mid v \in T_x \Sigma\} = T\Delta_{f_\Sigma}.$$

By assumption, y is not a critical point of f_Σ , so $d\pi_\Sigma$ induces an isomorphism. The decomposition of $T\tilde{\Delta}_{f_Y}$ now follows immediately from the definition of g and from Eq. (5.15). \square

Proof of Proposition 5.41. We consider first the case of

$$\tilde{\text{dev}}_Y : \mathcal{M}_{H,k,\mathbb{R}\times Y;k_-,k_+}^* ((A_1, \dots, A_N); J_Y) \rightarrow Y^{2N}.$$

Suppose that $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_N) \in \mathcal{M}_{H,k,\mathbb{R}\times Y;k_-,k_+}^* ((A_1, \dots, A_N); J_Y)$. For each $i = 1, \dots, N$, let $\tilde{y}_i = \tilde{v}_i(-\infty, 0) \in Y$ and $\tilde{x}_i = \tilde{v}_i(+\infty, 0) \in Y$, with

$$\tilde{y}_1 \in W_Y^s(\tilde{q}), \tilde{x}_N \in W_Y^u(\tilde{p})$$

$$(\tilde{x}_i, \tilde{y}_{i+1}) \in \tilde{\Delta}_{f_Y} \quad \text{for } 1 \leq i \leq N - 1.$$

Let $w_i = \pi_\Sigma(\tilde{v}_i)$ and $x_i = \pi_\Sigma(\tilde{x}_i)$, $y_i = \pi_\Sigma(\tilde{y}_i)$. Then it follows that

$$y_1 \in W_\Sigma^s(q), x_N \in W_\Sigma^u(p)$$

$$(x_i, y_{i+1}) \in \Delta_{f_\Sigma} \cup \{(p, p) \mid p \in \text{Crit}(f_\Sigma)\} \quad \text{for } 1 \leq i \leq N - 1.$$

Let $S \subset \Sigma^{2N-2}$ be the appropriate product of a number of copies of Δ_{f_Σ} and of $\{(p, p) \mid p \in \text{Crit}(f_\Sigma)\}$. By Proposition 5.35, the evaluation map on $\mathcal{M}_\Sigma((A_1, \dots, A_N); J_Y)$ is transverse to S .

Then by the previous lemma,

$$TS \subset d\pi_\Sigma \left(T_{\tilde{y}_0} W_Y^s(\tilde{q}) \times T_{(\tilde{x}_1, \tilde{y}_2)} \tilde{\Delta}_{f_Y} \times \dots \times T_{(\tilde{x}_{N-1}, \tilde{y}_N)} \tilde{\Delta}_{f_Y} \times T_{\tilde{x}_N} W_Y^u(\tilde{p}) \right).$$

It suffices, therefore, to obtain transversality in the vertical direction. Notice that by rotating by the action of the Reeb vector field on \tilde{v}_i , we obtain that the image of dev contains the subspace

$$\{(a_1 R, a_1 R, a_2 R, a_2 R, \dots, a_N R, a_N R) \mid (a_1, \dots, a_N) \in \mathbb{R}^N\} \subset (TY)^{2N}.$$

In the case of the chain of pearls in Σ , each of the spheres $w_i, i = 1, \dots, N$ must either be non-constant or have a non-trivial collection of augmentation punctures. Then, by Lemma 5.43, each punctured cylinder \tilde{v}_i has different multiplicities k_i^+, k_i^- at $\pm\infty$. The moduli space of k Floer cylinders has an $(S^1)^k$ -action by rotation of the domains of the cylinders. Linearizing this action, it follows that the image of $\text{dev}_Y(\tilde{v}_i)$ contains $(k_- R, k_+ R) \in T_{\tilde{y}_i} Y \oplus T_{\tilde{x}_i} Y$. While this holds for each $i = 1, \dots, N$, we only require such a vector for one cylinder. Then, by taking this in the case of $i = 1$, we see that the following $N + 1$ vertical vectors in $(\mathbb{R}R)^{2N} \subset TY^{2N}$ are in the image of the linearized evaluation map (the first two obtained by combining the two Reeb actions on \tilde{v}_1 , the remainder by the Reeb action on $\tilde{v}_i, i \geq 2$):

$$\begin{aligned} &(R, 0, 0, \dots, 0), \\ &(0, R, 0, \dots, 0), \\ &(0, 0, R, R, 0, 0, \dots, 0), \\ &(0, 0, 0, 0, R, R, 0, \dots, 0), \\ &\dots \\ &(0, 0, \dots, 0, R, R). \end{aligned}$$

By the previous lemma, the tangent space $T_{(\tilde{x}_i, \tilde{y}_{i+1})} \tilde{\Delta}_{f_Y}$ contains at least the vertical vector $(R, g_i R)$, where $g_i := g(\tilde{x}_i, \tilde{y}_{i+1}) > 0$, for each $1 \leq i \leq N - 1$. In the vertical direction, this then contains the following $N - 1$ vectors:

$$\begin{aligned} &(0, R, g_1 R, 0, \dots, 0), \\ &(0, 0, 0, R, g_2 R, 0, \dots, 0) \\ &\dots \\ &(0, \dots, 0, R, g_{N-1} R, 0). \end{aligned}$$

We now observe that this collection of $2N$ vectors spans $(\mathbb{R}R)^{2N}$. This establishes that $\tilde{e}v_Y$ defined on $\mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^* ((A_1, \dots, A_N); J_Y)$ is transverse to $W_Y^s(\tilde{q}) \times \tilde{\Delta}_{f_Y}^N \times W_Y^u(\tilde{p})$.

We now consider the case of

$$\tilde{e}v_{W,Y} : \mathcal{M}_{H,k,W; k_+}^* ((B; A_1, \dots, A_N); J_W) \rightarrow W \times Y^{2N+1}.$$

We will show this evaluation map is transverse to

$$\tilde{S} := W_W^u(x) \times \tilde{\Delta} \times \left(\tilde{\Delta}_{f_Y} \right)^{N-1} \times W_Y^u(\tilde{p}).$$

As before, it suffices to show transversality in a vertical direction, since, by Proposition 5.35, the projections to X, Σ are transverse. More precisely, let $S \subset W \times \Sigma \times \Sigma^{2N}$ be of the form $S = W_W^u(x) \times \Delta \times S' \times W_\Sigma^u(p)$, where $S' \subset \Sigma^{2N-2}$ is a product of some number of Δ_{f_Σ} and of $\{(p, p) \mid p \in \text{Crit}(f_\Sigma)\}$ so that $TS \subset Td\pi_\Sigma(\tilde{S})$. Proposition 5.35 gives transversality of $\tilde{e}v_{W,Y}$ to S .

Notice that the tangent space $T\tilde{S}$ contains at least the following vertical vectors (we put 0 in the first component since TW has no vertical direction):

$$\begin{aligned} &(0, R, R, 0, 0, \dots, 0) \\ &(0, 0, 0, R, g_1 R, 0, \dots, 0) \\ &\dots \\ &(0, \dots, 0, R, g_{N-1} R, 0). \end{aligned}$$

Let $(\tilde{v}_1, \tilde{v}_1, \dots, \tilde{v}_N) \in \mathcal{M}_{H,k,W; k_+}^*$. The plane \tilde{v}_1 converges to a Reeb orbit of multiplicity $l = B \bullet \Sigma$. Observe that domain rotation on the plane \tilde{v}_1 then gives that $(0, lR, 0, \dots, 0) \in TW \oplus TY \oplus TY^{2N}$ is in the image of $d\tilde{e}v_{W,Y}$.

As before, the Reeb rotation on each of the punctured cylinders $\tilde{v}_1, \dots, \tilde{v}_N$ gives that the following vertical vectors are in the image of $d\tilde{e}v_{W,Y}$:

$$\begin{aligned} &(0, 0, R, R, 0, 0, \dots, 0, 0) \\ &(0, 0, 0, 0, R, R, \dots, 0, 0) \\ &(0, 0, 0, 0, \dots, 0, R, R). \end{aligned}$$

We notice then that these vectors span $0 \oplus (\mathbb{R}R)^{2N-1}$, so it follows that the evaluation map is transverse to \tilde{S} .

Finally, the transversality of the evaluation maps at augmentation punctures comes from the fact that the augmentation evaluation maps

$$\begin{aligned} \tilde{ev}_\Sigma^a \times ev_\Sigma^a &: \mathcal{M}_{H,k,\mathbb{R} \times Y; k_-, k_+}^* ((A_1, \dots, A_N); J_Y) \times \mathcal{M}_X^* ((B_1, \dots, B_k); J_W) \\ &\rightarrow \Sigma^k \times \Sigma^k \\ \tilde{ev}_\Sigma^a \times ev_\Sigma^a &: \mathcal{M}_{H,k,W; k_+}^* ((B; A_1, \dots, A_N); J_W) \times \mathcal{M}_X^* ((B_1, \dots, B_k); J_W) \\ &\rightarrow \Sigma^k \times \Sigma^k \end{aligned}$$

factor through the evaluation maps

$$\begin{aligned} ev^a \times ev^a &: \mathcal{M}_{k,\Sigma}^* ((A_1, \dots, A_N); J_\Sigma) \times \mathcal{M}_X^* ((B_1, \dots, B_k); J_W) \rightarrow \Sigma^{2k} \quad \text{and} \\ ev^a \times ev^a &: \mathcal{M}_{k,(X,\Sigma)}^* ((B; A_1, \dots, A_N); J_W) \times \mathcal{M}_X^* ((B_1, \dots, B_k); J_W) \rightarrow \Sigma^{2k}. \end{aligned}$$

The required transversality for these maps is given by Proposition 5.35. Furthermore, these evaluation maps are invariant under the domain and Reeb rotations used to obtain transversality for \tilde{ev}_Y and for $\tilde{ev}_{W,Y}$ in the vertical directions, so the transversality follows immediately. \square

6. Monotonicity and the differential

The results of the previous section show that the moduli spaces of Floer cylinders with cascades that project to *simple* chains of pearls are transverse.

We will now assume that (X, Σ, ω) is a monotone triple, as in Definition 2.4, to show that these moduli spaces are sufficient for the purposes of defining the split Floer differential. Recall that this yields that (X, ω) is spherically monotone, with $c_1(TX) = \tau_X[\omega]$ on spherical homology classes for some $\tau_X > 0$, and $A \bullet \Sigma = K\omega(A)$ for some fixed $K > 0$ and every spherical homology class A in X . It is further assumed that $\tau_\Sigma := \tau_X - K > 0$, which implies that $(\Sigma, \omega_\Sigma = \omega|_\Sigma)$ is spherically monotone with monotonicity constant τ_Σ .

6.1. Index inequalities from monotonicity and transversality

First, we consider the Fredholm index contributions of a plane in W that could appear as an augmentation plane, to obtain some bounds on the possible indices.

Lemma 6.1. *If $v: \mathbb{C} \rightarrow W$ is a J_W holomorphic plane asymptotic to a given closed Reeb orbit γ in Y , the Fredholm index for the deformations of v (as an unparameterized curve) keeping γ fixed is $|\gamma|_0$ and it is non-negative. Furthermore, if v is multiply covered, this Fredholm index is at least 2.*

Proof. The fact that the Fredholm index $\text{Ind}(v)$ in the statement is given by $|\gamma|_0$ as in (3.6) can be seen using Theorem 5.18. On the other hand, thinking of v as giving a J_X -holomorphic sphere in homology class $B \in H_2(X; \mathbb{Z})$, with an order of contact $B \bullet \Sigma$ with Σ , we see that

$$\text{Ind}(v) = 2(\langle c_1(TX), B \rangle - B \bullet \Sigma - 1) = 2(\tau_X \omega(B) - K\omega(B) - 1).$$

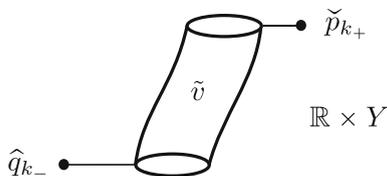


FIGURE 4. Option (1) in Proposition 6.2, call it Case 1

Since the plane is holomorphic, the class B has $\omega(B) > 0$. By our monotonicity assumptions, we have

$$\tau_X \omega(B) - K\omega(B) = (\tau_X - K)\omega(B) > 0.$$

Finally, since $\tau_X \omega(B) = \langle c_1(TX), B \rangle \in \mathbb{Z}$ and $K\omega(B) = B \bullet \Sigma \in \mathbb{Z}$, we have that $\tau_X \omega(B) - K\omega(B) > 0$ is an integer, and is thus at least 1.

It, therefore, follows that $\text{Ind}(v) \geq 0$.

Suppose now that v is a k -fold cover of an underlying simple holomorphic plane v_0 , representing classes $B = kB_0$ and B_0 , respectively. Then

$$\text{Ind}(v) + 2 = 2(\tau_X - K)\omega(kB_0) = k(\text{Ind}(v_0) + 2).$$

Hence, $\text{Ind}(v) \geq 2(k - 1)$. □

Proposition 6.2. *Any Floer cascade appearing in the differential, connecting two periodic orbits in $\mathbb{R} \times Y$, must be one of the following configurations:*

- (0) *An index 1 gradient trajectory in Y without any (non-constant) holomorphic components and without any augmentation punctures.*
- (1) *A smooth cylinder in $\mathbb{R} \times Y$ without any augmentation punctures and a non-trivial projection to Σ . The positive puncture converges to an orbit \check{p}_{k_+} and the negative puncture converges to an orbit \hat{q}_{k_-} . The difference in multiplicities of the orbits is given by $k_+ - k_- = K\omega(A)$, where $A \in H_2(\Sigma; \mathbb{Z})$ is the homology class represented by the projection of the cylinder to Σ . See Fig. 4.*
- (2) *A cylinder with one augmentation puncture and whose projection to Σ is trivial. The positive puncture converges to an orbit \check{p}_{k_+} and the negative puncture converges to an orbit \hat{q}_{k_-} . The augmentation plane has index 0. If $B \in H_2(X; \mathbb{Z})$ is the class represented by the augmentation plane, then the difference in multiplicities is given by $k_+ - k_- = K\omega(B)$. Furthermore, \check{p} and \hat{q} are critical points of f_Y contained in the same fibre of $Y \rightarrow \Sigma$, which we can write as $q = p$. See Fig. 5.*

Proof. Consider a cascade with N levels and k augmentation planes appearing in the differential $d\tilde{p}_{k_+} = \cdots + \tilde{q}_{k_-} + \cdots$. Let $A_1, \dots, A_N \in H_2(\Sigma)$ denote the homology classes of the projections to Σ , let $B_1, \dots, B_k \in H_2(X)$ denote the homology classes corresponding to the augmentation planes. Let $\gamma_i, i = 1, \dots, k$ denote the limits at the augmentation punctures, and let k_i denote their multiplicities. Let $A = \sum_{i=1}^N A_i$.

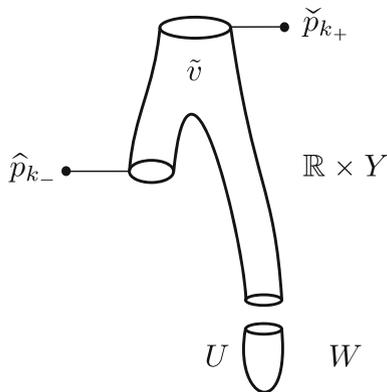


FIGURE 5. Option (2) in Proposition 6.2, call it Case 2

We, therefore, have $k_+ - k_- - \sum_{j=1}^k k_j = K\omega(A) = K \frac{\langle c_1(T\Sigma), A \rangle}{\tau_X - K}$. We also have $k_j = B_j \bullet \Sigma = K\omega(B_j)$. Notice then that $|\gamma_i|_0 = 2\langle c_1(TX), B_j \rangle - 2B_j \bullet \Sigma - 2$.

We, therefore, have

$$\begin{aligned}
 1 &= |\tilde{p}_{k_+}| - |\tilde{q}_{k_-}| = i(\tilde{p}) + M(p) - i(\tilde{q}) - M(q) \\
 &\quad + 2 \frac{\tau_X - K}{K} \left(k_+ - k_- - \sum_{j=1}^k k_j \right) + 2 \frac{\tau_X - K}{K} \sum_{j=1}^k k_j \\
 &= i(\tilde{p}) + M(p) - i(\tilde{q}) - M(q) + 2\langle c_1(T\Sigma), A \rangle + 2k + \sum_{j=1}^k |\gamma_j|_0. \tag{6.1}
 \end{aligned}$$

By Lemma 6.1, we have that for each $j = 1, \dots, k$, $|\gamma_j|_0 \geq 0$.

Consider the chain of pearls in Σ obtained by projecting the upper level of this split Floer trajectory to Σ . By Proposition 5.26, if this is a simple chain of pearls, it has Fredholm index

$$I_\Sigma := M(p) + 2\langle c_1(T\Sigma), A \rangle - M(q) + N - 1 + 2k.$$

If the chain of pearls is not simple, by monotonicity, we have that the index is at least as large as the index of the underlying simple chain of pearls.

Now let N_0 be the number of sublevels that project to constant curves in Σ and let N_1 be the number of sublevels that project to non-constant curves in Σ , $N = N_0 + N_1$. Note that by the stability condition, each cylinder that projects to a constant curve in Σ must have at least one augmentation puncture, so $N_0 \leq k$.

By transversality for simple chains of pearls (Proposition 5.26), we obtain the inequality

$$I_\Sigma \geq 2N_1 + 2k$$

by considering the two-dimensional automorphism group for the N_1 non-constant spheres and by considering the $2k$ -parameter family of moving augmentation marked points on the domains.

Combining with Eq. (6.1), we obtain

$$\begin{aligned}
 1 &= i(\tilde{p}) - i(\tilde{q}) + (I_\Sigma - N + 1) + \sum_{j=1}^k |\gamma_j|_0 \\
 1 &\geq (i(\tilde{p}) - i(\tilde{q}) + 1) + 2N_1 + 2k - N + \sum_{j=1}^k |\gamma_j|_0 \\
 1 &\geq (i(\tilde{p}) - i(\tilde{q}) + 1) + N_1 + k + (k - N_0) + \sum_{j=1}^k |\gamma_j|_0.
 \end{aligned}$$

Observe now that each term on the right-hand-side of the inequality is non-negative. In particular, there is at most one augmentation plane ($k \leq 1$) and if there is one, it must have $|\gamma_1|_0 = 2\langle c_1(TX), B_1 \rangle - 2B_1 \bullet \Sigma - 2 = 0$ (so the augmentation plane cannot be multiply covered, by Lemma 6.1).

We can further write

$$1 \geq (i(\tilde{p}) - i(\tilde{q}) + 1) + N_1 + k + (k - N_0).$$

Notice that $N_1 + 2k - N_0 \geq N$.

This inequality can be satisfied in one of the following ways:

- (0) $N = 0$. Then, either $i(\tilde{p}) = i(\tilde{q})$ or $\tilde{p} = \tilde{p}$ and $\tilde{q} = \hat{q}$. Since $N = 0$, this is a pure Morse differential term.
- (1) $N_1 = 1, N_0 = k = 0$ and $\tilde{p} = \tilde{p}, \tilde{q} = \hat{q}$. This case corresponds to a non-constant sphere in Σ without any augmentation punctures.
- (2) $N_1 = 0, k = 1, N_0 = 1$, and $\tilde{p} = \tilde{p}, \tilde{q} = \hat{q}$. In this case, the Floer cylinder has one augmentation puncture, and projects to a constant in Σ , so $q = p \in \Sigma$. □

We now consider the possible terms in the differential that connect non-constant Hamiltonian trajectories in $\mathbb{R} \times Y$ to Morse critical points in X .

Proposition 6.3. *Any Floer cascade appearing in the differential, connecting a non-constant Hamiltonian orbit $\tilde{p}_{k,+}$ in $\mathbb{R} \times Y$ to a Morse critical point x in W , consists of two levels. The upper level, in $\mathbb{R} \times Y$, projects to a point in Σ and is a cylinder asymptotic at $+\infty$ to an orbit $\check{p}_{k,+}$ and at $-\infty$ to a Reeb orbit γ in $\{-\infty\} \times Y$. This γ is the parametrized Reeb orbit associated with $\check{p}_{k,+}$.*

The lower level is a holomorphic plane in W converging to the parametrized orbit γ at ∞ and with 0 mapping to the descending manifold of the critical point x . As a parametrized curve, this has Fredholm index 1. See Fig. 6.

Proof. Suppose such a cascade occurs in the differential, connecting the non-constant orbit $\tilde{p}_{k,+}$ to the critical point x in the filling W .

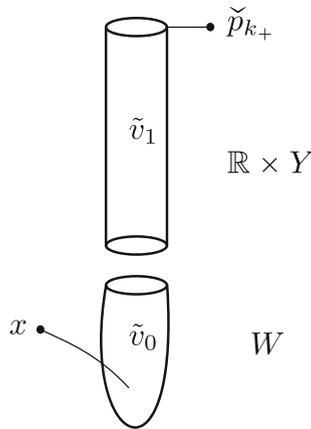


FIGURE 6. Configuration as in Proposition 6.3, call it Case 3

Let N be the number of cylinders in $\mathbb{R} \times Y$ that appear in the split Floer cylinder. Let $A_i \in H_2(\Sigma), i = 1, \dots, N$, denote the spherical classes represented by the projections of these cylinders to Σ . Let $A = \sum_{i=1}^N A_i$.

Let k be the number of augmentation planes, and let $B_j \in H_2(X), j = 1, \dots, k$ be the corresponding spherical homology classes in X . Let $\gamma_j, j = 1, \dots, k$, be the corresponding Reeb orbits with multiplicities $k_j = B_j \bullet \Sigma = K\omega(B_j)$.

Let $B \in H_2(X)$ be the spherical homology class in X represented by the lower level v_0 in W , connecting to the critical point x . Let $k_- = B \bullet \Sigma$ be the multiplicity of the orbit to which the plane v converges. As before, we have

$$k_+ - k_- - \sum_{j=1}^k k_j = K\omega(A).$$

We then have

$$\begin{aligned} 1 &= |\tilde{p}_{k_+}| - |x| \\ &= i(\tilde{p}) + M(p) + 1 - 2n + M(x) \\ &\quad + 2\frac{\tau_X - K}{K} \left(k_+ - k_- - \sum_{j=1}^k k_j + k_- + \sum_{j=1}^k k_j \right) \\ &= i(\tilde{p}) + M(p) + 1 - 2n + M(x) + 2\langle c_1(T\Sigma), A \rangle + 2\langle c_1(TX), B \rangle \\ &\quad - 2B \bullet \Sigma + 2k + \sum_{j=1}^k |\gamma_j|_0. \end{aligned} \tag{6.2}$$

Projecting to Σ , we obtain a chain of pearls with a sphere in X . Let N_0 be the number of constant spheres in Σ and let N_1 be the number of non-constant spheres in Σ , $N = N_0 + N_1$. Notice that each non-constant sphere in Σ has a two-parameter family of automorphisms, and each augmentation

marked point can be moved in a two-parameter family. Furthermore, the holomorphic sphere v_0 also has a two-parameter family of automorphisms. By passing to a simple underlying chain of pearls as necessary, and applying monotonicity and Proposition 5.26 (to $\mathcal{M}_{k,(X,\Sigma)}^*((B; A_1, \dots, A_N); x, p, J_W)$), we obtain

$$\begin{aligned} I_X &:= M(p) + 2\langle c_1(T\Sigma), A \rangle + 2(\langle c_1(TX), B \rangle - B \bullet \Sigma) \\ &\quad + M(x) - 2n + 1 + N + 2k \\ &\geq 2N_1 + 2k + 2. \end{aligned}$$

We now combine the inequality with Eq. (6.2):

$$\begin{aligned} 1 &= i(\tilde{p}) + I_X - N + \sum_{j=1}^k |\gamma_j|_0 \\ 1 &\geq i(\tilde{p}) + 2N_1 + 2k + 2 - N_0 - N_1 + \sum_{j=1}^k |\gamma_j|_0 \\ 0 &\geq i(\tilde{p}) + N_1 + k + (k + 1 - N_0) + \sum_{j=1}^k |\gamma_j|_0. \end{aligned}$$

Notice that we have $N_0 \leq k + 1$ since the first sphere in the chain of pearls with a sphere in X is allowed to be constant without any marked points. This observation together with Lemma 6.1 gives that each term on the right-hand-side of the inequality is non-negative. It follows, therefore, that each term must vanish: $N_1 = 0$, $N_0 = 1$, $k = 0$ and $\tilde{p} = \check{p}$. Notice that the Floer cylinder in $\mathbb{R} \times Y$ is contained in a single fibre of $\mathbb{R} \times Y \rightarrow \Sigma$, so the marker condition coming from \check{p} can be interpreted as a marker condition on the holomorphic plane v_0 (via the parametrized Reeb orbit γ in the statement). Without the marker condition, v_0 has Fredholm index 2, and thus with the marker constraint, it has index 1. \square

Remark 6.4. Similar analysis applied to continuation maps gives that our construction does not depend on the choices of almost complex structure J_Y , J_W or of the auxiliary Morse functions and gradient-like vector fields.

In general, $\partial^2 = 0$ is obtained through analyzing gluing and considering the boundary of 1-dimensional moduli spaces. In our situation, if additionally f_Σ and f_W are assumed to be lacunary (i.e. have no critical points of consecutive indices), all contributions to the differential of an orbit \check{p} are either of the form \hat{q} or constant orbits. This automatically gives that $\partial^2 = 0$ for split symplectic homology.

Case (2) in Proposition 6.2 allows for the existence of augmented configurations contributing to the symplectic homology differential. We will now adapt an argument originally due to Biran and Khanevsky [4] to show that if \overline{W} is a Weinstein domain (or equivalently, if W is a Weinstein manifold of finite-type), and Σ has minimal Chern number at least 2, then there can only be rigid augmentation planes if the isotropic skeleton has codimension at most 2 (in particular, $\dim_{\mathbb{R}} X = 2n \leq 4$).

Lemma 6.5. *If W is a Weinstein domain with isotropic skeleton of real codimension at least 3, then X is symplectically aspherical if and only if Σ is.*

Furthermore, any symplectic sphere in X is in the image of the inclusion

$$\iota_* : \pi_2(\Sigma) \rightarrow \pi_2(X).$$

Proof. The trivial direction is that if there exists a spherical class $A \in \pi_2(\Sigma)$ with $\omega(A) > 0$, then $\iota_*A \in \pi_2(X)$ and still has positive area.

We will now prove that any symplectic sphere in X is in the image of the inclusion. Let $C \subset W$ be the isotropic skeleton of W . Notice that by following the flow of the Liouville vector field on W , we obtain that $W \setminus C$ is symplectomorphic to a piece of the symplectization $(-\infty, a) \times Y$. Thus, we have that $X \setminus C$ is an open subset of a symplectic disk bundle over Σ (the normal bundle to Σ in X). We denote this bundle’s projection map by $\pi : X \setminus C \rightarrow \Sigma$.

Suppose $A \in \pi_2(X)$ is a spherical class with $\omega(A) > 0$. By hypothesis, the skeleton C is of codimension at least 3. We may, therefore, perturb A in a neighbourhood of the skeleton so that it does not intersect the skeleton C . If $\iota : \Sigma \rightarrow X$ and $j : X \setminus C \rightarrow X$ are the inclusion maps, then $\omega_\Sigma = \iota^*\omega$ and $\iota \circ \pi$ is homotopic to j . This implies that $\omega_X(A) = \omega_\Sigma(\pi_*A)$, and the result follows. □

Lemma 6.6. *Suppose W is a Weinstein domain with isotropic skeleton of real codimension at least 3 and Σ has minimal Chern number at least 2. Then, there do not exist any augmentation planes.*

Proof. Recall from Proposition 6.2 that an augmentation plane in the class B must have index 0, so $0 = 2(\langle c_1(TX), B \rangle - B \bullet \Sigma - 1)$. Now, $\langle c_1(TX), B \rangle - B \bullet \Sigma = (\tau_X - K)\omega(B) \geq 1$. Thus, the augmentation plane can only exist if there is a spherical class B with $(\tau_X - K)\omega(B) = 1$.

By applying Lemma 6.5, we have $B = \iota_*A$, where $A \in \pi_2(\Sigma)$ is a spherical class in Σ .

Now observe that $\langle c_1(T\Sigma), A \rangle + \langle c_1(N\Sigma), A \rangle = \langle c_1(TX), A \rangle$, so we have $\langle c_1(T\Sigma), A \rangle = (\tau_X - K)\omega_\Sigma(A)$. Hence, $1 = (\tau_X - K)\omega(A) = \langle c_1(T\Sigma), A \rangle$. This contradicts the assumption that the minimal Chern number of Σ is at least 2, so the augmentation plane cannot exist. □

Remark 6.7. Observe that this lemma applies more generally: if Σ has minimal Chern number at least 2, then an augmentation plane cannot represent a spherical class in the image of $\iota_* : \pi_2(\Sigma) \rightarrow \pi_2(X)$.

Additionally, we have that an augmentation plane cannot have image entirely contained in $\varphi(\bar{U})$. Indeed, any holomorphic sphere contained in $\varphi(\bar{U})$ will have index too high to be an augmentation plane: the J_X -holomorphic sphere with image in $\varphi(\bar{U})$ automatically comes in a two-parameter family (corresponding to the \mathbb{C}^* action on the normal bundle to Σ). To make this argument more precise, we use our index computations. Suppose a sphere in $\varphi(\bar{U})$ is an augmentation plane. It then represents a class ι_*A with $A \in H_2(\Sigma)$. By the same index argument as in Lemma 6.6, $1 = \langle c_1(T\Sigma), A \rangle$. Since the

image is assumed to be in $\varphi(\overline{\mathcal{U}})$, the projection of the curve to Σ is J_Σ -holomorphic. The index of this projection is given by $-4 + 2\langle c_1(T\Sigma), A \rangle = -2$. This must be non-negative; however, since the projection is J_Σ -holomorphic, and represents an indecomposable homology class. This contradiction then rules this possibility out.

Remark 6.8. The dichotomy between Σ with minimal Chern number equal to 1 and bigger than 1 is also explored in upcoming joint work of the first named author with D. Tonkonog, R. Vianna and W. Wu, studying the effect of the Biran circle bundle construction on superpotentials of monotone Lagrangian submanifolds [14].

7. Orientations

To orient our moduli spaces, we will take the point of view of coherent orientations, which is implemented in the Morse–Bott setting in [5, 9]. Some authors [36, 44] have used the alternative approach of canonical orientations. We find it more straightforward to use coherent orientations in our computations, especially since there are very few choices involved. Notice also that if one has a canonical orientation scheme, it is possible to extract a coherent orientation from this by making choices of preferred orientations of certain capping operators.

The geometry of our specific situation allows us to avoid some of the technical difficulties present in the general Morse–Bott situation. In particular, we have two key features that make our analysis more straightforward. First of all, we do not have any “bad” orbits appearing in our setting (recall from Sect. 3.1 that, if we take a “constant” trivialization, the Conley–Zehnder index does not depend on covering multiplicity). For another, the manifolds of orbits are all orientable, and are even oriented quite naturally by the symplectic/contact structures that exist on them.

We now recall the general method for obtaining signs in Floer homology, as first introduced in [18] and since generalized. First of all, over the space of all Fredholm Cauchy–Riemann operators, there is a determinant bundle. A choice of a section of this bundle then induces an orientation on moduli spaces of holomorphic curves. This (together with some additional choices in the Morse–Bott situation) allows us to orient all moduli spaces that occur in Floer homology. On the other hand, configurations that are counted in the differential have a natural \mathbb{R} -action on them by reparametrization, which also induces an orientation on these moduli spaces. The sign of such a term in the differential is positive if they agree and negative if they disagree.

7.1. Orienting the moduli spaces of curves

We now explain the first part of this method: how to orient the moduli spaces of Floer punctured cylinders, but without considering their constraints coming from evaluation maps. We begin by sketching the situation for the non-degenerate case and then discuss the modifications needed for the Morse–Bott situation.

First, consider all Cauchy–Riemann operators on Hermitian vector bundles over punctured spheres with fixed trivializations near the punctures $E \rightarrow \dot{S}$, as described in Sect. 5.2.1. For a given Hermitian vector bundle with fixed trivializations near the punctures and fixed, non-degenerate, asymptotic operators, the space of all Cauchy–Riemann operators with these asymptotic operators is contractible. Each such operator induces a Fredholm operator $D: W^{1,p}(\dot{S}, E) \rightarrow L^p(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$. There exists a line bundle over this space of Fredholm operators called the *determinant line bundle* and its fibre over an operator D is given by

$$\det D = (\Lambda^{\max} \ker D) \otimes_{\mathbb{R}} (\Lambda^{\max} \operatorname{coker} D)^*.$$

(See for instance [45].) An orientation corresponds to a nowhere vanishing continuous section of this determinant bundle over the space of Cauchy–Riemann operators (topologized in a way compatible with the discrete topology on the space of asymptotic operators).

In the case of non-degenerate operators, an orientation is *coherent* if it respects the gluing operation on Cauchy–Riemann operators, considered as operators $D: W^{1,p}(\dot{S}, E) \rightarrow L^p(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$. Indeed, given two such operators

$$D: W^{1,p}(\dot{S}, E) \rightarrow L^p(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

and

$$D': W^{1,p}(\dot{S}', E') \rightarrow L^p(\dot{S}', \Lambda^{0,1}T^*\dot{S}' \otimes E')$$

that have a matching asymptotic operator at a positive puncture for D and a negative puncture for D' , we may form a glued surface $\dot{S} \# \dot{S}'$, a glued bundle $E \# E' \rightarrow \dot{S} \# \dot{S}'$, and a glued operator

$$D \# D': W^{1,p}(\dot{S} \# \dot{S}', E \# E') \rightarrow L^p(\dot{S} \# \dot{S}', \Lambda^{0,1}T^*(\dot{S} \# \dot{S}') \otimes (E \# E')).$$

This operator is not unique, but depends on a contractible family of choices, in particular on a gluing parameter. If the operators D and D' are both surjective, this is explicitly constructed by a map $\ker D \oplus \ker D' \rightarrow \ker(D \# D')$, which we take to be orientation preserving. After stabilizing operators that are not surjective, we obtain a map $\det D \otimes \det D' \rightarrow \det(D \# D')$, which we require to be orientation preserving in a coherent orientation scheme. (See, for instance, [7] and [18, Section 3].) Thus, an orientation of D and an orientation of D' induce an orientation of $D \# D'$. We will refer to this as the *gluing property for coherent orientations*.

In our setting, we also require the coherent orientation to have the following two properties:

- the orientation of the direct sum of two operators is the tensor product of their orientations,
- the orientation of a complex linear operator is its canonical orientation.

Finally, we extend this coherent orientation to Cauchy–Riemann operators with possibly degenerate asymptotics, but acting on weighted spaces so

they are still Fredholm. Let δ be a vector of weights, $\delta: \Gamma \cup \{\pm\infty\} \rightarrow \mathbb{R}$ so that

$$D: W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

is a Fredholm Cauchy–Riemann operator. The operator is then conjugate to a non-degenerate operator $D^\delta: W^{1,p}(\dot{S}, E) \rightarrow L^p(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$ as given in Definition 5.15. For fixed δ , recall that D^δ is not unique, but depends on a contractible family of choices (of cut-off functions). The resulting orientation of D is then independent of the choices involved.

An orientation of $D^\delta: W^{1,p}(\dot{S}, E) \rightarrow L^p(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$ induces an orientation of $D: W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$ by this conjugation.

From [7, 18], [16, Section 1.8], a coherent orientation of the determinant bundle over non-degenerate Cauchy–Riemann operators exists and may be specified by choosing a preferred section of the determinant bundle over certain *capping operators*. These are operators whose domain is the once punctured sphere \mathbb{C} (where the puncture is positive), with a trivial Hermitian vector bundle over them and specified asymptotic operator. To achieve the two properties listed above, it suffices to enforce them on these capping operators since the linear gluing operation described in [18] respects direct sums and complex linearity.

We now describe how we orient capping operators for the relevant asymptotic operators. By Lemma 5.22, the linearized operator associated with a Floer cylinder \tilde{v} is a compact perturbation of a split operator $D_{\tilde{v}}^{\mathbb{C}} \oplus \dot{D}_w^{\Sigma}$, where $w = \pi_{\Sigma} \circ \tilde{v}$. There is also a corresponding splitting of the asymptotic operators at the asymptotic limits. In particular, \dot{D}_w^{Σ} has complex linear asymptotic operators, and thus is a compact perturbation of a complex linear Cauchy–Riemann operator. Hence, its orientation is induced by the canonical one, and is independent of choice of trivialization or of capping operator (which may always be taken to be complex linear).

We are left with the task of orienting operators with the same asymptotic operators as $D_{\tilde{v}}^{\mathbb{C}}$. By Lemma 5.24, if \tilde{v} converges at both $\pm\infty$ to a closed Hamiltonian orbit, with $\delta > 0$ sufficiently small (see Remark 5.21), the operator

$$D_{\tilde{v}}^{\mathbb{C}}: W_{\mathbf{V}_0}^{1,p,\delta}(\mathbb{R} \times S^1, \mathbb{C}) \rightarrow L^{p,\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

has Fredholm index 1, is surjective and its kernel contains an element that can be identified with the Reeb vector field. We may identify the kernel (and cokernel) of this operator with those of

$$D_{\tilde{v}}^{\mathbb{C}}: W^{1,p,-\delta}(\mathbb{R} \times S^1, \mathbb{C}) \rightarrow L^{p,-\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C})).$$

At $\pm\infty$, the $-\delta$ -perturbed asymptotic operators (see Definition 5.15) associated with $D_{\tilde{v}}^{\mathbb{C}}$ are

$$\mathbf{A}_{\pm} := - \left(J \frac{d}{dt} + \begin{pmatrix} h''(e^{b\pm})e^{b\pm} \pm \delta & 0 \\ 0 & \pm \delta \end{pmatrix} \right). \tag{7.1}$$

(The asymptotic operator at a Reeb orbit at $-\infty$ is just $-J \frac{d}{dt}$ and is $-\delta$ perturbed to give $-(J \frac{d}{dt} - \delta)$.)

We now choose capping operators for the \mathbf{A}_\pm , which determines an orientation of $D_v^{\mathbb{C}}$ by the coherent orientation scheme.

Lemma 7.1. *Let $\delta > 0$ be sufficiently small. There is a choice of capping operators with orientations for the asymptotic operators \mathbf{A}_\pm above, such that the orientation induced on*

$$D_v^{\mathbb{C}}: W^{1,p,-\delta}(\mathbb{R} \times S^1, \mathbb{C}) \rightarrow L^{p,-\delta}(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

identifies the Reeb vector field as positively oriented. (Recall that we have identified $\mathbb{R}\partial_r \oplus \mathbb{R}R$ with \mathbb{C} .)

Proof. Recall that for each $b_k > 0$ satisfying $h'(e^{b_k}) = k \in \mathbb{Z}_+$, we have a Y -parametric family of 1-periodic Hamiltonian orbits. We can associate to each of these orbits two operators A_\pm , as in (7.1). We will define capping operators

$$\Phi_k^\pm: W^{1,p}(\mathbb{C}, \mathbb{C}) \rightarrow L^p(\text{Hom}^{0,1}(T(\mathbb{C}), \mathbb{C}))$$

with these asymptotic operators.

We first define two families of auxiliary Fredholm operators. For each $k > 0$,

$$\Psi_k: W^{1,p}(\mathbb{R} \times S^1, \mathbb{C}) \rightarrow L^p(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

is an operator given by

$$\Psi_k(F)(\partial_s) = F_s + iF_t + \begin{pmatrix} a(s) - \delta & 0 \\ 0 & -\delta \end{pmatrix} F,$$

where the function $a: \mathbb{R} \rightarrow \mathbb{R}$ is such that $\lim_{s \rightarrow -\infty} a(s) = h''(e^{b_1})e^{b_1}$ and $\lim_{s \rightarrow +\infty} a(s) = h''(e^{b_k})e^{b_k}$. Let now

$$\Xi_k: W^{1,p}(\mathbb{R} \times S^1, \mathbb{C}) \rightarrow L^p(\text{Hom}^{0,1}(T(\mathbb{R} \times S^1), \mathbb{C}))$$

be an operator given by

$$\Xi_k(F)(\partial_s) = F_s + iF_t + \begin{pmatrix} h''(e^{b_k})e^{b_k} + \delta(s) & 0 \\ 0 & \delta(s) \end{pmatrix} F,$$

where $\delta: \mathbb{R} \rightarrow \mathbb{R}$ is such that $\lim_{s \rightarrow -\infty} \delta(s) = -\delta < 0$ and $\lim_{s \rightarrow +\infty} \delta(s) = \delta$.

The operators Ψ_k are isomorphisms (in particular, they are canonically oriented). This follows from an argument analogous to the proof of Lemma 5.24. A version of the same argument implies that the operators Ξ_k are Fredholm of index 1 and surjective, and that their kernels contain elements that can be identified with the Reeb vector field.

Now, pick an arbitrary capping operator Φ_1^- . Define Φ_k^- for $k > 1$ by gluing $\Phi_1^- \# \Psi_k$. Define Φ_k^+ for all $k > 0$ by gluing $\Phi_k^- \# \Xi_k$. For these choices of capping operators, $D_v^{\mathbb{C}}$ are oriented in the direction of the Reeb flow, as wanted. \square

We will now discuss how to orient

$$D_\delta: W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, \mathbb{C}) \rightarrow L^{1,p,\delta}(\dot{S}, T^*\dot{S} \otimes \mathbb{C})$$

for $\delta > 0$ and small, where \mathbf{V} is the collection of kernels of the asymptotic operators at the punctures. On the one hand, we have an orientation for

$$D_{-\delta}: W^{1,p,-\delta}(\dot{S}, \mathbb{C}) \rightarrow L^{1,p,-\delta}(\dot{S}, T^*\dot{S} \otimes \mathbb{C}),$$

where D_δ is the same Cauchy–Riemann operator, i.e. it is the restriction of $D_{-\delta}$ to its domain. From Lemma 5.20, determinant bundles of D_δ and $D_{-\delta}$ are isomorphic, which then induces an orientation on D_δ . On the other hand, D_δ is a finite dimensional stabilization of $D_\delta|_{W_0^{1,p,\delta}}$ (stabilized by $\oplus \mathbf{V}_z$), also inducing an orientation of the determinant bundle of D_δ . In the remainder of this section, we will verify these orientations are the same for suitable choices. This will then prove to be useful in analyzing the situation with constraints in Sect. 7.2.

Lemma 7.2. *Let \mathbf{A} be a degenerate asymptotic operator, let V be its kernel and let $\delta > 0$ be chosen small enough that $[-\delta, \delta] \cap \sigma(\mathbf{A}) = \{0\}$.*

Then, the kernel of $\frac{\partial}{\partial s} - \mathbf{A}: W^{1,p,-\delta}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow L^{p,-\delta}(\mathbb{R} \times S^1, \mathbb{C}^n)$ consists of constant maps with values in V , and its cokernel is trivial.

In particular, an orientation of this operator corresponds to an orientation of V .

Proof. The proof follows from expanding $L^2(S^1, \mathbb{C}^n)$ in a Hilbert basis given by eigenvectors of the asymptotic operator A seen as an elliptic self-adjoint unbounded operator on $L^2(S^1, \mathbb{C}^n)$. Then, the kernel of $\frac{\partial}{\partial s} - \mathbf{A}$ is spanned by solutions of the form $e^{\lambda s} v(t)$, where $v(t)$ is an eigenfunction for the eigenvalue λ . Since we require exponential growth of rate δ , this forces $-\delta < \lambda < \delta$. The result for the kernel now follows since 0 is the only such eigenvalue.

The statement about the cokernel now follows from similar analysis of the adjoint operator. Indeed, the adjoint is $-\frac{\partial}{\partial s} - \mathbf{A}$, and so non-zero elements of its kernel will take a similar form, but with $\lambda < -\delta$ and $\lambda > \delta$, showing no such element exists. □

We thank Chris Wendl and Richard Siefring for suggesting this argument. See also [31, Theorem 10.4.19].

Up to this point, we have coherent orientations for Cauchy–Riemann operators acting on spaces with weights

$$D: W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E).$$

We now consider orientations for operators free to move in subspaces of the kernels of the relevant asymptotic operators. Let D be a Cauchy–Riemann operator on the punctured cylinder $\dot{S} = \mathbb{R} \times S^1 \setminus \Gamma$. Let δ be a vector of sufficiently small weights that for each puncture $z_0 \in \Gamma \cup \{\pm\infty\}$, $[-|\delta_{z_0}|, |\delta_{z_0}|] \cap \sigma(\mathbf{A}_{z_0}) \subset \{0\}$. Fix a collection of subspaces \mathbf{V} so at each puncture $z_0 \in \{\pm\infty\} \cup \Gamma$ with $\delta_{z_0} > 0$, $V_{z_0} \subset \ker \mathbf{A}_{z_0}$. If $\delta_{z_0} \leq 0$, set $V_{z_0} = 0$. Then, the corresponding operator

$$D_\delta: W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

is Fredholm. This is then the operator that corresponds to having exponential convergence to an element of V_z at the puncture $z \in \Gamma \cup \{\pm\infty\}$ (and with appropriate behaviour at punctures with $\delta_z \geq 0$).

Let $D_\delta^0: W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*S \otimes E)$ be the restriction of this operator to the subspace where all sections decay to 0 at punctures for which $\delta_z > 0$.

Given a choice of ordering of the punctures in $\Gamma = \{z_1, \dots, z_m\}$, we have the following map:

$$\det V_{-\infty} \otimes \prod_{i=1}^m \det V_{z_i} \otimes \det D_\delta^0 \otimes \det V_{+\infty} \rightarrow \det D_\delta, \tag{7.2}$$

where $\det V_{z_0} = \Lambda^{\max} V_{z_0}$.

Definition 7.3. Suppose $\delta, \mathbf{V}, D_\delta$ and D_δ^0 as above. Given orientations of $V_{\pm\infty}$ and assuming that V_z is a complex linear vector space for each $z \in \Gamma$, an orientation of D_δ^0 induces an orientation on D_δ . We define then the orientation on D_δ and hence on D to be such that this map is orientation preserving.

Finally, we verify that this orientation of D_δ is consistent with the one induced from Lemma 5.20.

Lemma 7.4. *Let D be a Cauchy–Riemann operator on the punctured cylinder $\dot{S} = \mathbb{R} \times S^1 \setminus \Gamma$, and let $z_0 \in \{\pm\infty\} \cup \Gamma$, δ, δ' and \mathbf{V} as in Lemma 5.20. Suppose furthermore that the asymptotic operator at each puncture in Γ is complex linear. For each puncture $z \in \Gamma \cup \{\pm\infty\}$, let $V_z = \ker \mathbf{A}_z$ be oriented by Lemma 7.2.*

Then, for a coherent orientation, the orientation of

$$D_\delta: W_{\mathbf{V}}^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

from Definition 7.3 agrees with the orientation of the operator

$$D_{\delta'}: W^{1,p,\delta'}(\dot{S}, E) \rightarrow L^{p,\delta'}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$$

via the identification of kernel and cokernel given by Lemma 5.20.

Proof. For notational simplicity, we will consider the case $z_0 = +\infty$. The other cases will be similar, aside from a reordering of the terms.

$$\text{Let } D_+ = \frac{\partial}{\partial s} - \mathbf{A}_{+\infty}: W^{1,p,-\delta}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow L^{p,-\delta}(\mathbb{R} \times S^1, \mathbb{C}^n).$$

Let $D_\delta^0: W^{1,p,\delta}(\dot{S}, E) \rightarrow L^{p,\delta}(\dot{S}, \Lambda^{0,1}T^*\dot{S} \otimes E)$ be the restriction of D_δ to the space of sections decaying to 0 at each puncture with $\delta_z > 0$.

Observe now that $D_{\delta'}$ is homotopic to the glued operator $D_\delta^0 \# D_+$. By Lemma 7.2, the determinant bundle of D_+ is $\det V_{+\infty}$. By the gluing property for coherent orientations, the gluing map

$$\det D_\delta^0 \otimes \det D_+ \rightarrow \det D_{\delta'}$$

is orientation preserving.

By Definition 7.3, we have the following map is orientation preserving:

$$\det D_\delta^0 \otimes \det V_{+\infty} \rightarrow \det D_\delta.$$

The result now follows since, by hypothesis and Lemma 7.2, $\det V_{+\infty}$ is assumed to be oriented by $\det D_+$. □

We remark here that our asymptotic operators are either complex linear or have a kernel that is naturally identified with the Reeb vector field or with the tangent space to the contact manifold Y . The Reeb vector field and the contact form on Y induce orientations on these asymptotic operators, which are compatible with the choices of orientations of capping operators in Lemma 7.1.

We have now oriented the operators $D_v^{\mathbb{C}}$ and \dot{D}_w^{Σ} acting on spaces of sections free to move in their Morse–Bott families. Since the linearized Floer operator is a compact perturbation of their direct sum, we get induced orientations on the transverse moduli spaces of Floer cylinders with punctures.

7.2. Orientations with constraints

We have now explained how to orient all of the moduli spaces of punctured cylinders with ends free to move in the corresponding Morse–Bott families of orbits. This is not yet sufficient to orient our moduli spaces of cascades. The additional ingredient necessary is to orient moduli spaces of holomorphic curves with *constraints* on their asymptotic evaluation maps, with the asymptotic evaluation map constrained to lie in stable/unstable manifolds of critical points of the auxiliary Morse functions, or, in the case of multilevel cascades, constrained to lie in [flow] diagonals in manifolds of orbits.

Let us begin by stating the convention in [9] for how to orient a fibre sum (which agrees with [28, Convention 7.2.(b)]).

Definition 7.5. Given linear maps between oriented vector spaces $f_i: V_i \rightarrow W$, $i = 1, 2$, such that $f_1 - f_2: V_1 \oplus V_2 \rightarrow W$ is surjective, the *fibre sum orientation* on $V_1 \oplus_{f_i} V_2 = \ker(f_1 - f_2)$ is such that

- (1) $f_1 - f_2$ induces an isomorphism $(V_1 \oplus V_2)/\ker(f_1 - f_2) \rightarrow W$ which changes orientations by $(-1)^{\dim V_2 \cdot \dim W}$,
- (2) where a quotient U/V of oriented vector spaces is oriented in such a way that the isomorphism $V \oplus (U/V) \rightarrow U$ (associated with a section of the quotient short exact sequence) preserves orientations.

One key property of this orientation convention for fibre sums is that it is associative (this property specifies the orientation convention almost uniquely, as explained in [28, Remark 7.6.iii] and [34]; this was pointed out to us by Maksim Maydanskiy).

To orient our constrained moduli spaces, we follow the point of view in the literature [3, 5, 9, 20, 36]. Specifically, we begin by orienting moduli spaces of unconstrained Floer cylinders as in the previous section, by the chosen coherent orientation of Cauchy–Riemann operators with free asymptotics. We also fix orientations on all stable and unstable manifolds of the relevant manifolds of orbits (see the next section for more details), as well as on the relevant diagonals and flow diagonals. Then, we orient the moduli spaces of Floer cylinders with cascades by the rule that the asymptotic constraints are obtained as fibre products over descending and ascending manifolds of the Morse functions in the manifolds of orbits Y_k and W and as fibre products over flow diagonals and diagonals in $Y_k \times Y_k$ and in $Y \times Y$. The fibre products are oriented using Definition 7.5. For this scheme to induce a differential,

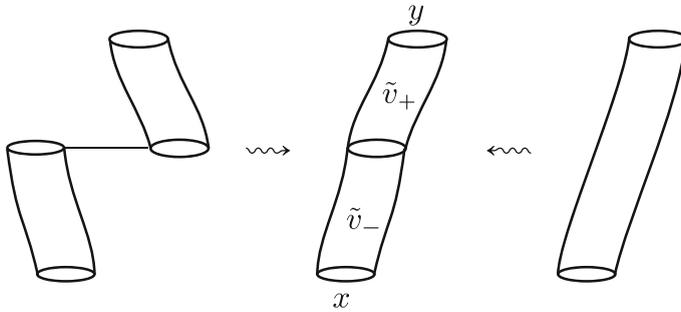


FIGURE 7. An interior point of the moduli space of cascades

we then need the orientations of the various boundary components of these moduli spaces of cascades to be consistent.

Observe that in a general Morse–Bott situation, there are additional orientation difficulties that are not present in our problem. Specifically, [5, 9, 36] have to deal with parametric families of asymptotic operators that move in the space of asymptotic operators of fixed degeneracy. In our problem, the asymptotic operators of $D_{\tilde{v}}^C$ are constant on each Morse–Bott family of orbits, dramatically simplifying the problem to consider.

We also notice another key feature regarding cascades: suppose that \tilde{v}_- and \tilde{v}_+ are two (punctured) cylinders so the asymptotic limit of \tilde{v}_- at $+\infty$ matches the asymptotic limit of \tilde{v}_+ at $-\infty$, as in the centre of Fig. 7. Let x denote the limit of \tilde{v}_- at $-\infty$ and let y denote the limit of \tilde{v}_+ at $+\infty$. This configuration can arise in two ways. It appears in the compactification of the space of cylinders with negative end in the Morse–Bott family of orbits that includes x and positive end in the Morse–Bott family of orbits that includes y (right in the figure). It also appears as the limit of a two level cascade as the length of the finite length flow line goes to 0 (left in the figure). The key point of a Morse–Bott orientation scheme is that the orientations should be such that the broken configuration of $(\tilde{v}_-, \tilde{v}_+)$ should appear as an interior point of the moduli space. We now sketch the key point that is developed in greater detail in [36, Section 8.4] (and in greater generality, and with totally real boundary conditions):

Lemma 7.6. *The coherent orientation of the Cauchy–Riemann operator that corresponds to gluing \tilde{v}_- and \tilde{v}_+ is the opposite of the orientation induced as the boundary of the fibre sum orientation of \tilde{v}_\pm over the flow diagonal.*

Sketch of proof. The linearized operator that describes the tangent space to the moduli space of pairs $(\tilde{v}_-, \tilde{v}_+)$ with matching asymptotic $\tilde{v}_-(+\infty) = \tilde{v}_+(-\infty) \in S_0$ is naturally given by:

$$\begin{aligned}
 D_{\tilde{v}_-} \oplus D_{\tilde{v}_+} : W^{1,p,\delta}(\dot{S}_-, E_-) \oplus \Delta \oplus W^{1,p,\delta}(\dot{S}_+, E_+) \\
 \longrightarrow L^{p,\delta}(\dot{S}_-, \Lambda^{0,1}T^*\dot{S}_- \otimes E_-) \oplus L^{p,\delta}(\dot{S}_+, \Lambda^{0,1}T^*\dot{S}_+ \otimes E_+),
 \end{aligned}
 \tag{7.3}$$

where $\delta > 0$ imposes exponential decay (for a weight chosen smaller than the spectral gap, as in Remark 5.21) and where $\Delta \subset TS_0 \oplus TS_0$ is the diagonal.

Notice that Δ is naturally isomorphic to TS_0 and can be oriented as the image of the map $x \rightarrow (x, x)$. Let \mathbf{A} be the (degenerate) asymptotic operator at the shared orbit.¹ After the conjugation described in Definition 5.15, we obtain non-degenerate operators $\hat{D}_{\tilde{v}_-}$ and $\hat{D}_{\tilde{v}_+}$ that have asymptotic operators $\mathbf{A} + \delta$ and $\mathbf{A} - \delta$, respectively. If we consider now a δ -perturbed Cauchy–Riemann operator coming from the linearization at a trivial orbit cylinder, $T := \partial_s - \mathbf{A} + f(s)$, with $f(s) = +\delta$ near $-\infty$ and $f(s) = -\delta$ near $+\infty$, we obtain an operator with trivial cokernel and whose kernel is identified with TS_0 (by Lemma 7.2). This allows us to identify the $D_{\tilde{v}_-} \oplus D_{\tilde{v}_+}$ in (7.3) with the triple $(\hat{D}_{\tilde{v}_-}, T, \hat{D}_{\tilde{v}_+})$. This triple can be glued to obtain the linearization of the space of cylinders with one fewer cascade, so this is naturally oriented as the boundary.

Similarly, by taking the fibre product over the diagonal, we see this as oriented with the opposite orientation of the boundary of the flow diagonal, which is oriented as $[0, \infty) \times \Delta$. This allows us to conclude that our orientation scheme is coherent with respect to the additional breakings that appear in the Morse–Bott setting. \square

7.3. A calculation of signs

Having now explained the general framework of our orientations, let us now give an explicit description of the signs associated with a Floer cylinder with cascades contributing to the differential. By Propositions 6.2 and 6.3, there are four types of contributions to the differential, referred to as Cases 0 through 3. We will explain how to determine the signs in each case.

In Case 0, we have gradient flow lines of Morse–Smale pairs (f, Z) on manifolds of orbits, which can either be (f_Y, Z_Y) on Y or $(-f_W, -Z_W)$ on W (see Definition 4.4). Let us stipulate the orientation conventions for Morse homology that we will use. The Morse complex of a Morse–Smale pair (f, Z) on a manifold S is generated by critical points of f and the differential ∂_f is such that, given $p \in \text{Crit}(f)$,

$$\partial_f(p) = \sum_{\substack{q \in \text{Crit}(f) \\ \text{ind}_f(p) - \text{ind}_f(q) = 1}} \#((W_S^s(q) \cap W_S^u(p)) / \mathbb{R}) q. \tag{7.4}$$

In this formula, we use the notation of (3.1) for critical manifolds of Z . Note that they intersect transversely, by the Morse–Smale assumption.

We need to make sense of the signed count in the formula. We will be interested in the cases where S is Σ , Y or W , all of which are oriented (by their chosen symplectic, contact and symplectic forms, respectively). If we fix an orientation on a critical manifold at a critical point p , then we get an induced orientation on the other critical manifold, by imposing that the splitting

$$T_p W_S^s(p) \oplus T_p W_S^u(p) \cong T_p S \tag{7.5}$$

¹Technically, to define this requires a trivialization of ξ along this orbit. In our setting the asymptotic operator is complex linear in the ξ direction, so this choice does not matter.

preserves orientations. Pick orientations on all unstable manifolds of Σ and W . For all $p \in \text{Crit}(f_\Sigma)$, we will assume that the orientations on critical submanifolds of Σ and Y are such that the restrictions of $\pi_\Sigma: Y \rightarrow \Sigma$ to

$$W_Y^u(\tilde{p}) \rightarrow W_\Sigma^u(p) \quad \text{and} \quad W_Y^s(\tilde{p}) \rightarrow W_\Sigma^s(p) \tag{7.6}$$

are orientation-preserving diffeomorphisms.

If $\gamma: \mathbb{R} \rightarrow S$ is a rigid flow line from q to p (critical points of consecutive indices), then it induces a diffeomorphism onto its image

$$\gamma(\mathbb{R}) \subset W_S^s(q) \cap W_S^u(p).$$

The source \mathbb{R} has its usual orientation, corresponding to increasing values of time in the orbit γ , and the image $W_S^s(q) \cap W_S^u(p)$ is a transverse intersection, and can be oriented in such a way that the splitting

$$TW_S^u(p) \cong T(W_S^s(q) \cap W_S^u(p)) \oplus TW_S^u(q) \tag{7.7}$$

is orientation-preserving (see [26, Equation (2)] for a similar convention). The flow line γ contributes to $\#((W_S^s(q) \cap W_S^u(p))/\mathbb{R})$ in (7.4) positively iff it preserves orientations. The Case 0 contribution of a flow line to the symplectic homology differential is the same as its contribution to the Morse differential.

Let us now consider Case 1, which is more interesting. Recall that such configurations consist of a Floer cylinder without augmentation punctures, together with two flow lines of Z_Y at the ends. Suppose that the Floer cylinder converges to orbits of multiplicities k_\pm at $\pm\infty$. Such a cylinder is an element of the space $\mathcal{M}_{H,0,\mathbb{R} \times Y; k_-, k_+}^*(A; J_Y)$, for some $A \in H_2(\Sigma; \mathbb{Z})$.

Cylinders with cascades that contribute to the differential in Case 1 are elements of spaces $\mathcal{M}_{H,1}^*(\widehat{q}_{k_-}, \check{p}_{k_+}; J_W)$, for $p \neq q \in \text{Crit}(f_\Sigma)$ (recall the notation in (5.11)). These spaces are unions of fibre products

$$W_Y^s(\widehat{q}) \times_{\text{ev}} \mathcal{M}_{H,0,\mathbb{R} \times Y; k_-, k_+}^*(A; J_Y) \times_{\text{ev}} W_Y^u(\check{p}) \tag{7.8}$$

defined with respect to the inclusion maps

$$W_Y^s(\widehat{q}), W_Y^u(\check{p}) \rightarrow Y$$

and the evaluation maps from (5.8)

$$e\check{v}_Y: \mathcal{M}_{H,0,\mathbb{R} \times Y; k_-, k_+}^*(A; J_Y) \rightarrow Y \times Y,$$

for appropriate $A \in H_2(\Sigma; \mathbb{Z})$. Recall the discussion of Case 0 above, which included a specification of orientations on all critical submanifolds of Y . We use the fibre sum convention (in Definition 7.5) to orient the fibre product (7.8).

Observe now that if $\mathcal{M}_{H,1}^*(\widehat{q}_{k_-}, \check{p}_{k_+}; J_W)$ is one-dimensional, then its tangent space at every point is generated by the infinitesimal translation in the s -direction on the domain of the Floer cylinder. This induces an orientation on $\mathcal{M}_{H,1}^*(\widehat{q}_{k_-}, \check{p}_{k_+}; J_W)$. Comparing this orientation with the one defined above with the fibre sum rule, we get the sign of such a contribution to the split symplectic homology differential.

We adapt the argument above to associate a sign to a contribution to the differential in Case 2. Such cascades are elements of spaces

$\mathcal{M}_{H,1}^*(\widehat{p}_{k_-}, \check{p}_{k_+}; J_W)$, for $p \in \text{Crit}(f_\Sigma)$. The analogue of (7.8) is now (using the notation of (5.10)):

$$W_Y^s(\widehat{p}) \times_{\text{ev}} \left(\mathcal{M}_X^*(B; J_W) \times_{\text{ev}} \mathcal{M}_{H,1,\mathbb{R} \times Y; k_-, k_+}^*(0; J_Y) \right) \times_{\text{ev}} W_Y^u(\check{p}). \tag{7.9}$$

Notice that in this case we also need capping operators for augmentation punctures. The asymptotic operators at such punctures are

$$\mathbf{A} = -J \frac{d}{dt},$$

which are complex linear. We may, therefore, take (canonically oriented) complex linear capping operators at these punctures. We can now orient the fibre product (7.9) using the fibre sum convention. The sign of such a contribution to the differential is obtained by comparing this orientation with the one induced by s -translation on the domain of the punctured Floer cylinder in $\mathcal{M}_{H,1,\mathbb{R} \times Y; k_-, k_+}^*(0; J_Y)$.

Finally, Case 3 Floer cylinders with cascades that contribute to the differential are elements of $\mathcal{M}_{H,1}^*(x, \check{p}_{k_+}; J_W)$, which are unions of fibre products

$$W_W^u(x) \times_{\text{ev}} \left(\mathcal{M}_{H,0,W; k_+}^*((B; 0); J_W) \times_{\text{ev}} \widetilde{\Delta} \right) \times_{\text{ev}} W_Y^u(\check{p}).$$

The relevant evaluation maps are given by factors of $\text{ev}_{W,Y}$ in (5.9). It will be useful to write an alternative description of this fibre product. Recall that $\mathcal{M}_{H,0,W; k_+}^*((B; 0); J_W)$ is a space of pairs $(\tilde{v}_0, \tilde{v}_1)$ with the following properties. The simple J_W -holomorphic cylinder $\tilde{v}_0: \mathbb{R} \times S^1 \rightarrow W$ has a removable singularity at $-\infty$, defining a pseudoholomorphic sphere in X in class $B \in H_2(X; \mathbb{Z})$, with order of contact $k_+ = B \bullet \Sigma$ at ∞ . Denote the space of such cylinders by $\mathcal{M}_H^*(B; J_W)$. The Floer cylinder $\tilde{v}_1: \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times Y$ converges at $+\infty$ to a Hamiltonian orbit of multiplicity k_+ and at $-\infty$ to a Reeb orbit of the same multiplicity in $\{-\infty\} \times Y$. It projects to a constant in Σ . Denote the space of such cylinders by $\mathcal{M}_{H,k_+}^*(0; J_Y)$. We have evaluation maps

$$\begin{aligned} (\text{ev}_+^1, \text{ev}_+^1): \mathcal{M}_H^*(B; J_W) &\rightarrow W \times Y & \text{and} \\ (\text{ev}_-^2, \text{ev}_+^2): \mathcal{M}_{H,k_+}^*(0; J_Y) &\rightarrow Y \times Y \end{aligned}$$

and can write

$$\mathcal{M}_{H,0,W; k_+}^*((B; 0); J_W) = \mathcal{M}_H^*(B; J_W) \times \mathcal{M}_{H,k_+}^*(0; J_Y)$$

and

$$\mathcal{M}_{H,0,W; k_+}^*((B; 0); J_W) \times_{\text{ev}} \widetilde{\Delta} = \mathcal{M}_H^*(B; J_W)_{\text{ev}_+^1} \times_{\text{ev}_-^2} \mathcal{M}_{H,k_+}^*(0; J_Y).$$

We now rewrite the Case 3 contributions to the differential as

$$W_W^u(x) \times_{\text{ev}_-^1} \left(\mathcal{M}_H^*(B; J_W)_{\text{ev}_+^1} \times_{\text{ev}_-^2} \mathcal{M}_{H,k_+}^*(0; J_Y) \right) \times_{\text{ev}_+^2} W_Y^u(\check{p}), \tag{7.10}$$

which is oriented using coherent orientations on the spaces of cylinders and the fibre sum orientation convention.

The space $\mathcal{M}_H^*(B; J_W)_{\text{ev}_+^1} \times_{\text{ev}_-^2} \mathcal{M}_{H,k_+}^*(0; J_Y)$ has an action of $\mathbb{R}_1 \times \mathbb{R}_2$, where the one-dimensional real vector space \mathbb{R}_1 acts by s -translation on the

domain of the cylinder in W and \mathbb{R}_2 acts by s -translation on the domain of the cylinder in $\mathbb{R} \times Y$. The sign of a Case 3 contribution to the differential is obtained by comparing the coherent/fibre product orientation on (7.10) with the usual orientation on $\mathbb{R}_1 \times \mathbb{R}_2$, corresponding to s -translation on the domain of \tilde{v}_0 followed by s -translation on the domain of \tilde{v}_1 .

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